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Realization of nonlinear composite systems

by

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Abstract: The paper studies the realization problem for series and parallel connections of nonlinear single-input single-output systems, described by higher order differential equations. Necessary and sufficient conditions are given for the existence of the classical state space realization in both cases. It is proved that post- and parallel compensators are of no help in overcoming non-realizability. Results are illustrated by an example.

Keywords: nonlinear control systems, realization, composite systems.

1. Introduction

It is well known that, unlike the linear systems, not all nonlinear higher order input-output (i/o) differential equations can be transformed into the classical state space form. The realization problem for continuous-time nonlinear systems has been extensively studied in van der Schaft (1987), Crouch, Lamnabhi-Lagarrigue (1988, 1995), Delaleau, Respondek (1995), Moog, Zheng, Liu (2002), Glad (1988) and in Kotta, Mullari (2005) the different approaches have been compared. For other topics, studied for nonlinear composite systems, see Willems (1997), Kawski (2000), Rudolph (2000), Delaleau, Rudolph (1998) and Sontag, Ingalls (2002).

This paper studies the series and parallel connections of two subsystems described by nonlinear i/o differential equations with respect to realizability and realization in the classical state space form. Unlike many other topics, this problem was first studied in the discrete-time domain. In Nõmm (2003) the necessary and sufficient realizability conditions for series and parallel connections of two subsystems were given together with the constructive procedures (up to finding the integrating factors and integrating the one-forms) to find the state equations. Nõmm et el. (2004) demonstrated how to construct a post-compensator

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or a feedback to overcome non-realizability of the control system and to make the composite system realizable. Finally, in Nõmm, Kotta, Tõnso (2005) implementation of the procedures in the computer algebra system *Mathematica* was described that allows one to find either pre-, post-, or parallel compensators that make a composite system realizable if the original control system is not.

Our goal is to extend a part of those results to the continuous-time domain. The results presented will demonstrate once again that similarity between nonlinear continuous- and discrete-time systems is limited and in some area the results surprisingly differ from each other.

The paper is organized as follows. Section 2 describes the series and parallel connections of two subsystems and gives the problem statement. It also recalls the mathematical tools and earlier results on realizability that will be used to study realizability of composite systems. Necessary and sufficient realizability conditions, together with procedures to find the state equations are developed in Section 3. In Section 4 we discuss the possibility to use pre-, post- or parallel compensators to overcome non-realizability. An example is discussed in Section 5. Conclusions are drawn in the last section.

2. Problem statement and mathematical tools

Consider a nonlinear single-input single-output system described by the following differential equation

$$\Sigma_{obj} : y^{(n)} = \phi((y, \ldots, y^{(n-1)}, u, \ldots, u^{(s)}) \tag{1}$$

where u is a real-valued scalar input, y is a real-valued scalar output, ϕ is a meromorphic function defined on $\mathbb{R}^{n+s+1}$, n and s are nonnegative integers, $n > s$. A classical state-space representation of the form

$$\begin{array}{rcl} \dot{x} &=& f(x, u), \quad x \in \mathbb{R}^n, \\ y &=& h(x) \end{array} \tag{2}$$

is called a realization of (1) if the external behavior of two systems coincides, where the behavior of (1) or (2) is the set of all pairs (u, y) that satisfy (1) or (2) (for some trajectory x), respectively. An input-output system is said to be realizable if there exists a realization of the form (2).

For a series connection of two systems, (1) and

$$\Sigma_{pst} : \tilde{y}^{(m)} = \psi((\tilde{y}, \ldots, \tilde{y}^{(m-1)}, y, \ldots, y^{(p)}) \tag{3}$$

we understand a composite system so that the output of system (1) is the input for system (3). In (3), y is a real-valued scalar input, $\tilde{y}$ is a real-valued scalar output, ψ is a meromorphic function defined on $I\!R^{m+p+1}$, m and p are nonnegative integers, $m > p$. A series connection of a system and a post-compensator will be denoted by Σ_S. A series connection of a system with pre-compensator will be denoted Σ_{SP}. Composition of systems (1) and (3) is shown schematically in Fig. 1.

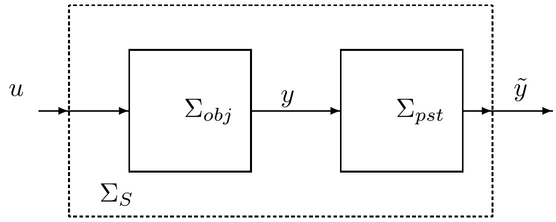

Figure 1. Series connection of a system and a post-compensator

For a parallel connection of two systems, (1) and

$$\Sigma_{par} : \bar{y}^{(m)} = \gamma(\bar{y}, \dots, \bar{y}^{(m-1)}, u, \dots, u^{(p)}) \tag{4}$$

we understand a composite system with the same input u and with the output $\hat{y} = y + \bar{y}$ being the sum of the outputs of systems Σ_{obj} and Σ_{par}. In (4), u is a real-valued scalar input, $\bar{y}$ is a real-valued scalar output, γ is a meromorphic function defined on $\mathbb{R}^{m+p+1}$, m and p are nonnegative integers, $m > p$. Parallel connection of two systems will be denoted by Σ_P. Composition of two systems (1) and (4) is shown schematically in Fig. 2.

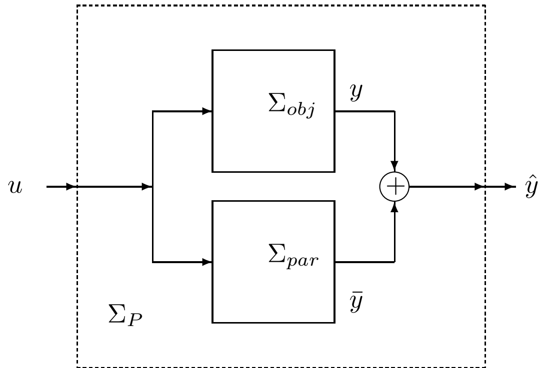

Figure 2. Parallel connection of two systems

The following problems will be studied in this paper.

1. Find the necessary and sufficient conditions for the series connection of systems (1) and (3) to be realizable.
2. Find the necessary and sufficient conditions for the parallel connection of systems (1) and (4) to be realizable.
3. For a non-realizable system Σ_{obj}, find, if possible, a pre-, post- or parallel compensator such that the respective composition of two systems is realizable.

The realization problem for the single-input single-output nonlinear continuous-time system in van der Schaft (1987) is studied using the language of vector fields. We recall now this result and use it later. Let F denote the vector field associated to system (1)

$$F = y^{(1)}\frac{\partial}{\partial y} + \ldots + \phi\frac{\partial}{\partial y^{(n-1)}} + u^{(1)}\frac{\partial}{\partial u} + \ldots + v\frac{\partial}{\partial u^{(s)}} \tag{5}$$

where $v = u^{(s+1)}$. Let $\mathcal{K}$ denote the field of meromorphic functions on a finite number of variables $\{y, \ldots y^{(n-1)}, u \ldots, u^{(s)}, v^{(k)}, k \geq 0\}$. The increasing sequence of distributions $\{S_k\}$ of

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\Big\{\frac{\partial}{\partial y}, \ldots, \frac{\partial}{\partial y^{(n-1)}}, \frac{\partial}{\partial u}, \ldots, \frac{\partial}{\partial u^{(s+1)}}\Big\}$$

is defined by

$$\begin{aligned} S_1 &= \mathrm{span}_{\mathcal{K}}\{\frac{\partial}{\partial u^{(s+1)}}\} \\ S_{k+1} &= \bar{S}_k + [F, \bar{S}_k \cap \ker \mathrm{d}y \cap \ker \mathrm{d}u], \quad k = 1, \ldots, s+1, \end{aligned} \tag{6}$$

where $\bar{S}_k$ denotes the involutive closure of the distribution S_k, and $[F, S_k]$ denotes the distribution spanned by all Lie brackets $[F, X]$, with X a vector field, contained in S_k. The following theorem states the necessary and sufficient conditions for system (1) to be realizable.

THEOREM 1 (van der Schaft, 1987) *The i/o differential equation (1) is locally realizable in the classical state space form iff all the distributions $S_1, \ldots, S_{s+2}$ are involutive.*

While the result of this theorem is valid only locally it can be easily generalized to hold generically i.e. to hold almost everywhere, except on a set of measure zero.

3. Realization of composite systems

Consider a series connection Σ_S of system (1) and a post-compensator (3), depicted in Fig. 1. To simplify the notation, we suppose that $p \leq s$. Associate with Σ_S an extended state-space system Σ_{Se} with input $v(t) = u^{(s+1)}$ and state $\theta(t) = [y, \ldots, y^{(n-1)}, \tilde{y}(t), \ldots, \tilde{y}^{(m-1)}, u, \ldots u^{(s)}]$, defined as

$$\dot{\theta} = f_e(\theta, v) \tag{7}$$

where

$$f_e = (\theta_2, \ldots, \theta_{n-1}, \phi(\cdot), \theta_{n+2}, \ldots, \theta_{n+m-1}, \psi(\cdot), \theta_{n+m+2}, \ldots, \theta_{n+m+s+1}, v)\ .$$

The realizability conditions of composite system Σ_S will be formulated in terms of the iterative Lie brackets of the vector fields

$$\tilde{F} = y^{(1)}\frac{\partial}{\partial y} + \ldots + \phi\frac{\partial}{\partial y^{(n-1)}} + \tilde{y}^{(1)}\frac{\partial}{\partial \tilde{y}} + \ldots \\ + \psi\frac{\partial}{\partial \tilde{y}^{(m-1)}} + u^{(1)}\frac{\partial}{\partial u} + \ldots + u^{(s)}\frac{\partial}{\partial u^{(s-1)}} + v\frac{\partial}{\partial u^{(s)}}, \quad (8)$$

defined by the extended system (7) and $\partial/\partial u^{(s+1)}$. Let $\tilde{\mathcal{K}}$ denote now the field of meromorphic functions in a finite number of variables $\{\theta, v^{(k)}, k \geq 0\}$. Define for system (7) the sequence of distributions $\tilde{S}_k$ of

$$\tilde{\mathcal{E}} = \mathrm{span}_{\tilde{\mathcal{K}}}\left\{\frac{\partial}{\partial y}, \ldots, \frac{\partial}{\partial y^{(n-1)}}, \frac{\partial}{\partial \tilde{y}}, \ldots, \frac{\partial}{\partial \tilde{y}^{(m-1)}}, \frac{\partial}{\partial u}, \ldots, \frac{\partial}{\partial u^{(s+1)}}\right\}$$

in analogy with (6):

$$\begin{aligned} \tilde{S}_1 &= \mathrm{span}_{\tilde{\mathcal{K}}}\left\{\frac{\partial}{\partial u^{(s+1)}}\right\} \\ \tilde{S}_{k+1} &= \bar{\tilde{S}}_k + \Big[\tilde{F}, \bar{\tilde{S}}_k \cap \ker \mathrm{d}\tilde{y} \cap \ker \mathrm{d}u\Big], \quad k = 1, \ldots, s+1. \end{aligned} \quad (9)$$

THEOREM 2 *The system Σ_S admits generically a state-space realization iff for $1 \leq k \leq s+2$ the distributions $\tilde{S}_k$ defined by (9) for the extended system Σ_{Se} are involutive. For the realizable system Σ_S, the state coordinates $x_i(\theta)$, $i = 1, \ldots, n+m$, can be found as the functions whose differential $\mathrm{d}x_i$ annihilate the vector fields $L^k_{\tilde{F}}\frac{\partial}{\partial u^{(s)}}$ and $\frac{\partial}{\partial u^{(s+1)}}$:*

$$\begin{aligned} &\langle \mathrm{d}x_i, L^k_{\tilde{F}}\frac{\partial}{\partial u^{(s)}}\rangle = 0, \quad k = 0, \ldots, s \\ &\langle \mathrm{d}x_i, \frac{\partial}{\partial u^{(s+1)}}\rangle \equiv 0. \end{aligned} \quad (10)$$

Proof. Sufficiency. Assume that the distributions $\tilde{S}_1, \ldots, \tilde{S}_{s+2}$ are involutive. If the distribution $\tilde{S}_{s+2}$ is involutive, then it can be expressed as follows (van der Schaft, 1990)

$$\tilde{S}_{s+2} = \mathrm{span}_{\tilde{\mathcal{K}}}\left\{L^s_{\tilde{F}}\frac{\partial}{\partial u^{(s)}}, \ldots, L_{\tilde{F}}\frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s+1)}}\right\}. \quad (11)$$

Because of (11), the functions x_i defined by (10) are the invariants of the distribution $\tilde{S}_{s+2}$:

$$\langle \mathrm{d}x_i, \tilde{S}_{s+2}\rangle \equiv 0. \quad (12)$$

Choose $n+m$ functionally independent functions x_i, satisfying condition (12). This can be always done since the number of functionally independent invariants of any involutive distribution equals the dimension $n+m+s+2$ of the manifold (where the distribution is defined) minus the dimension $s+2$ of the distribution. According to (9), $\tilde{S}_{s+1} \subset \tilde{S}_{s+2}$ which, together with (12), implies

$$\langle \mathrm{d}x_i, \tilde{S}_{s+1} \rangle \equiv 0. \tag{13}$$

Taking the time derivative of equation (13), we obtain

$$\langle \mathrm{d}\dot{x}_i, \tilde{S}_{s+1} \rangle = -\langle \mathrm{d}x_i, \dot{\tilde{S}}_{s+1} \rangle, \tag{14}$$

where the dot over a symbol means the Lie derivative with respect to the vectorfield $\tilde{F}$[1]. From (9), $\dot{\tilde{S}}_{s+1} \subset \tilde{S}_{s+2}$ and taking also into account (12), the right hand side of (14) equals zero. The latter yields

$$\langle \mathrm{d}\dot{x}_i, \tilde{S}_{s+1} \rangle = 0. \tag{15}$$

According to van der Schaft (1990), in the new coordinates $\big(x, u, \ldots, u^{(s+1)}\big)$,

$$\tilde{S}_{s+1} = \operatorname{span}_{\tilde{\mathcal{K}}} \Big\{ \frac{\partial}{\partial u^{(1)}}, \ldots, \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s+1)}} \Big\}$$

and together with (15) the latter means that $\mathrm{d}\dot{x}_i$ annihilates the vector fields $\partial/\partial u^{(k)}$, $k = 1, \ldots, s$, which is possible if and only if $\dot{x}_i$, $i = 1, \ldots, n+m$, do not depend on $u^{(1)}, \ldots, u^{(s)}$.

Necessity. Assume that Σ_S has a classical state space realization. Consequently, there exist an extended coordinate transformation for Σ_{Se} that preserves the control related coordinates:

$$\Phi(y, \ldots, y^{(n-1)}, \tilde{y}, \ldots, \tilde{y}^{(m-1)}, u, \ldots, u^{(s)}) = (x(\theta), u, \ldots, u^{(s)}) \tag{16}$$

such that

$$\dot{x}_i = f_i(x, u).$$

In coordinates (16) the vector field (8) for the extended system (7) takes the following form:

$$\tilde{F} = \sum_{i=n}^{n+m} f_i \frac{\partial}{\partial x_i} + u^{(1)} \frac{\partial}{\partial u} + \ldots + v \frac{\partial}{\partial u^{(s)}},$$

[1] Due to the fact that vector field $\tilde{F}$ is an operator of total time derivative, the Lie derivative $L_{\tilde{F}} G$ and time derivative $\dot{G}$ are identical where G is an arbitrary tensor object.

and therefore

$$L^k_{\tilde{F}} \frac{\partial}{\partial u^{(s)}} = (-1)^k \frac{\partial}{\partial u^{(s-k)}}, \quad k = 1, \ldots s.$$

From direct computation (see (9)), in the new coordinates, for $k = 1, \ldots, s$

$$\tilde{S}_{k+2} = \mathrm{span}_{\tilde{\mathcal{K}}} \Big\{ (-1)^k \frac{\partial}{\partial u^{(s-k)}}, \ldots, \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s+1)}} \Big\},$$

see also van der Schaft (1990). Obviously, distributions $\tilde{S}_1, \ldots, \tilde{S}_{s+2}$ are involutive. ∎

Consider now the parallel connection Σ_P of systems (1) and (4), depicted in Fig. 2. To simplify the notation, we suppose that $p \leq s$. Associate with Σ_p an extended state-space system Σ_{Pe} with input $v(t) = u^{(s+1)}$ and state $\theta(t) = [y, \ldots, y^{(n-1)}, \bar{y}(t), \ldots, \bar{y}^{(m-1)}, u, \ldots u^{(s)}]$ defined as

$$\dot{\theta} = f_e(\theta, v) \tag{17}$$

where

$$f_e = (\theta_2, \ldots, \theta_{n-1}, \phi(\cdot), \theta_{n+2}, \ldots, \theta_{n+m-1}, \gamma(\cdot), \theta_{n+m+2}, \ldots, \theta_{n+m+s+1}, v)\ .$$

The realizability condition for the composite system Σ_p will be formulated in terms of the iterative Lie brackets of the vector fields

$$\hat{F} = y^{(1)} \frac{\partial}{\partial y} + \ldots + \phi \frac{\partial}{\partial y^{(n-1)}} + \bar{y}^{(1)} \frac{\partial}{\partial \bar{y}} + \ldots + \gamma \frac{\partial}{\partial \bar{y}^{(m-1)}} + u^{(1)} \frac{\partial}{\partial u} + \ldots + v \frac{\partial}{\partial u^{(s)}}$$

defined by the extended system (17), and $\partial/\partial u^{(s+1)}$. Let $\hat{\mathcal{K}}$ denote now the field of meromorphic functions in a finite number of variables $\{\theta, v^{(k)}, k \geq 0\}$. Define for system (17) the sequence of distributions $\hat{S}_k$ of

$$\hat{\mathcal{E}} = \mathrm{span}_{\hat{\mathcal{K}}} \{ \frac{\partial}{\partial y}, \ldots, \quad \frac{\partial}{\partial y^{(n-1)}}, \frac{\partial}{\partial \bar{y}}, \ldots, \frac{\partial}{\partial \bar{y}^{(m-1)}}, \frac{\partial}{\partial u}, \ldots, \frac{\partial}{\partial u^{(s+1)}} \}$$

in analogy with (6)

$$\begin{aligned} \hat{S}_1 &= \mathrm{span}_{\hat{\mathcal{K}}} \left\{ \tfrac{\partial}{\partial u^{(s+1)}} \right\} \\ \hat{S}_{k+1} &= \bar{\hat{S}}_k + [\hat{F}, \bar{\hat{S}}_k \cap \ker \mathrm{d}\hat{y} \cap \ker \mathrm{d}u], \quad k = 1, \ldots, s+1\ . \end{aligned} \tag{18}$$

THEOREM 3 *The system Σ_P admits generically a state-space realization iff for $1 \leq k \leq s+2$ the distributions $\hat{S}_k$ defined by (18) for the extended system Σ_{Pe} are involutive. For the realizable system Σ_P, the state coordinates $x_i(\theta)$,*

$i = 1, \dots, n+m$ can be found as the functions whose differential $\mathrm{d}x_i$ annihilate the vector fields $L^k_{\hat{F}} \dfrac{\partial}{\partial u^{(s)}}$ and $\dfrac{\partial}{\partial u^{(s+1)}}$:

$$\langle \mathrm{d}x_i, L^k_{\hat{F}} \frac{\partial}{\partial u^{(s)}} \rangle = 0, \quad k = 0, \dots, s$$
$$\langle \mathrm{d}x_i, \frac{\partial}{\partial u^{(s+1)}} \rangle = 0.$$

The proof of this theorem is similar to that of Theorem 2 and is therefore omitted.

4. An attempt to overcome non-realizability

Suppose system (1) does not admit a classical state space realization. An interesting question is if it is possible to find either a pre-, post- or parallel compensator (see Figs. 3, 1 and 2, respectively) such that the composite system is realizable in the classical state space form. In the discrete-time case, for an arbitrary non-realizable higher order i/o difference equation it is always possible to construct a pre-, post- or a parallel compensator such that the resulting series composition will be realizable (Nõmm, Kotta, Tõnso, 2005). Furthermore, the series connection that results from pre-compensation is i/o equivalent to the series connection that results from post-compensation. As we will demonstrate below, this is not the case for continuous-time systems. While it is always possible to find a pre-compensator that will make the composite system realizable, there is no way to overcome non-realizability by applying a post- or parallel compensator. This shows a deep difference between the continuous and discrete time cases and is due to the difference between the properties of derivative and shift operators.

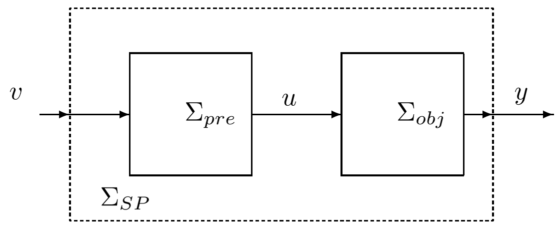

Figure 3. Series connection of the pre-compensator and the system

4.1. Case of the series connection

THEOREM 4 *For any non-realizable system of the form (1) there always exists a pre-compensator*

$$u^{(r)} = v, \quad r \leq s$$

such that the series connection is realizable.

Proof. Note that though we cannot realize every i/o equation in the classical state space form, one can always write down the so-called extended state equations that depend, besides the inputs, also on their r time derivatives. In the worst case we can take $r = s$ which will guarantee that the i/o system that relates v to y will be realizable. The realization in this case is of course, nothing else than the extended system that corresponds to the i/o equation (1). The actual value of r depends on the first noninvolutive distribution in the sequence of distributions S_k. The proof comes from applying Delaleau and Respondek (1995) result on lowering the time derivatives of inputs in the extended state space equation. In case one can lower the highest input derivative up to order r, one needs only a pre-compensator of order r. ■

THEOREM 5 *If system (1) is not realizable, then there does not exist a post-compensator of the form*

$$\tilde{y}^{(r)} = y, \quad r \in \mathbb{N} \tag{19}$$

such that the series connection is realizable.

Proof. We will demonstrate that if S_l is the first non-involutive distribution of the original system, then $\tilde{S}_l$ is also the first non-involutive distribution of the composite system and therefore, adding the post-compensator is of no help in making the system realizable.

Define by M the manifold with coordinates $\{y, \ldots, y^{(n-1)}, u, \ldots, u^{(s)}\}$ and let $\tilde{M}$ be the manifold with coordinates $\{\tilde{y}, \ldots, \tilde{y}^{(n+r-1)}, u, \ldots, u^{(s)}\}$. Constructing the series connection via the post-compensator (19) corresponds to defining the immersion $\Phi : M \to \tilde{M}$ that retains the input coordinates

$$\Phi : u^{(\lambda)} \longmapsto u^{(\lambda)} \quad \lambda = 0, \ldots, s \tag{20}$$

and relates the new and old output coordinates as follows

$$\Phi : y^{(i)} \longrightarrow \tilde{y}^{(r+i)}, \quad i = 0, \ldots, n-1.$$

The vector field (8) for the composite system takes now the form

$$\tilde{F} = F^* + T\Phi(F), \tag{21}$$

where

$$F^* := \tilde{y}^{(1)} \frac{\partial}{\partial \tilde{y}} + \ldots + \tilde{y}^{(y)} \frac{\partial}{\partial \tilde{y}^{r-1}},$$

and the image of vector field F onto manifold $T\tilde{M}$

$$T\Phi(F) = \tilde{y}^{(r+1)} \frac{\partial}{\partial \tilde{y}^{(r)}} + \ldots + \phi \circ \Phi \frac{\partial}{\partial y^{(n+r-1)}} + u^{(1)} \frac{\partial}{\partial u} + \ldots + v \frac{\partial}{\partial u^{(s)}}.$$

In a similar manner as for the original system, we can define the distributions S_k for the composite system

$$\tilde{S}_{k+1} = \bar{\tilde{S}}_k + \Big[\big(F^* + T\Phi(F)\big), \big(\bar{\tilde{S}}_k \cap \ker \mathrm{d}\tilde{y} \cap \ker \mathrm{d}u\big)\Big]. \tag{22}$$

Vector fields P and $\tilde{P}$ defined on tangent bundles of the manifolds M and $\tilde{M}$ respectively, are said to be Φ-related, if the tangent map $T\Phi : TM \to T\tilde{M}$ maps the vector field P uniquely to vector field $\tilde{P}$.

We will prove next by induction that the vector fields $L_F^k \frac{\partial}{\partial u^{(s)}}$ and $L_{\tilde{F}}^k \frac{\partial}{\partial u^{(s)}}$ are Φ-related for $k = 1, \ldots, s+1$. Due to (21),

$$L_{\tilde{F}} \frac{\partial}{\partial u^{(s)}} = L_{F^*} \frac{\partial}{\partial u^{(s)}} + L_{T\Phi(F)} \frac{\partial}{\partial u^{(s)}}, \quad L_{F^*} \frac{\partial}{\partial u^{(s)}} = 0$$

and therefore, from (20)

$$L_{\tilde{F}} \frac{\partial}{\partial u^{(s)}} = L_{T\Phi(F)} \Big(T\Phi\big(\frac{\partial}{\partial u^{(s)}}\big)\Big) = T\Phi\Big(L_F \frac{\partial}{\partial u^{(s)}}\Big) .$$

The latter equality comes from

$$T\Phi[P_1, P_2] = [T\Phi(P_1), \qquad T\Phi(P_2)] \tag{23}$$

(see proposition (2.30) in Nijmeijer and van der Schaft, 1990). Therefore, for $k = 1$, the relation holds. Assuming now that

$$L_{\tilde{F}}^k \frac{\partial}{\partial u^{(s)}} = T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big) \tag{24}$$

holds for k, we will demonstrate that relation (24) holds also for $k+1$. From direct computation and (21)

$$\begin{aligned} L_{\tilde{F}}^{k+1} \frac{\partial}{\partial u^{(s)}} &= L_{\tilde{F}}\Big(L_{\tilde{F}}^k \frac{\partial}{\partial u^{(s)}}\Big) = \Big[\tilde{F}, T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big)\Big] \\ &= \Big[F^*, T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big)\Big] + \Big[T\Phi(F), T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big)\Big] \end{aligned} \tag{25}$$

where because of (21)

$$\Big[F^*, T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big)\Big] = 0.$$

Therefore (25) yields

$$L_{\tilde{F}}^{k+1} \frac{\partial}{\partial u^{(s)}} \quad = \quad \Big[T\Phi(F), T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big)\Big] = T\Phi\Big(L_F^{k+1} \frac{\partial}{\partial u^{(s)}}\Big). \tag{26}$$

Assume now, that S_l is the first non-involutive distribution. If S_{l-1} is involutive, then (see van der Schaft, 1990)

$$S_l = S_{l-1} + \operatorname{span}_{\mathcal{K}}\{L_F^{l-2} \frac{\partial}{\partial u^{(s)}}\}$$

where

$$S_{l-1} = \operatorname{span}_{\mathcal{K}}\{L_F^{l-3} \frac{\partial}{\partial u^{(s)}}, \dots, L_F \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s+1)}}\}.$$

Since, van der Schaft (1990)

$$L_F^{\xi} \frac{\partial}{\partial u^{(s)}} = (-1)^{\xi} \frac{\partial}{\partial u^{(s-\xi)}} + \sum_{j=1}^{\xi} \gamma_{\xi,j}(\cdot) \frac{\partial}{\partial u^{(n-j)}},$$

the direct computation shows that non-zero Lie brackets of the vector fields $L_F^{\xi} \frac{\partial}{\partial u^{(s)}}, \xi = 1, \dots, l-2$, can not be the linear combination of these vector fields, and therefore, they do not belong to S_l. So, if a distribution S_l is non-involutive, then necessarily

$$\Big[L_F^{l-2} \frac{\partial}{\partial u^{(s)}}, L_F^q \frac{\partial}{\partial u^{(s)}}\Big] \neq 0 \tag{27}$$

for some $q = 0, \dots, l-3$.

From (22) we obtain

$$\tilde{S}_{k+2} = \operatorname{span}_{\tilde{\mathcal{K}}}\Big\{T\Phi\Big(L_F^k \frac{\partial}{\partial u^{(s)}}\Big), \dots, T\Phi\Big(L_F \frac{\partial}{\partial u^{(s)}}\Big), T\Phi\Big(\frac{\partial}{\partial u^{(s)}}\Big), \frac{\partial}{\partial u^{(s+1)}}\Big\}$$

$$k = 0, 1, \dots, l-2.$$

We next prove by induction that for $k = 1, \dots, l-2$ the distribution $\tilde{S}_{k+2}$ is the image $T\Phi(S_{k+2})$ of distribution S_{k+2} :

$$\tilde{S}_{k+2} = T\Phi\Big(S_{k+2}\Big). \tag{28}$$

Obviously, $\tilde{S}_1$ and $\tilde{S}_2$ are involutive and $\tilde{S}_1 = T\Phi(S_1)$, $\tilde{S}_2 = T\Phi(S_2)$. Therefore, (28) holds for $k = 0$. Assuming now that (28) holds for k, and $\tilde{S}_k$ is involutive we will show next that (28) also holds for $k+1$. Really, according to (22)

$$\begin{aligned}&\tilde{S}_{k+1} = \tilde{S}_k + \big[(F^* + T\Phi(F)), \tilde{S}_k\big] = \\ &T\Phi(S_k) + \big[F^*, T\Phi(S_k)\big] + \big[T\Phi(F), T\Phi(S_k)\big] = \\ &\qquad\qquad\qquad = T\Phi\big(S_k + [F, S_k]\big) = T\Phi(S_{k+1})\end{aligned}$$

since

$$\big[F^*, T\Phi(S_k)\big] = 0.$$

Now, if S_l is a first non-involutive distribution, then the distribution $\tilde{S}_l = T\Phi(S_l)$, as its image, can neither be involutive, since according to (23) and (27) for some $q = 0, \ldots, l-3$

$$\Big[L_{\tilde{F}}^{l-2}\frac{\partial}{\partial u^{(s)}}, L_{\tilde{F}}^{q}\frac{\partial}{\partial u^{(s)}}\Big] = T\Phi\Big(\big[L_F^{l-2}\frac{\partial}{\partial u^{(s)}}, L_F^{q}\frac{\partial}{\partial u^{(s)}}\big]\Big) \neq 0.$$

■

4.2. Case of parallel connection

Unlike the discrete-time case, where for any non-realizable system there always exists a parallel compensator such that the corresponding composite system is realizable (Nõmm, Kotta, Tõnso, 2005), continuous-time systems do not enjoy this property. The following theorem formalizes this fact.

THEOREM 6 *For a non-realizable system of the form (1) there does not exist a compensator of the form (4) such that their parallel connection Σ_p is realizable.*

Proof. Since system (1) is non-realizable, there exists an integer l, $3 \leq l \leq s+2$, such that the distribution S_l is not involutive, van der Schaft (1987). The sequence of distributions $\hat{S}_k$ for the parallel connection Σ_P of systems (1) and (4) is defined by (18). We will demonstrate that if S_l is the first non-involutive distribution of the original system, then $\hat{S}_l$ is also the first non-involutive distribution of the composite system and therefore, adding the parallel compensator is of no help in making the system realizable. According to (18), under the assumption that $\hat{S}_{l-1}$ is involutive the distribution $\hat{S}_l$ is given by

$$\hat{S}_l \quad = \quad \mathrm{span}_{\hat{\mathcal{K}}}\Big\{L_{\hat{F}}^{l-2}\frac{\partial}{\partial u^{(s)}}, \ldots, \frac{\partial}{\partial u^{(s)}}, \frac{\partial}{\partial u^{(s+1)}}\Big\}.$$

Distribution $\hat{S}_l$ is not involutive iff for some $k = 1, \ldots, l-3$

$$\Big[L_{\hat{F}}^{l-2}\frac{\partial}{\partial u^{(s)}}, L_{\hat{F}}^{k}\frac{\partial}{\partial u^{(s)}}\Big] \quad \neq \quad 0.$$

Due to the structure of functions ϕ in (1) and γ in (4), the vector field $L_{\hat{F}}^k \dfrac{\partial}{\partial u^{(s)}}$, $k = 1, \ldots l-3$ has the following form:

$$L_{\hat{F}}^k \frac{\partial}{\partial u^{(s)}} = (-1)^k \frac{\partial}{\partial u^{(s-k)}} + \sum_{i=1}^{k} \Xi_{n-i,k}\Big(y, ..., y^{(n-1)}, u, ..., u^{(s)}\Big) \frac{\partial}{\partial y^{(n-i)}} + \\ + \sum_{i=1}^{k} \bar{\Xi}_{m-i,k}\Big(\bar{y}, ..., \bar{y}^{(m-1)}, u, ..., u^{(s)}\Big) \frac{\partial}{\partial \bar{y}^{(m-i)}},$$

where Ξ and $\bar{\Xi}$ are certain meromorphic functions of their arguments. In other words, the Lie derivatives are linear combinations of the vector fields $\dfrac{\partial}{\partial u^{(s-k)}}$, $\dfrac{\partial}{\partial y^{(n-i)}}$, and $\dfrac{\partial}{\partial \bar{y}^{(m-i)}}$, $i = 1, ..., k$ and therefore

$$\left[L_{\hat{F}}^{l-2} \frac{\partial}{\partial u^{(s)}}, L_{\hat{F}}^k \frac{\partial}{\partial u^{(s)}}\right] = \left[L_F^{l-2} \frac{\partial}{\partial u^{(s)}}, L_F^k \frac{\partial}{\partial u^{(s)}}\right] + \\ +(-1)^{l-2}\left[\frac{\partial}{\partial u^{(s-l+2)}}, \sum_{i=1}^{k} \bar{\Xi}_{m-i,k}(\cdot) \frac{\partial}{\partial \bar{y}^{(m-i)}}\right] + \\ +(-1)^k \left[\frac{\partial}{\partial u^{(s-k)}}, \sum_{i=1}^{l-2} \bar{\Xi}_{m-i,k}(\cdot) \frac{\partial}{\partial \bar{y}^{(m-i)}}\right] + \\ + \left[\sum_{i=1}^{l-2} \bar{\Xi}_{m-i,k}(\cdot) \frac{\partial}{\partial \bar{y}^{(m-i)}}, \ \sum_{i=1}^{k} \bar{\Xi}_{m-i,k}(\cdot) \frac{\partial}{\partial \bar{y}^{(m-i)}}\right]. \tag{29}$$

The non-involutivity of the distribution S_l means, that for some k the first term in the right-hand side of equation (29) is non-zero. Consequently, the right-hand side can be zero only if the sum of the remaining three terms would give the same vector field with the opposite sign. The first term of expression (29) is the linear combination of vector fields $\left\{\dfrac{\partial}{\partial y^{(n-k)}}, ..., \dfrac{\partial}{\partial y^{(n)}}, \dfrac{\partial}{\partial u}, ..., \dfrac{\partial}{\partial u^{(s)}}\right\}$, but the sum of three remaining terms is the linear combination of vector fields $\left\{\dfrac{\partial}{\partial \bar{y}^{(m-k)}}, ..., \dfrac{\partial}{\partial \bar{y}^{(m)}}, \dfrac{\partial}{\partial u}, ..., \dfrac{\partial}{\partial u^{(s)}}\right\}$. Consequently, the sum on the right hand side of equation (29) can never be zero. The latter means that distribution $\hat{S}_l$ is not involutive. ■

5. Example

Consider the system

$$y^{(3)} = (y^{(1)} - u)(u^{(1)})^2 + u^{(2)}. \tag{30}$$

The vector field (5), associated to system (30) is

$$F = y^{(1)}\frac{\partial}{\partial y} + y^{(2)}\frac{\partial}{\partial y^{(1)}} + \left[(y^{(1)} - u)(u^{(1)})^2 + u^{(2)}\right]\frac{\partial}{\partial y^{(2)}} + u^{(1)}\frac{\partial}{\partial u} + u^{(2)}\frac{\partial}{\partial u^{(1)}} + u^{(3)}\frac{\partial}{\partial u^{(2)}}.$$

Compute

$$L_F\frac{\partial}{\partial u^{(2)}} = -\frac{\partial}{\partial u^{(1)}} - \frac{\partial}{\partial y^{(2)}},$$
$$L_F^2\frac{\partial}{\partial u^{(2)}} = \frac{\partial}{\partial u} + 2u^{(2)}(y^{(1)} - u)\frac{\partial}{\partial y^{(2)}} + \frac{\partial}{\partial y^{(1)}}$$

and

$$S_3 = \mathrm{span}_{\mathcal{K}}\{L_F\frac{\partial}{\partial u^{(2)}}, \frac{\partial}{\partial u^{(2)}}, \frac{\partial}{\partial u^{(3)}}\},$$
$$S_4 = \mathrm{span}_{\mathcal{K}}\left\{\frac{\partial}{\partial u^{(3)}}, \frac{\partial}{\partial u^{(2)}}, L_F\frac{\partial}{\partial u^{(2)}}, L_F^2\frac{\partial}{\partial u^{(2)}}\right\}.$$

Obviously,

$$\left[\frac{\partial}{\partial u^{(2)}}, L_F\frac{\partial}{\partial u^{(2)}}\right] = 0$$

which means that S_3 is involutive, but since

$$\left[L_F\frac{\partial}{\partial u^{(2)}}, L_F^2\frac{\partial}{\partial u^{(2)}}\right] = -2(y^{(1)} - u)\frac{\partial}{\partial y^{(1)}} \notin S_4, \tag{31}$$

S_4 is non-involutive. Therefore, system (30) is not realizable. Since S_3 is involutive, we can choose the generalized state coordinates

$$\begin{aligned} x_1 &= y \\ x_2 &= y^{(1)} \\ x_3 &= y^{(2)} - u^{(1)} \end{aligned}$$

such that $\langle \mathrm{d}x_i, S_k\rangle = 0$ holds for all $i = 1, 2, 3$ and $k = 0, 1$. This yields the generalized state equations

$$\begin{aligned} x_1^{(1)} &= x_2 \\ x_2^{(1)} &= x_3 + u^{(1)} \\ x_3^{(1)} &= (x_2 - u)(u^{(1)})^2 \end{aligned}$$

that depend on $u^{(1)}$ but not anymore on $u^{(2)}$. Therefore, according to the proof of Theorem 3 we can introduce the pre-compensator

$$v = u^{(1)}.$$

The choice of state coordinates $z_1 = y,\ z_2 = \dot{y},\ z_3 = \ddot{y} - \dot{u},\ z_4 = u$ yields the following state equations for the composite system

$$\begin{aligned} z_1^{(1)} &= z_2 \\ z_2^{(1)} &= z_3 + v \\ z_3^{(1)} &= (z_2 - z_4)v^2 \\ z_4^{(1)} &= v. \end{aligned}$$

Applying to system (30) the post-compensator $\tilde{y}^{(r)} = y$, we get the composite system with equations

$$\tilde{y}^{(r+3)} = \big(\tilde{y}^{(r+1)} - u\big)(u^{(1)})^2 + u^{(1)}. \tag{32}$$

The vector field (8), associated to composite system (32) is

$$\tilde{F} = \tilde{y}^{(1)}\frac{\partial}{\partial \tilde{y}} + \ldots + \Big[\big(\tilde{y}^{(r+1)} - u\big)(u^{(1)})^2 + u^{(2)}\Big]\frac{\partial}{\partial \tilde{y}^{(r+2)}} + u^{(1)}\frac{\partial}{\partial u} + u^{(2)}\frac{\partial}{\partial u^{(1)}} + v\frac{\partial}{\partial \ddot{u}}.$$

Compute

$$\begin{aligned} L_{\tilde{F}}\frac{\partial}{\partial u^{(2)}} &= -\frac{\partial}{\partial u^{(1)}} - \frac{\partial}{\partial \tilde{y}^{(r+2)}} \\ L^2_{\tilde{F}}\frac{\partial}{\partial u^{(2)}} &= \frac{\partial}{\partial u} + 2u^{(1)}\big(\tilde{y}^{(r+1)} - u\big)\frac{\partial}{\partial \tilde{y}^{(r+2)}} + \frac{\partial}{\partial y^{(r+1)}}\ . \end{aligned}$$

Distribution

$$\tilde{S}_4 = \mathrm{span}_{\tilde{\mathcal{K}}}\Big\{\frac{\partial}{\partial u^{(3)}}, \frac{\partial}{\partial u^{(2)}}, L_{\tilde{F}}\frac{\partial}{\partial u^{(2)}}, L^2_{\tilde{F}}\frac{\partial}{\partial u^{(2)}}\Big\}$$

is not involutive, since

$$\Big[L_{\tilde{F}}\frac{\partial}{\partial u^{(3)}}, L^2_{\tilde{F}}\frac{\partial}{\partial u^{(2)}}\Big] = -2\big(\tilde{y}^{(r+1)} - u\big)\frac{\partial}{\partial \tilde{y}^{(r+2)}} \notin \tilde{S}_4.$$

6. Conclusions

This paper studies the realization problem for continuous-time composite systems. Necessary and sufficient conditions for the series and parallel connection to be realizable are presented together with the procedure to obtain the state equations. It was proved that unlike the discrete-time case, non-realizability cannot be overcome by using a post- or parallel compensator.

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