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Singularities of the stationary domain for polydynamical systems

by

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Abstract: For k-parameter families of polydynamical systems on a 1-dimensional manifold, we classify the generic singularities of the stationary domain, when $k \leq 3$.

Keywords: singularities, generic classification, polydynamical system, stationary domain.

1. Introduction

A *polydynamical system* on a smooth manifold M is defined by a finite number of vector fields $v_1, \cdots, v_n$, $n \geq 2$. These vector fields are called *admissible velocities*. Locally, near a point of the phase space, the system can be written in the classical form

$$\dot{x} = v(x, u)$$

where x is a local smooth coordinate, $\dot{x} = dx/dt$, $v(x, u)$ is the velocity of the motion at the point x under the value u of the control parameter, $v(\cdot, i) = v_i$, $1 \leq i \leq n$, $u \in \{1, \cdots, n\}$, $n \geq 2$.

An *admissible motion* $x : t \mapsto x(t) \in M$ passing through a point $x_0 \in M$ of its trajectory is defined as a solution of the Cauchy problem

$$\dot{x} = v(x, u(t)), \qquad x(t_0) = x_0$$

where $u : t \mapsto u(t) \in \{1, \cdots, n\}$ is a piecewise constant function called an *admissible control*.

When the admissible velocities depend additionally on a parameter, we get a *family of polydynamical systems*. Its *admissible velocities* are such families of vector fields.

Given a family of polydynamical systems, a point of the product space of the phase space by the parameter space is a *stationary strategy point* if the zero velocity belongs to the convex hull of all the admissible velocities at that point. The *stationary domain* is the set of all such points.

The stationary domain plays an important role on the study of various control problems. For example, in a generic 2-dimensional case, its interior (closure) coincides with the interior (closure, respectively) of local transitivity zones, Davydov (1994, 1995). In the case of time averaged optimization on a 1-dimensional manifold, for any point of this domain there exists an admissible motion providing an averaged profit that equals the value of the density at that point (Moreira, 2005). That motivates the epithet "stationary strategy" for such a point. Besides, in time averaged optimization on the circle, stationary strategies can provide the optimal profit (Arnold, 2002) and belong to one of two types (stationary strategies and periodic motions) inside which the optimal strategy can be found (Davydov, Mena-Matos, 2007).

For 1-parameter families of control systems and of profit densities on a 1-dimensional manifold, the classification of generic singularities of the stationary domain was obtained in Moreira (2005, 2006) for polydynamical case and in Davydov, Mena-Matos (to appear) for the general case. Here we describe this classification for families of polydynamical sytems with 2- and 3-dimensional parameter and generalize the results in Moreira (2005, 2006) concerning the generic singularities of the stationary domain for any natural k. Although the proofs are based on the same ideas, it becomes more complex to prove the transversality conditions and to identify equivalent stationary domains. Besides, this generalization involves a heavier framework on the statement of the results.

2. Singularities of the stationary domain

The product space of the phase space M by the parameter space is naturally fibred over the parameter, that is, with fibres $\mathcal{F}_p = M \times \{p\}$, for every parameter value p. Two objects of the same nature defined on a fibred space are $\mathcal{F}-$*equivalent* if one of them can be carried out to the other by a fibred diffeomorphism, i. e., by a diffeomorphism that sends fibres to fibres.

Consider a family of polydynamical systems on a 1-dimensional manifold and denote by Z the union of the zeros of all the admissible velocities on the product space. A point of this set is called a *point type* A_{I_j} with $I_j = (i_1, \cdots, i_j)$, (all $j, i_1, \cdots, i_j$ are nonnegative integers and $0 \le i_1 \le \cdots \le i_j$), if at this point the germ of the set Z is $\mathcal{F}$-equivalent to the germ at the origin of the set

$$\left(x^{i_1+1} + \sum_{l=1}^{i_1} p_l x^{i_1-l}\right) \prod_{l=2}^{j} \left(x^{i_l+1} + \sum_{m=|I_l|-i_l}^{|I_l|} p_m x^{|I_l|-m}\right) = 0$$

where x and $p_1,\ p_2, \ldots$ are local coordinates along the phase space and the parameter space, respectively, $I_l = (i_1, \cdots, i_l)$, $1 \le l \le j$ and $|I_l| = l - 1 + i_1 + \cdots + i_l$.

On the space of our objects (families of vector fields, families of polydynamical systems, etc.) we introduce the fine smooth Whitney topology. A property is *generic* (or *holds generically*) if it holds for any element of some open everywhere dense subset.

THEOREM 1 *Generically, every point of the set Z of a k-parameter family of polydynamical systems on a 1-dimensional manifold is of one of the types A_{I_j} with $|I_j| \leq k$.*

THEOREM 2 *Consider the space of k-parameter families of polydynamical systems on a 1-dimensional manifold, with $k \leq 3$. Generically, the germ of the stationary domain of a family at any boundary point is, up to $\mathcal{F}$-equivalence, the germ at the origin of one of the sets from the second column of:*
- *Table 1, if $k = 1$,*
- *Tables 1 and 2, if $k = 2$,*
- *Tables 1, 2 and 3, if $k = 3$.*

Moreover, the germs of the stationary domains of a generic family, and of any other sufficiently close to it, can be reduced one to another by $\mathcal{F}$-equivalence close to the identity.

In these tables, the third and the fourth columns show the type of the point and the restriction on the number of admissible velocities, respectively.

Table 1.

N.	Singularities	Type	n
1	$x \leq 0$	A_0	≥ 2
$2_\pm$	$\pm(x^2 + p_1) \leq 0$	A_1	≥ 2
$3_\pm$	$\pm x(x + p_1) \leq 0$	$A_{0,0}$	2
$4_\pm$	$x \leq 0 \vee \pm(x + p_1) \leq 0$		≥ 3

Table 2.

N.	Singularities	Type	n
5	$x^3 + p_1 x + p_2 \leq 0$	A_2	≥ 2
6	$x(x^2 + p_1 x + p_2) \leq 0$	$A_{0,1}$	2
$7_\pm$	$x \leq 0 \vee \pm(x^2 + p_1 x + p_2) \leq 0$		≥ 3
8_*	$x(x + p_1) \leq 0 \vee x(x + p_2) \leq 0$	$A_{0,0,0}$	3
$8_\pm$	$\pm x(x + p_1) \leq 0 \vee x(x + p_2) \geq 0$		
$9_\pm$	$x \leq 0 \vee x + p_1 \leq 0 \vee \pm(x + p_2) \leq 0$		≥ 4

Observe that Tables 2 and 3 correspond to the singularities of the stationary domain at points type A_{I_j} with $|I_j| = 2$ and 3, respectively. Naturally, for any $k > 1$, the generic Table k consists of singularities of the stationary domain at points type A_{I_j} with $|I_j| = k$.

Table 3.

N.	Singularities	Type	n
$10_\pm$	$\pm(x^4+p_1x^2+p_2x+p_3)\leq 0$	A_3	≥ 2
$11_\pm$	$\pm x(x^3+p_1x^2+p_2x+p_3)\leq 0$	$A_{0,2}$	2
$12_\pm$	$x\leq 0\vee\pm(x^3+p_1x^2+p_2x+p_3)\leq 0$		≥ 3
$13_\pm$	$\pm(x^2+p_1)(x^2+p_2x+p_3)\leq 0$	$A_{1,1}$	2
14_*	$x^2+p_1\leq 0\vee x^2+p_2x+p_3\leq 0$		≥ 3
$14_\pm$	$\pm(x^2+p_1)\leq 0\vee x^2+p_2x+p_3\geq 0$		
$15_\pm$	$\pm x(x+p_1)\leq 0\vee x(x^2+p_2x+p_3)\leq 0$	$A_{0,0,1}$	3
$16_{\pm\pm}$	$x\leq 0\vee\pm(x+p_1)\leq 0\vee\pm(x^2+p_2x+p_3)\leq 0$		≥ 4
$17_{\pm\pm}$	$x(x+p_1)\leq 0\vee\pm x(x+p_2)\leq 0\vee\pm x(x+p_3)\leq 0$	$A_{0,0,0,0}$	4
18_*	$x\leq 0\vee x+p_1\leq 0\vee x+p_2\leq 0\vee x+p_3\leq 0$		≥ 5
$18_\pm$	$x\leq 0\vee x+p_1\leq 0\vee\pm(x+p_2)\leq 0\vee x+p_3\geq 0$		

3. Proofs

In this section, the main results are proved.

3.1. Proof of Theorem 1

The following statement is useful.

LEMMA 1 *Generically, every equilibrium point*[1] *of a k-parameter family of vector fields on a 1-dimensional manifold is an equilibrium point type A_l with $0\leq l\leq k$.*

Proof. In a fixed coordinate system we can consider a vector field as a function. $\mathcal{F}$-equivalence acts differently on the field and on the respective function but preserves their zero levels. But in a generic case the germ of a k-parameter family of smooth functions on the line at any point of its zero level is $\mathcal{F}$-equivalent to the germ at the origin of either x or $x^{l+1}+p_1x^{l-1}+\cdots+p_l$, $1\leq l\leq k$ (Arnold, Varchenko, Gusein-Zade, 1985). Consequently, in a generic case every equilibrium point of a k-parameter family of vector fields on a 1-dimensional manifold is a point of one of the types A_l with $0\leq l\leq k$ ■

Now let us prove the theorem. For a k-parameter family of polydynamical systems with n admissible velocities, any point of the set Z has to be an equilibrium point of exactly j admissible velocities $v_{\alpha_1},\cdots,v_{\alpha_j}$, where $j,\alpha_1,\cdots,\alpha_j$ are natural numbers not greater than n. Due to this lemma, generically this point is of type A_{i_1} for v_{α_1}, ..., A_{i_j} for v_{α_j}, where $i_1,\cdots,i_j$ are nonnegative integers. We can always suppose that $0\leq i_1\leq\cdots\leq i_j$.

[1]an equilibrium point of a family of vector fields is a point where the family vanishes

Thus, by applying the Mather Division Theorem (Golubitsky, Guillemin, 1986), we reduce this set near this point by $\mathcal{F}$-equivalence to the set

$$\left(x^{i_1+1}+\sum_{l=1}^{i_1} p_l x^{i_1-l}\right)\prod_{l=2}^{j}\left(x^{i_l+1}+\sum_{m=|I_l|-i_l}^{|I_l|} r_m(p)\, x^{|I_l|-m}\right)=0$$

near the origin ($x=0, p=0$); where all r_m are smooth functions vanishing at the origin.

Finally, Thom Transversality Theorem (Golubitsky, Guillemin, 1986) implies that $|I_j| \le k$ and that the map

$$p \mapsto (p_1, \cdots, p_{i_1}, r_{i_1+1}(p), \cdots, r_{|I_j|}(p), p_{|I_j|+1}, \cdots, p_k)$$

has maximum rank at the origin. Consequently, by selecting new coordinates

$$p_l = r_l(p), \qquad i_1+1 \le l \le |I_j|$$

we get the needed normal form of the set Z near the point under consideration, proving Theorem 1.

3.2. Proof of Theorem 2

At a boundary point of the stationary domain at least one of the admissible velocities has to vanish and all the other that do not vanish must have the same direction (Moreira, 2006).

When there is exactly one admissible velocity vanishing at the point under consideration then, in a generic case, this point is of one of the types $A_l = 0$, $0 \le l \le k$, due to Lemma 1. That leads to the singularities 1, $2_\pm$, 5 and $10_\pm$ of Tables 1, 2 and 3.

If the considered point is an equilibrium point of exactly $j \ge 2$ admissible velocities then all the singularities are obtained as an immediate consequence of Theorem 1 and by the following reasoning: if there are just j admissible velocities then, in some coordinate system, the stationary domain is the set of points where at least two admissible velocities have opposite directions; if there are more admissible velocities, we must include in the former stationary domain the region where the direction of the previous velocities is opposite to the one of all the other admissible velocities at that point.

The stability of the stationary domain up to small perturbations of a generic family of polydynamical systems follows immediately from the transversality conditions used to conclude the genericity of Theorem 1

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