

Assets/liabilities portfolio immunization as an optimization problem

by

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Abstract: The aim of this paper is to present bond portfolio immunization strategies in the case of multiple liabilities, based on single-risk or multiple-risk measure models under the assumption of multiple shocks in the term structure of interest rates referring, in particular, to Fong and Vasicek (1984), Nawalkha and Chambers (1996), Balbás and Ibáñez (1998) and Hürlimann (2002). Immunization problem is formulated as a constrained optimization problem under a fixed open loop strategy. New risk measures associated with changes of the term structure are also defined.

Keywords: immunization, optimization problem, single risk measures, multiple risk measure.

1. Introduction

Management of interest rate risk, control of changes in value of a stream of future cash flows as a result of changes in interest rates are important issues for an investor. Therefore, many researchers have examined the immunization problem for a portfolio when the investor is in debt and obliged to pay it off in a fixed horizon date. An ideal situation is when the portfolio present value is equal to the discounted worth of investor's liability at the present moment and does not fall below the target value (the terminal value of the portfolio under the scenario of no change in the interest rate) at prespecified time. Early work on immunization is based upon the Macaulay definition of duration and it is shown independently by Samuelson (1945) and Redington (1952) that if the Macaulay durations of assets and liabilities are equal, the portfolio is protected against a local parallel change in the yield curve. Fisher and Weil (1971) formalize the traditional theory of immunization, defining the conditions under which the value of an investment in a bond portfolio is hedged against any parallel shifts

in the forward rates. The main result of this theory is that immunization is achieved if the Fisher-Weil duration of the portfolio is equal to the length of the investment horizon (see Rządkowski and Zaremba, 2000). Unfortunately, this traditional approach has serious limitation since it implies arbitrage opportunity inconsistent with the rules of modern finance theory. To overcome it, the problem of immunization is formulated as a maxmin (see Bierwag and Khang, 1979) or the Bayesian strategy (see Kondratiuk-Janyska and Kałuska, 2005). Unfortunately, the derived results strongly depend on the class of shocks and they are not duration strategies in most cases. The pioneer work of Fong and Vasicek (1984) indicates new direction in studying immunization. They propose to determine the lower bound of the change in a portfolio value which leads to a risk controlling strategy. Nawalkha and Chambers (1996), Balbás and Ibáñez (1998), Balbás et al. (2002), Nawalkha et al. (2003) and Kałuska and Kondratiuk-Janyska (2004ab) follow their approach considering a single liability and a single shock (a change) in the term structure of interest rates (TSIR for short). This is far from reality since investors have to deal with multiple liabilities under multiple shocks in the TSIR. Obviously, multiple liabilities can be handled as an extension of a single liability case by separately immunizing each of the liability cash flows. However, this might not be the optimal solution. Therefore we set different lower bounds on the change of portfolio value throughout the paper but extending it to the fixed income portfolio with a given liability structure (see Hürlimann, 2002) under multiple shocks. We choose one of the open loop strategies and formulate the immunization problem as a constrained optimization problem. In consequence, we present immunization strategies based on single-risk measure models (see Section 3) or multiple-risk measure models (see Section 4). New risk measures associated with changes of the term structure are also defined. As a by-product, we generalize the risk measure defined by Nawalkha and Chambers (1996). In the end, we briefly sketch the problem of immunization when the open loop strategy is changed. No attempt has been made here to study closed loop strategies. Some preliminary results can be found in Ghezzi (1997, 1999, 2000).

2. Preliminary notations

Denote by $[0, T]$ the time interval with $t = 0$ the present moment, and let H be an investor planning horizon, $0 < H < T$, when the portfolio is rebalanced. The portfolio consists of bond inflows $A_t \geq 0$ occurring at fixed time $t \leq T$ to cover multiple liabilities L_t due at dates $t \leq T$ ($0 < t_1 < t_2 < \dots < t_d = T$). This is a typical situation e.g. when an insurance company has to discharge its random liabilities and invests the money in acquiring an immunized bond portfolio. Denote the set of available bonds by \mathcal{A} . Generally, this is an arbitrary subset of $[0, \infty)^d$ that might be nonconvex since we do not assume that the market is complete and bonds are infinitely divisible. Additionally, we assume that liabilities are nonnegative random variables. Consequently, $N_t = A_t - L_t$

is the net cash flow at time t . By $f(t, s)$ we mean an instantaneous forward rate over the time interval $[t, s]$ and therefore we can write that investing 1 at time t in a zero coupon-bond we get $\exp\left(\int_t^s f(t, u)du\right)$ at time s . The set of instantaneous forward rates $\{f(t, s) : 0 < t \leq s\}$ determines a random term structure of interest rates defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At time $t = 0$, $s \rightarrow f(0, s)$ is deterministic. Hence

- $a_t = A_t \exp\left(\int_t^H f(0, u)du\right)$ is the time- H value of A_t ,
- $l_t = L_t \exp\left(\int_t^H f(0, u)du\right)$ is the time- H value of L_t ,
- $n_t = a_t - l_t$ is the time- H value of net worth,
- $A(t) = \sum_{s \leq t} a_s$ is an accumulated value of assets,
- $L(t) = \sum_{s \leq t} l_s$ is an accumulated value of liabilities,
- $N(t) = A(t) - L(t)$,
- $V(0) = \mathbf{E} \sum_t n_t = \mathbf{E} \int_0^T dN(t)$ is the time- H average value of the portfolio of asset and liability flows if forward rates equal future spot rates.

A decision problem of an investor is to design the stream of bonds to cover a stream of liabilities. If among available bonds there are such that $N_t = 0$ for all t , then the portfolio is immunized. In reality, the market is incomplete which excludes an ideal adjustment of assets to liabilities. An investor constructing a bond portfolio meets two kinds of risks: *reinvestment* and *price*. The first one is connected with the way of reinvesting coupons paid before an investment horizon. The other appears by pricing bonds before their expiry dates. Since a portfolio value at time H depends on the reinvestment strategy, we require the following open-loop strategy:

- (a) If $t < H$ then the value of N_t at time H is equal to

$$N_t \exp\left(\int_t^H f(t, s)ds\right).$$

That means that if $N_t = A_t - L_t > 0$ for $0 < t < H$, an investor purchases $(H-t)$ -year strip bonds. Otherwise, he sells short $(H-t)$ -year strip bonds.

- (b) If $t > H$, the value of N_t at H equals

$$N_t \exp\left(-\int_H^t f(H, s)ds\right) = N_t \exp\left(\int_t^H f(H, s)ds\right),$$

which means that at time H the portfolio priced according to the TSIR is sold by the investor.

As a consequence, the value of a portfolio at H equals

$$\sum_t N_t \exp\left(\int_t^H f(t \wedge H, s)ds\right) = \sum_t n_t \exp(k(t)),$$

where

$$k(t) = \int_t^H [f(t \wedge H, s) - f(0, s)] ds \quad (1)$$

is a shock in the instantaneous forward rate and $a \wedge b = \min(a, b)$. From the investor's standpoint, the average time- H value of his portfolio under assumptions (a)-(b) is given by

$$V(k) = \mathbf{E} \left(\int_0^T \exp k(t) dN(t) \right). \quad (2)$$

The classical immunization problem consists in finding a portfolio such that $V(k) \geq V(0)$ for all $k \in \mathcal{K}$, where \mathcal{K} stands for a feasible class of shocks. Our aim is to find a lower bound on $\inf_{k \in \mathcal{K}} V(k)$ which is dependent only on bond portfolio proportions. Next, we select at $t = 0$ such a portfolio among available bonds on the market that this lower bound is maximal.

3. Single risk measure models

3.1. M-Absolute as a risk measure

The linear cash flow dispersion measure, the M-Absolute defined by Nawalkha and Chambers (1996),

$$M_{NCh} = \frac{\int_0^T |t - H| dA(t)}{\int_0^T dA(t)}$$

is an immunization risk measure designed to build immunized bond portfolios in the case of a single liability. In the case of multiple liabilities, we define the generalized M-Absolute of Nawalkha and Chambers by

$$M = \int_0^T |A(t) - A(T) + \mathbf{E}(L(T) - L(t))| dt.$$

LEMMA 3.1 *In the case of a single nonrandom liability at time H such that $N(T) = 0$,*

$$M = A(T)M_{NCh}.$$

Proof. Since $L(t) = 0$ for $t < H$ and $L(t) = L$ for $t \geq H$ thus integrating by parts we get

$$\begin{aligned} M &= \int_0^T |A(t) - L(t)| dt = \int_0^H (t - H)' A(t) dt + \int_H^T (t - H)' (A(T) - A(t)) dt \\ &= \int_0^T |t - H| dA(t), \end{aligned}$$

which is the desired conclusion. ■

Introduce a class of shocks related to Nawalkha and Chambers (1996):

$$\mathcal{K}_{NCh} = \{k(\cdot) : k_1 \leq (\mathbf{E} \exp(k(t)))' \leq k_2 \text{ for all } t \in [0, T]\},$$

where k_1, k_2 are arbitrary numbers. We assume that the function

$$t \rightarrow (\mathbf{E} \exp(k(t)))'$$

is continuous for all $k(\cdot) \in \mathcal{K}_{NCh}$. Here and subsequently, a prime denotes a partial derivative with respect to t .

We will make the following assumption:

A1. A random variable l_t is independent from the TSIR for every $t > 0$.

PROPOSITION 3.1 *Under assumption A1, a lower bound on the post-shifts change in the portfolio expected time- H value is given by*

$$\inf_{k \in \mathcal{K}_{NCh}} V(k) - V(0) \geq -k_3 M, \quad (3)$$

where $k_3 = \max(-k_1, k_2)$.

Proof. From assumption **A1**, we get

$$\begin{aligned} V(k) &= \sum_t \mathbf{E} \left[n_t e^{k(t)} \right] = \int_0^T \mathbf{E} e^{k(t)} d\mathbf{E}N(t) \\ &= \mathbf{E} e^{k(T)} \mathbf{E}N(T) - \int_0^T \mathbf{E}N(t) \left(\mathbf{E} e^{k(t)} \right)' dt \\ &= \int_0^T (\mathbf{E}N(T) - \mathbf{E}N(t)) \left(\mathbf{E} e^{k(t)} \right)' dt + \mathbf{E}N(T). \end{aligned}$$

As $\mathbf{E}N(T) = V(0)$, we have

$$\inf_{k \in \mathcal{K}_{NCh}} V(k) - V(0) \geq -k_3 \int_0^T |\mathbf{E}N(t) - \mathbf{E}N(T)| dt = -k_3 M,$$

as desired. ■

As a corollary of Proposition 3.1 we get the following immunization strategy:

$$\min_{(A_t) \in \mathcal{A}} \int_0^T |A(T) - A(t) + \mathbf{E}(L(t) - L(T))| dt.$$

EXAMPLE 3.1 Suppose that d kinds of zero coupon bonds are available on the market. The face value of the bond at the maturity date t is B_t . Denote

by $b_t = B_t \exp \int_t^H f(0, u) du$ the time- H value of B_t . An investor builds an immunized portfolio to discharge his liabilities L_t under the assumption that $\mathbf{E}N(T) = 0$. Denoting by x_t the amount of purchased t -year bond units, the immunization problem should be solved according to the model:

$$\min_{(x_t)} \int_0^T \left| \sum_{s \leq t} x_s b_s - \mathbf{E}L(t) \right| dt \quad (4)$$

subject to $\sum_t x_t b_t = \mathbf{E}L(T)$, $x_t \geq 0$ for all t .

REMARK 3.1 *If the sequence (N_t) is nonrandom, it is necessary to assume that $k_1 < 0$ and $k_2 > 0$ in order to exclude the arbitrage opportunity. Then the lower bound in (3) is negative.*

Proposition 3.1 holds under assumption **A1**. On the other hand, in the portfolio consisting of options for interest rates or other derivative instruments whose values depend on TSIR, this requirement is not satisfied. However, by omitting it we get the following result:

$$\inf_{k \in \mathcal{K}_{NC^h}^{mod}} V(k) - V(0) \geq -k_3 M_{mod}, \quad (5)$$

where

- $\mathcal{K}_{NC^h}^{mod} = \{k(\cdot) : k_1 \leq (e^{k(t)})' \leq k_2 \text{ for all } t \in [0, T], \omega \in \Omega\}$
and $k_1, k_2 \in \mathbb{R}$,
- $M_{mod} = \mathbf{E} \int_0^T |N(t) - N(T)| dt$.

3.2. Duration gap as a risk measure

Duration is the most commonly used by practitioners measure of risk in bond investing. We define

$$D_A = \int_0^T t dA(t) = \sum_t t a_t \quad \text{and} \quad D_L = \int_0^T t dL(t) = \sum_t t l_t$$

as durations of the asset and liability cash flows, respectively. It is easy to check that

$$D_A = A(T) D_A^{FW} \quad \text{and} \quad D_L = L(T) D_L^{FW},$$

where

$$D_A^{FW} = \frac{\sum_t t A_t \exp\left(-\int_0^t f(0, s) ds\right)}{\sum_t A_t \exp\left(-\int_0^t f(0, s) ds\right)}$$

is the Fisher and Weil time-honored duration.

Following Fong and Vasicek (1984) define the class of shocks:

$$\mathcal{K}_{FV} = \left\{ k(\cdot) : (\mathbf{E}e^{k(t)})' \leq k_1 \text{ for all } t \in [0, T] \right\}, \text{ where } k_1 > 0.$$

The appropriate strategy would be to hold a portfolio of assets whose schedule of cash flow at least covered the pattern of liabilities. But one might be interested in more careful scenario, that is $A(t) \geq L(t) + c(t)$ for all t , where $c(t)$ is a nonnegative function. Thus, it is worth considering immunization among portfolios belonging to the set:

$$\mathcal{A}_{\mathbf{E}}(c) = \{A(\cdot) : A(t) \geq \mathbf{E}(N(T) + L(t)) + c(t) \text{ for all } t \in [0, T]\}. \quad (6)$$

PROPOSITION 3.2 For $A(\cdot) \in \mathcal{A}_{\mathbf{E}}(c)$ and under assumption **A1**,

$$\inf_{k \in \mathcal{K}_{FV}} V(k) - V(0) \geq k_1 (D_A - \mathbf{E}D_L). \quad (7)$$

Proof. By the proof of Proposition 3.1 we get

$$\begin{aligned} V(k) - V(0) &= \int_0^T (\mathbf{E}N(T) - \mathbf{E}N(t)) (\mathbf{E}e^{k(t)})' dt \\ &\geq k_1 \int_0^T (\mathbf{E}N(T) - \mathbf{E}N(t) + c(t)) dt - \int_0^T c(t) (\mathbf{E}e^{k(t)})' dt \\ &= k_1 \mathbf{E} \int_0^T (N(T) - N(t)) dt + \int_0^T c(t) \left(k_1 - (\mathbf{E}e^{k(t)})' \right) dt. \end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned} \mathbf{E} \int_0^T (N(T) - N(t)) dt &= -\mathbf{E} \int_0^T t d(N(T) - N(t)) = \mathbf{E} \int_0^T t dN(t) \\ &= D_A - \mathbf{E}D_L, \end{aligned}$$

which completes the proof. ■

EXAMPLE 3.2 Let assume that a sequence of liabilities is nonrandom. Proposition 3.2 yields

$$\inf_{k \in \mathcal{K}_{FV}} V(k) - V(0) \geq k_1 (D_A - D_L). \quad (8)$$

Since for all $A(\cdot) \in \mathcal{A}_{\mathbf{E}}(0)$

$$D_A - D_L = \int_0^T (N(T) - N(t)) dt \leq 0$$

then the right-hand side of inequality (8) is negative and the arbitrage opportunity is excluded. Immunization in a class of feasible portfolios consists in finding a portfolio whose duration gap is the smallest.

REMARK 3.2 *The generalized M-Absolute and duration gap $D_A - \mathbf{E}D_L$ are related by*

$$|D_A - \mathbf{E}D_L| \leq M.$$

This means that the portfolio for which M is minimized has small duration gap. The proof is straightforward.

Introduce

$$\mathcal{A}(c) = \{A(\cdot) : A(t) \geq N(T) + L(t) + c(t) \text{ for all } t \in [0, T], \omega \in \Omega\}.$$

Upon dropping of assumption **A1**, the following Proposition holds:

PROPOSITION 3.3 *If $A(\cdot) \in \mathcal{A}(c)$, then*

$$\inf_{k \in \mathcal{K}_0} V(k) - V(0) \geq k_1(D_A - \mathbf{E}D_L), \quad (9)$$

where $\mathcal{K}_0 = \{k(\cdot) : \sup_{t \in [0, T]} (e^{k(t)})' \leq k_1 \text{ for all } \omega \in \Omega\}$ with k_1 being a real number.

Proof. The proof is extremely similar to that for Proposition 3.2 and we omit it. ■

3.3. Other single risk measures

In this subsection we present different lower bounds on $V(k) - V(0)$ and define new single risk measures not presented in the literature so far.

PROPOSITION 3.4 *Let assumption **A1** holds. Then for every $A(\cdot)$*

$$\inf_{k \in \mathcal{K}_1} V(k) - V(0) \geq -k_1 \left(\int_0^T [\mathbf{E}N(T) - \mathbf{E}N(t)]^2 dt \right)^{\frac{1}{2}}, \quad (10)$$

where $\mathcal{K}_1 = \left\{ k(\cdot) : \int_0^T [(\mathbf{E}e^{k(t)})']^2 dt \leq k_1^2 \right\}$ with k_1 being a positive number. *If assumption **A1** is not satisfied, then*

$$\inf_{k \in \mathcal{K}_2} V(k) - V(0) \geq -k_2 \left(\int_0^T \mathbf{E} (N(T) - N(t))^2 dt \right)^{\frac{1}{2}}, \quad (11)$$

where $\mathcal{K}_2 = \left\{ k(\cdot) : \int_0^T \mathbf{E} [(e^{k(t)})']^2 dt \leq k_2^2 \right\}$, $k_2 > 0$.

Proof. By the Cauchy-Schwarz inequality we get

$$\begin{aligned} V(k) - V(0) &= \int_0^T \mathbf{E} [N(T) - N(t)] \left(\mathbf{E} e^{k(t)} \right)' dt \\ &\geq - \left[\int_0^T \left[\left(\mathbf{E} e^{k(t)} \right)' \right]^2 dt \right]^{\frac{1}{2}} \left[\int_0^T [\mathbf{E} N(T) - \mathbf{E} N(t)]^2 dt \right]^{\frac{1}{2}}, \end{aligned}$$

which completes the proof of (10). The proof for (11) is quite similar. ■

Inequality (10) implies the following immunization problem:

find a portfolio which minimizes

$$\int_0^T [A(T) - A(t) + \mathbf{E}(L(t) - L(T))]^2 dt. \quad (12)$$

EXAMPLE 3.3 Under the assumptions like in Example 3.1, strategy (12) leads to the following optimization problem:

$$\begin{aligned} \min_{(x_t)} \int_0^T \left[\sum_{s \leq t} x_s b_s - \mathbf{E} L(t) \right]^2 dt \\ \text{subject to } \sum_t x_t b_t = \mathbf{E} L(T), \quad x_t \geq 0 \text{ for all } t. \end{aligned} \quad (13)$$

Since all the functions occurring in (13) are convex with respect to x_t , the Karush-Kuhn-Tucker conditions are necessary and sufficient for the optimality of a portfolio (see e.g. Panjer, 1998, p. 426).

REMARK 3.3

If the Hölder or Young inequalities are applied instead of the Schwarz inequality, one gets other measures in the form of $\int_0^T |\mathbf{E} N(T) - \mathbf{E} N(t)|^p dt$ for $p > 1$. However, these problems become so complicated that numerical methods are needed to obtain the optimal portfolio composition.

4. Second-order duration risk measures

The goal of researchers is to provide the simplest applicable models since investors need simplicity and power. Single-risk models such as duration are relatively easy to implement and therefore are worth studying. On the other hand, multiple-risk-measure models have been developed to improve performance relative to single-risk-measure models. Higher order duration risk measures can capture large shifts in different shape parameters of the TSIR. However, they can be difficult to implement. To keep the balance between complexity and

satisfactory adjustment to reality, we concentrate on second-order duration risk measures. Define

$$C_A = \int_0^T t^2 dA(t) = \sum_t t^2 a_t \quad \text{and} \quad C_L = \int_0^T t^2 dL(t) = \sum_t t^2 l_t$$

as convexities of the asset and liability cash flows at time H , respectively. It is easy to check that

$$C_A = A(T)C_A^{(0)} \quad \text{and} \quad C_L = L(T)C_L^{(0)},$$

where

$$C_A^{(0)} = \frac{\sum_t t^2 A_t \exp\left(-\int_0^t f(0, s) ds\right)}{\sum_t A_t \exp\left(-\int_0^t f(0, s) ds\right)}$$

is a traditionally defined assets convexity.

Introduce a set of admissible investment strategies

$$\mathcal{A}_2 = \left\{ A(\cdot) : \int_0^t (A(s) - A(T) + \mathbf{E}(L(T) - L(s))) ds \geq 0 \text{ for all } t \in [0, T] \right\}.$$

Note that \mathcal{A}_2 includes $A_{\mathbf{E}}(0)$ (see (6)).

PROPOSITION 4.1 *Under assumption **A1** and for all $A(\cdot) \in \mathcal{A}_2$*

$$\begin{aligned} \inf_{k \in \mathcal{K}_3} V(k) - V(0) &\geq \inf_{k \in \mathcal{K}_3} \left[(D_A - \mathbf{E}D_L) \left(\mathbf{E} \left(e^{k(T)} \right)' - k_1 T \right) \right] \\ &\quad + \frac{1}{2} k_1 (C_A - \mathbf{E}C_L), \end{aligned} \quad (14)$$

where $\mathcal{K}_3 = \left\{ k(\cdot) : \inf_{t \in [0, T]} \mathbf{E} \left[\left(e^{k(t)} \right)'' \right] \geq k_1 \right\}$ with k_1 being a real number.

Proof. Assumption **A1** and integration by parts lead to

$$\begin{aligned} V(k) - V(0) &= \mathbf{E} \int_0^T [N(T) - N(t)] \left(e^{k(t)} \right)' dt \\ &= \mathbf{E} \int_0^T (N(T) - N(s)) ds \mathbf{E} \left(e^{k(T)} \right)' \\ &\quad + \int_0^T \mathbf{E} \left(e^{k(t)} \right)'' \int_0^t (\mathbf{E}N(s) - \mathbf{E}N(T)) ds dt \\ &\geq (D_A - \mathbf{E}D_L) \mathbf{E} \left(e^{k(T)} \right)' + k_1 \int_0^T \int_0^t [\mathbf{E}N(s) - \mathbf{E}N(T)] ds dt. \end{aligned} \quad (15)$$

Note that

$$\begin{aligned}
 \int_0^T \int_0^t [\mathbf{E}N(s) - \mathbf{E}N(T)] ds dt &= \mathbf{E} \left[t \int_0^t (N(s) - N(T)) ds \Big|_0^T \right. \\
 &\quad \left. - \int_0^T t [N(t) - N(T)] dt \right] \\
 &= -T\mathbf{E}(D_A - D_L) - \frac{1}{2}\mathbf{E} \left[t^2 (N(t) - N(T)) \Big|_0^T \right] + \frac{1}{2}\mathbf{E} \int_0^T t^2 dN(t) \\
 &= -T(D_A - \mathbf{E}D_L) + \frac{1}{2}(C_A - \mathbf{E}C_L). \tag{16}
 \end{aligned}$$

From (15) and (16) we get (14), which is the desired conclusion. \blacksquare

As a corollary of Proposition 4.1 we get the following immunization strategy:

$$\text{choose a portfolio which minimizes } C_A \tag{17}$$

$$\text{subject to } D_A = \mathbf{E}D_L, \mathbf{E} \int_0^t [N(s) - N(T)] ds \geq 0 \text{ for all } t \in [0, T]$$

under the condition that $k_1 < 0$. Hence an investor's aim is to find a duration-matching portfolio with the lowest convexity. In this case, a solution to (17) is a bullet portfolio. On the other hand, if $k_1 > 0$ an investor should

$$\text{construct a portfolio which maximizes } C_A \tag{18}$$

$$\text{subject to } D_A = \mathbf{E}D_L, \mathbf{E} \int_0^t [N(s) - N(T)] ds \geq 0 \text{ for all } t \in [0, T].$$

It is well-known that a barbell portfolio has the highest convexity (Zaremba, 1998, Zaremba and Smoleński, 2000ab). A selection of a unique bond portfolio corresponding to the strategy (17) or (18), respectively, under the conditions like in Example 1 consists in solving an equivalent problem:

$$\min_{(x_t)} \sum_t t^2 x_t b_t \tag{19}$$

$$\text{subject to } \sum_t t x_t b_t = \mathbf{E}D_L, \sum_t x_t b_t = \mathbf{E}L(T),$$

$$\int_0^t \sum_{u \leq s} x_u b_u ds \geq \int_0^t \mathbf{E}L(s) ds, \quad x_t \geq 0, \text{ for all } t \in [0, T].$$

Since the function occurring in (19) is linear, the Karush-Kuhn-Tucker conditions are necessary and sufficient for the optimality of a portfolio.

REMARK 4.1 *If it is impossible to construct a bond portfolio such as $D_A = \mathbf{E}D_L$, then this condition should be replaced by $D_A - \mathbf{E}D_L = G$, where G is a fixed duration gap. Since $0 \leq \int_0^T [\mathbf{E}N(s) - \mathbf{E}N(T)] ds = \mathbf{E}D_L - D_A$ holds, G must be a negative number.*

Proposition 4.1 holds for any sequence (N_t) such as $\int_0^t (\mathbf{E}N(s) - \mathbf{E}N(T)) ds \geq 0$ for all t . Now we drop this constraint. Put

$$\mathcal{K}_4 = \left\{ k(\cdot) : \mathbf{E} \int_0^T \left[\left(e^{k(t)} \right)'' \right]^2 dt \leq k_1^2 \right\}, \quad k_1 > 0.$$

PROPOSITION 4.2 *Under assumption **A1** and for an arbitrary sequence (N_t)*

$$\begin{aligned} \inf_{k \in \mathcal{K}_4} V(k) - V(0) &\geq \inf_{k \in \mathcal{K}_4} \left[(D_A - \mathbf{E}D_L) \mathbf{E} \left(e^{k(T)} \right)' \right] \\ &\quad - k_1 \left[\mathbf{E} \int_0^T \left(\int_0^t (N(s) - N(T)) ds \right)^2 dt \right]^{\frac{1}{2}}. \end{aligned} \quad (20)$$

Proof. Analysis similar to that in the proof of Proposition 4.1 shows that

$$V(k) - V(0) \geq \mathbf{E} \left[(D_A - D_L) \left(e^{k(T)} \right)' \right] + \mathbf{E} \int_0^T \left(e^{k(t)} \right)'' \int_0^t (N(s) - N(T)) ds dt.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbf{E} \int_0^T \left(e^{k(t)} \right)'' \int_0^t (N(s) - N(T)) ds dt &\geq - \left[\mathbf{E} \int_0^T \left(\left(e^{k(t)} \right)'' \right)^2 dt \right]^{\frac{1}{2}} \\ &\quad \times \left[\mathbf{E} \int_0^T \left(\int_0^t (N(s) - N(T)) ds \right)^2 dt \right]^{\frac{1}{2}}, \end{aligned}$$

which is our claim. ■

From Proposition 4.2 we obtain the optimization problem:

$$\begin{aligned} \min_{(A_t) \in \mathcal{A}} \mathbf{E} \int_0^T \left(\int_0^t (A(s) - L(s)) ds - t(A(T) - L(T)) \right)^2 dt \quad (21) \\ \text{subject to } D_A = \mathbf{E}D_L, \end{aligned}$$

where \mathcal{A} is a class of available asset cash flows.

EXAMPLE 4.1 Reconsider assumptions in Example 3.1 by adding conditions that $L(t)$ is nonrandom and $A(T) = L(T)$. Thus, strategy (21) leads to the

following optimization problem:

$$\min_{(x_t)} \int_0^T \left(\int_0^t \left(\sum_{u \leq s} x_u b_u - L(s) \right) ds \right)^2 dt \quad (22)$$

$$\text{subject to } \sum_t x_t b_t = L(T), \sum_t t x_t b_t = D_L, x_t \geq 0 \text{ for all } t.$$

REMARK 4.2 *The results of the paper can be easily modified to different open loop strategies. We may, e.g., choose assets satisfying the following conditions:*

C1. $N_{t_1} \geq 0$ which states that L_{t_1} is discharged,

C2. $N_{t_2} + N_{t_1} \exp\left(\int_{t_1}^{t_2} f(0, s) ds\right) \geq 0$ which states that L_{t_2} is discharged and so on, till $t_k = H$ when the portfolio is sold. The above conditions can be rewritten as follows: for every $t \leq H$,

$$N(t) \geq 0.$$

Under this scenario, the time- H value of the portfolio is given by

$$V(k) = \mathbf{E} \left(\int_0^T \exp(\tilde{k}(t)) dN(t) \right),$$

where

$$\tilde{k}(t) = \begin{cases} \sum_{t \leq t_i \leq H} \int_{t_i}^{t_{i+1}} (f(t_i, s) - f(0, s)) ds & \text{for } t < H \\ - \int_H^t (f(H, s) - f(0, s)) ds & \text{for } t \geq H \end{cases}$$

Obviously, $\tilde{k}(t) = k(t)$ for $t > H$ but $\tilde{k}(t)$ might be different from $k(t)$ defined in (1). Nevertheless, by appropriately modifying the class of shocks in Propositions 1-6, one can derive their counterparts for the new reinvestment strategy. However, the condition: $N(t) \geq 0$ for all $t \leq H$, should be added to the constrained set of optimization problems.

5. Conclusions

The traditional approach to the problem of immunization is to construct a portfolio such that $V(k) \geq V(0)$ for all $k \in \mathcal{K}$, where \mathcal{K} is a feasible class of shocks. Unfortunately, this implies arbitrage opportunity inconsistent with the rules of modern finance theory. One of the way to overcome it is to view the immunization as a maxmin strategy (see Bierwag and Khang, 1979) guaranteeing the highest return of a portfolio. However, finding direct solutions is very difficult on the incomplete market. Hence, Fong and Vasicek (1984) propose to set the lower bound of the change in a portfolio value which lead to a risk controlling

strategy. We follow their approach by maximizing lower bounds of the change in a portfolio value considering the case of multiple liabilities and shocks in the TSIR in a model of discrete time. This approach implies new strategies of immunization that consist in maximizing either single-risk or multiple-risk measures under a fixed open loop strategy. We briefly discuss the problem of immunization when the open loop strategy is changed.

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Note added in proof

During the review process we found a paper by Gajek (2005) concerning the problem of asset and liability immunization. However, the problem is formulated from a different standpoint. Asset and liability streams are hedged on time 0 whereas we consider time $H > 0$. Moreover, the results obtained under a martingale structure assumption cannot be reduced to ours and vice versa.

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