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# Optimal control for elasto-orthotropic plate

by

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**Abstract:** The optimal control problems and a weight minimization problem are considered for elastic three-layered plate with inner obstacle and friction condition on a part of the boundary. The state problem is represented by a variational inequality and the design variables influence both the coefficients and the set of admissible state functions. We prove the existence of a solution to the abovementioned problem on the basis of a general theorem on the control of variational inequalities. Next, the approximate optimization problem is proved on the basis of the general theorem for the continuous problem. When the mesh/size tends to zero, then any sequence of appropriate solutions converges uniformly to a solution of the continuous problem. Finally, the application to the optimal design of unilaterally supported of rotational symmetrical load elastic annular plate is presented.

**Keywords:** control of variational inequalities, elasto-orthotropic plate, optimal design, weight minimization, approximate optimization problem.

## 1. Introduction

Plates and shells are main elements of many advanced structures. One of the most important characteristic of a construction is its weight, which determines the consumption of material needed for production of the construction as well as some of its operating features.

We shall deal with an optimization problem for the unilateral contact between an elastic three-layered plate and inner obstacle. The model of three-layered plate ignores shears in the middle layer. We assume that a homogeneous and orthotropic plate occupying a domain $\Omega \times (-[H_{\langle 0 \rangle} + \mathcal{O}], [H_{\langle 0 \rangle} + \mathcal{O}])$ of the space $\mathbb{R}^3$ is loaded by a transversal distributed force $p(x, y)$ perpendicular to the plane XY. The plate is supported unilaterally by an inner rigid obstacle (punch). Here, on the part of boundary $\partial\Omega$ we have the displacement with friction of the points of $\partial\Omega_{\text{CONTACT}}$. The role of control variable is played by the thickness of exterior layer (appearing also in the right-hand side) and the friction bound (slip limit), respectively. The control variables have to belong to a set of Lipschitz continuous functions. The inner obstacle and the variable thickness (the exterior layer) imply that the convex set of admissible states depends on the control parameters. The cost functional represents a weight of the three-layered plate. Into the weight minimization problem, we introduce constraints, which express bounds for some mean values of the intensity of stress field. Moreover we consider another cost functional which represents the intensity of shear forces (the von Misess yield criterion). The state problem is modelled by a variational inequality (fourth order elliptic variational inequality) where the control variable influences both the coefficients of the linear monotone operator and the set of admissible state functions. On the basis of the general existence theorem for a class of optimization problems with variational inequalities, we prove the existence of at least one solution to the weight minimization (is treated via a penalty method). We deduce the continuous dependence of the deflection on the control variable (thickness of the plate and slip limit). Next we define a finite element discretization of the penalized optimal control problem and prove its solvability. Here, any sequence of approximate solutions, with mesh size decreasing to zero, contains a subsequence, converging to a solution of the penalized control problem. From here, taking into account a sequence of solutions with the penalization parameter tending to zero, any limit point is proved to coincide with a solution of the original weight minimization problem. Finally, this theoretical advance we apply to the shape optimization of elastic axisymmetric circular plate with annular opening.

## 2. Basic relations

A three-layered plate consists of two exterior layers, which are made of a strong material (the so-called carrier layers), and of a comparatively light, non-strong middle layer (the so-called filler), the latter ensuring the joint work of the exterior layers. Consider the three-layered plate whose middle layer is of thickness $H_{\langle 0\rangle}(x,y)$ and two exterior layers are of thickness $\mathcal{O}(x,y)$. We suppose that $\mathcal{O}$ is much less than $H_{\langle 0\rangle}$ ($\mathcal{O} \ll H_{\langle 0\rangle}(x,y)$) and that the material of the middle layer is much more flexible that material of the exterior layers. In this case, the shearing stresses perceive mainly the middle layer and the bending stresses perceive mainly the exterior ones.

Suppose also that, in the transversal direction, the elasticity modulus of the material of the middle layer is infinitely large. The material of the middle layer is usually light, so that the mass of the plate is concentrated in the exterior layers. This is why, in solving optimisation problems for three-layered plates, the control is usually the function $\mathcal{O}(x,y)$ determining the thickness of the carrier layers. In what follows, we assume that the equality: $H_{\langle 0\rangle} + \mathcal{O}$ = constant, determining the parallelism of the midplanes of the carrier layer, holds. The Kirchhoff hypotheses are supposed to be fulfilled for the three-layered plate as

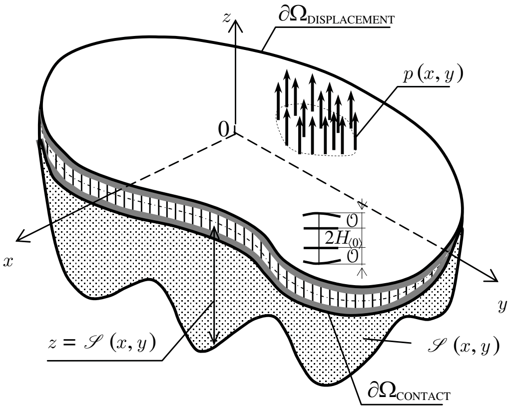


Figure 1.

whole. Then the strain components $\langle \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \rangle$ are expressed by the formulas

$$\begin{cases} \varepsilon_{xx}(x,y,z) = \partial\xi(x,y,z)/\partial x = -z\partial^2 v(x,y)/\partial x^2, \\ \varepsilon_{yy}(x,y,z) = \partial\eta(x,y,z)/\partial y = -z\partial^2 v(x,y)/\partial y^2, \\ \varepsilon_{xy} = \partial\xi(x,y,z)/\partial y + \partial\eta(x,y,z)/\partial x = -2z(\partial^2 v(x,y)/\partial x\partial y), \end{cases}$$

where under Kirchhoff hypotheses, the components $\xi(x,y,z)$ and $\eta(x,y,z)$ of the vector of displacements of points of the plate in the directions of the X and Y axes have the form

$$\begin{cases} \xi(x,y,z) = -z\partial v(x,y)/\partial x, \\ \eta(x,y,z) = -z\partial v(x,y)/\partial y, \end{cases}$$

where $v(x,y)$ denote displacements of points of the midplane along the Z axis.

For an orthotropic plate the stress components $\langle \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \rangle$ are determined by the relations:

$$\begin{cases} \sigma_{xx}(x,y,z) & = E_{11}\varepsilon_{xx}(x,y,z) + E_{12}\varepsilon_{yy}(x,y,z) \\ & = -E_{11}z(\partial^2 v(x,y)/\partial x^2) - E_{12}z(\partial^2 v(x,y)/\partial y^2), \\ \sigma_{yy}(x,y,z) & = E_{12}\varepsilon_{xx}(x,y,z) + E_{22}\varepsilon_{yy}(x,y,z) \\ & = -E_{21}z(\partial^2 v(x,y)/\partial x^2) - E_{22}z(\partial^2 v(x,y)/\partial y^2), \\ \sigma_{xy}(x,y,z) & = G\varepsilon_{xy}(x,y,z) = -2Gz(\partial^2 v(x,y)/\partial x\partial y), \end{cases} \tag{2.1}$$

where

$$\begin{aligned} & E_{11} = E_1/(1-\mu_{12}\mu_{21}), \\ & E_{22} = E_2/(1-\mu_{12}\mu_{21}), \qquad\qquad ((2.2),1^{\text{o}}) \\ & E_{12} = E_{21} = \mu_{21}E_{11} = \mu_{12}E_{22}, \end{aligned}$$

$[E_1,\ E_2,\ G,\ \mu_{12},\ \mu_{21}]$ being the elasticity characteristics of the material.

In this formulae:
$E_{11}$ is the Young modulus in direction $x$,
$E_{22}$ is the Young modulus in direction $y$,
$\mu_{12}$ is the Poisson coefficient of contraction in direction $y$, due to a traction in direction $x$,
$\mu_{21}$ is the Poisson coefficient of contraction in direction $x$, due to a traction in direction $y$.
$G$ is the tangetial shear modulus.

Suppose that

$$\begin{cases} E_1, E_2, G \ \text{ are positive numbers,} \\ \mu_1 \text{ and } \mu_2 \text{ are constants, } 0 \leqslant \mu_{12} < 1, \text{ or } 0 \leqslant \mu_{21} < 1, \\ H_{\langle 0 \rangle} \geqslant \ \text{constant}_{\langle A \rangle} > 0, \\ H_{\langle 0 \rangle} + \mathcal{O} = \ \text{constant}_{\langle B \rangle}, \end{cases} \tag{(2.2),2$^o$}$$

where $\text{constant}_{\langle A \rangle}$ and $\text{constant}_{\langle B \rangle}$ are positive numbers.

The layers are assumed to be made of orthotropic materials, so that the relation (2.1) between stresses and strains are valid, and moreover $E_{11} = 0$, $E_{22} = 0$, $E_{12} = 0$, $G = 0$, for the inner layer, and the elasticity characteristics of the exterior layers coincide.

For the bending moments and torque, we have in view of (2.1)

$$\begin{cases} M_{xx}(x,y) = \int\limits_{-[H_{\langle 0 \rangle}+\mathcal{O}]}^{[H_{\langle 0 \rangle}+\mathcal{O}]} z\sigma_{xx}(x,y,z)\mathrm{d}z \\ \qquad \approx \ -[(2H_{\langle 0 \rangle} + \mathcal{O})/2]\mathcal{O}\sigma_{xx}(x,y,-[(2H_{\langle 0 \rangle} + \mathcal{O})/2]) \\ \qquad\quad +[(2H_{\langle 0 \rangle} + \mathcal{O})/2]\mathcal{O}\sigma_{xx}(x,y,[(2H_{\langle 0 \rangle} + \mathcal{O})/2]) \\ \qquad = D_{11}(\mathcal{O})(\partial^2 v/\partial x^2) + D_{12}(\mathcal{O})(\partial^2 v/\partial y^2), \\ \text{where} \\ D_{11}(\mathcal{O}) = (E_{11}(2H_{\langle 0 \rangle} + \mathcal{O})^2\mathcal{O})/2, \ D_{12}(\mathcal{O}) = (E_{12}(2H_{\langle 0 \rangle} + \mathcal{O})^2\mathcal{O})/2, \\ E_{11}, \ E_{12} \text{ are the elasticity characteristics of exterior layers} \\ \text{for which } ((2.2),1^o) \text{ holds true.} \end{cases} \tag{2.3a}$$

Similarly, we have

$$\begin{cases} M_{yy}(x,y) = \int\limits_{-[H_{\langle 0 \rangle}+\mathcal{O}]}^{[H_{\langle 0 \rangle}+\mathcal{O}]} z\sigma_{yy}(x,y,z)\mathrm{d}z \\ \qquad \approx D_{21}(\mathcal{O})(\partial^2 v(x,y)/\partial x^2) + D_{22}(\mathcal{O})(\partial^2 v(x,y)\partial y^2), \\ M_{xy}(x,y) = \int\limits_{-[H_{\langle 0 \rangle}+\mathcal{O}]}^{[H_{\langle 0 \rangle}+\mathcal{O}]} z\sigma_{xy}(x,y,z)\mathrm{d}z \approx D_{33}(\mathcal{O})(\partial^2 v(x,y)/\partial x \partial y). \end{cases} \tag{2.3b}$$

Here one has

$$\begin{cases} D_{21}(\mathcal{O}) = (E_{21}(2H_{\langle 0\rangle} + \mathcal{O})^2\mathcal{O})/2 = D_{12}(\mathcal{O}), \\ D_{22}(\mathcal{O}) = (E_{22}(2H_{\langle 0\rangle} + \mathcal{O})^2\mathcal{O})/2, \\ D_{33}(\mathcal{O}) = G(2H_{\langle 0\rangle} + \mathcal{O})^2\mathcal{O}, \end{cases} \tag{2.4}$$

and $[E_{21}, E_{22}, G]$ are the elasticity characteristics of the exterior layers.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ and let $\mathscr{S}(x, y)$ be a smooth function in $\bar{\Omega}$. We denote the standard Sobolev function spaces by $H^k(\Omega) \equiv W_2^k(\Omega)$, $k = 1, 2$. In the following $L_2(\Omega)$ and $L_\infty(\Omega)$ denote the space of Lebesgue-square integrable functions on $\Omega$ and the space of essentially bounded functions on $\Omega$ with the standard norms $\|.\|_{L_2(\Omega)}$ and $\|.\|_{L_\infty(\Omega)}$ respectively. The inner product in $L_2(\Omega)$ will be denoted by $\langle ., . \rangle_{L_2(\Omega)}$.

Let the plate be clamped at a part $\partial\Omega_{\text{DISPLACEMENT}}$ of the boundary $\partial\Omega$ whereas on the remaining part $\partial\Omega_{\text{CONTACT}}$ of the plate subjected to a contact with friction: $\partial\Omega = \partial\Omega_{\text{DISPLACEMENT}} \cup \partial\Omega_{\text{CONTACT}}$, be a mutually disjoint (non-overlaping) decomposition of the boundary ($\text{MEAS}\ \partial\Omega_{\text{DISPLACEMENT}} > 0$, $\text{MEAS}\ \partial\Omega_{\text{CONTACT}} > 0$). Then, we have

$$\begin{aligned} V(\Omega) = &\{v \in H^2(\Omega) : \mathscr{M}_0 v = 0 \text{ on } \partial\Omega_{\text{DISPLACEMENT}}, \\ &\quad \mathscr{M}_1 v = 0 \text{ on } \partial\Omega_{\text{DISPLACEMENT}} \text{ in sense of traces}\}, \end{aligned}$$

$\mathscr{M}_0 v$ is the restriction of $v$ to $\partial\Omega_{\text{DISPLACEMENT}}$, $\mathscr{M}_1 v = \partial v/\partial n$ is the normal derivative on $\partial\Omega_{\text{DISPLACEMENT}}$, where $[\mathscr{M}_0, \mathscr{M}_1]$ are trace (or boundary) operators.

On the part $\partial\Omega_{\text{CONTACT}}$ of the boundary $\partial\Omega$ we have displacement with friction on points of $\partial\Omega_{\text{CONTACT}}$. If the reaction force $|V^*_{nz}(v)|$ is below a certain value, the friction is not overcome and there is no displacement $v$, above this value, there is a displacement in a direction opposite to the force. This means that on the part $\partial\Omega_{\text{CONTACT}}$ we prescribe a slip limit $\mathscr{F}$ and the following friction conditions: either the surface force $|V^*_{nz}(v)|$ is less that the slip limit $\mathscr{F}$ and the plate remains in its original position, or $|V^*_{nz}(v)|$ equals $\mathscr{F}$ and the plate can slip into a new equilibrium position in the opposite direction to the friction force. As a consequence, we have the following condition on $\partial\Omega_{\text{CONTACT}}$

$$\begin{cases} |V^*_{nz}(v)| \leqslant \mathscr{F},\ |V^*_{nz}(v)| < \mathscr{F} \Rightarrow v = 0, \\ |V^*_{nz}(v)| = \mathscr{F} \Rightarrow \text{There exists } \lambda \geqslant 0 \text{ such that } v = -\lambda V^*_{nz}(v). \end{cases}$$

Let us recall some connections between continuous functionals and the Radon measures. We denote by $C_{\text{COMPACT}}(\Omega)$ the space of all continuous functions with compact support in $\Omega$. A sequence $\{\theta_n\}_{n\in N}$, $\theta_n \in C_{\text{COMPACT}}(\Omega)$ converges to $\theta \in C_{\text{COMPACT}}(\Omega)$, if the supports of the functions $\theta_n$ belong to a compact subset of $\Omega$ and $\{\theta_n\}_{n\in N}$ converges to $\theta$ uniformly on $\Omega$. Due to the representation theorem every continuous linear functional $\mathscr{V}$ over $C_{\text{COMPACT}}(\Omega)$ can be represented

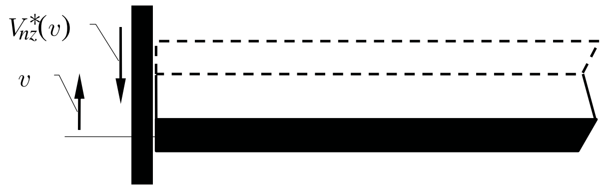


Figure 2.

by the integral

$$\langle \mathscr{V}, \theta \rangle_{C_{\mathsf{COMPACT}}}(\Omega) = \int_\Omega \theta \mathrm{d}\mu \quad \text{for any} \quad \theta \in C_{\mathsf{COMPACT}}(\Omega), \tag{2.5}$$

where $\mu$ belongs to the set $\mathscr{M}(\Omega)$ (the set of all measures defined on $\Omega$). A linear continuous functional $\mathscr{V}$ on $C_{\mathsf{COMPACT}}(\Omega)$ is said to be positive, if $\mathscr{V} \geqslant 0$ for all $\theta \in C_{\mathsf{COMPACT}}(\Omega)$, $\theta(x, y) \geqslant 0$. Positive functionals on $C_{\mathsf{COMPACT}}(\Omega)$ possess an important property: linear and positive functional $\mathscr{V}$ on $C_{\mathsf{COMPACT}}(\Omega)$ is continuous and can be represented in the form (2.5) with a nonnegative measure $\mu$.

The thickness $\mathcal{O}$ will be sought in the following set of admissible functions

$$\begin{aligned}\mathscr{U}_{ad}^{\mathcal{O}}(\Omega) :=&\{\mathcal{O} \in C^{(0),1}(\bar{\Omega}) : \mathcal{O}_{\text{MIN}} \leqslant \mathcal{O}(x, y) \leqslant \mathcal{O}_{\text{MAX}},\\ &|\partial\mathcal{O}/\partial x| \leqslant \text{constant}_{\langle 1\rangle}, |\partial\mathcal{O}/\partial y| \leqslant \text{constant}_{\langle 2\rangle}\\ &(\text{or } |\partial\mathcal{O}/\partial\xi| \leqslant \text{constant}_{\langle\xi\rangle},\ |\partial\mathcal{O}/\partial\eta| \leqslant \text{constant}_{\langle\eta\rangle})\},\end{aligned}$$

where $C^{(0),1}(\bar{\Omega})$ denotes the set of Lipschitz functions $\mathcal{O}_{\text{MIN}}$, $\mathcal{O}_{\text{MAX}}$ and [$\text{const}_{\langle 1\rangle}$, $\text{const}_{\langle 2\rangle}, \text{const}_{\langle\xi\rangle}, \text{const}_{\langle\eta\rangle}$] are positive parameters, $[\xi, \eta]$ some skew coordinates. Due to Arzela theorem (Litvinov, 2000), $\mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$ is a compact subset of $\mathscr{U}^{\mathcal{O}}(\Omega)$ $(= C(\bar{\Omega}))$.

Taking into consideration the relations (2.1) to (2.4), we get the following expression for the strain energy of the orthotropic plate as a functional of $v(x, y)$ say

$$\begin{aligned}\mathcal{E}(\mathcal{O}, v) =& \frac{1}{2}\int_\Omega \left( D_{11}(\mathcal{O})\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2D_{12}(\mathcal{O})\frac{\partial^2 v}{\partial^2 x}\frac{\partial v^2}{\partial y^2}\right.\\ &\left.+ D_{22}(\mathcal{O})\left(\frac{\partial^2 v}{\partial y^2}\right)^2 + 2D_{33}(\mathcal{O})\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2\right)\mathrm{d}\Omega.\end{aligned} \tag{2.6}$$

The convex function $\mathcal{E}(\mathcal{O}, v)$ is weakly or Gâteaux differentiable on $V(\Omega)$, the corresponding element from $\mathscr{D}^*(\Omega)$ will by denoted by $\mathbf{grad}_v \mathcal{E}(\mathcal{O}, v)$. We have (the main variation of the strain (internal) energy)

$$
\begin{aligned}
&\langle \mathbf{grad}_v \mathcal{E}(\mathcal{O}, v), w\rangle_{V(\Omega)} = \lim_{\lambda\to 0}[\mathcal{E}(\mathcal{O}, v+\lambda w) - \mathcal{E}(\mathcal{O}, v)]/\lambda \\
&= \int\limits_\Omega \Big[\frac{\partial^2}{\partial x^2}\Big(D_{11}(\mathcal{O})\frac{\partial^2 v}{\partial x^2}\Big) + \frac{\partial^2}{\partial y^2}\Big(D_{22}(\mathcal{O})\frac{\partial^2 v}{\partial y^2}\Big) + \frac{\partial^2}{\partial x^2}\Big(D_{12}(\mathcal{O})\frac{\partial^2 v}{\partial y^2}\Big) \\
&+\frac{\partial^2}{\partial y^2}\Big(D_{12}(\mathcal{O})\frac{\partial^2 v}{\partial x^2}\Big) + 2\frac{\partial^2}{\partial x\partial y}\Big(D_{33}(\mathcal{O})\frac{\partial^2 v}{\partial x\partial y}\Big)\Big] w\mathrm{d}\Omega,
\end{aligned} \tag{2.7}
$$

for any $w \in \mathscr{D}(\Omega)$.

Thus, according to (2.6) and (2.7), the following bilinear form on $[V(\Omega)\times V(\Omega)]$ corresponds to the strain energy of the ortohotropic plate (the variation of the intenal energy or virtual work equation)

$$
\begin{aligned}
a(\mathcal{O}, v, z) = \int\limits_\Omega \Big[&D_{11}(\mathcal{O})\left(\frac{\partial^2 v}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial x^2}\right) + D_{22}(\mathcal{O})\left(\frac{\partial^2 v}{\partial y^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) + \\
&D_{12}(\mathcal{O})\left(\left(\frac{\partial^2 v}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) + \left(\frac{\partial^2 v}{\partial y^2}\right)\left(\frac{\partial^2 z}{\partial x^2}\right)\right) + \\
&2D_{33}(\mathcal{O})\left(\left(\frac{\partial^2 v}{\partial x\partial y}\right)\left(\frac{\partial^2 z}{\partial x\partial y}\right)\right)\Big]\mathrm{d}\Omega, \quad [v,z]\in V(\Omega).
\end{aligned} \tag{2.8}
$$

Moreover for fixed $v \in V(\Omega)$ we consider the application: $z \to a(\mathcal{O}, v, z)$, $\mathscr{D}(\Omega) \to \mathrm{R}$ (the linear continuous on $\mathscr{D}(\Omega)$). Then there exists the element $\mathsf{Q}(\mathcal{O})v \in \mathscr{D}^*(\Omega)$ such that

$$\langle \mathsf{Q}(\mathcal{O})v, z\rangle_{\mathscr{D}(\Omega)} = a(\mathcal{O}, v, z) \quad \text{for any} \quad z \in \mathscr{D}(\Omega), \text{ or } z \in H_0^2(\Omega). \tag{2.9}$$

Here the relation (2.9) defines the linear continuous operator $\mathsf{Q}(\mathcal{O}) : V(\Omega) \to \mathscr{D}^*(\Omega)$. If $\mathscr{D}(\Omega)$ is dense in $V(\Omega)$ the operator $\mathsf{Q}(\mathcal{O})$ coincides with the operator $\mathscr{A}(\mathcal{O})$: $V(\Omega) \to V^*(\Omega)$ defined by the equation

$$\langle \mathscr{A}(\mathcal{O})v, z\rangle_{V(\Omega)} = a(\mathcal{O}, v, z) \quad \text{for } [v,z]\in V(\Omega),\ \mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega). \tag{2.10}$$

(It is advantageous to consider the equivalent formulation when $\mathscr{A}(\mathcal{O})$ is generated by a bilinear, control dependent bounded functional $a(\mathcal{O}, \cdot, \cdot) : V(\Omega) \times V(\Omega) \to R$.)

In the following we denote by $\mathcal{A}(\mathcal{O})$ the restriction of $\mathsf{Q}(\mathcal{O})$ to: $D(\mathcal{A}(\mathcal{O})) = \{v \in V(\Omega) : \mathsf{Q}(\mathcal{O})v \in L_2(\Omega), a(\mathcal{O}, v, z) = \langle \mathsf{Q}(\mathcal{O})v, z\rangle_{L_2(\Omega)}$ for any $z \in V(\Omega)\} = \{v \in V(\Omega)$ there exists the element $\mathcal{H}(\mathcal{O})v \in L_2(\Omega)\}$ such that: $\langle \mathcal{H}(\mathcal{O})v, z\rangle_{L_2(\Omega)} = a(\mathcal{O}, v, z)$ for any $z \in V(\Omega)\}$. Thus, $D(\mathcal{A}(\mathcal{O}))$ is the set of the elements $v \in V(\Omega)$ such that the linear form: $z \to a(\mathcal{O}, v, z)$, which is defined on $V(\Omega)$ can be prolonged to the linear continuous form on $L_2(\Omega)$ consequently: $D(\mathcal{A}(\mathcal{O})) = D(A(\mathcal{O}))$ and the operator $\mathcal{A}(\mathcal{O})$ coincides with the operator $A(\mathcal{O})$ which is defined by the triplet $[V(\Omega), L_2(\Omega), a(\mathcal{O}, v, z)]$.

The loading of the homogenous orthotropic plate (e.g. a concrete plate reinforced by welded ribs) is given by

$$\begin{cases} 1^\circ & \text{The surface (traction) forces } p(x,y). \\ 2^\circ & \text{The body forces (within the plate) } 2[\omega_1 H_{\langle 0\rangle} + \omega_2 \mathcal{O}], \\ & \text{where } \omega_i = \text{constant}_{\langle \mathrm{i} \rangle},\ i=1,2 \text{ is a given specific weight.} \end{cases}$$

Let us consider the following virtual work of external loading

$$\langle \mathrm{L}(\mathbf{e}), v\rangle_{V(\Omega)} = \int_\Omega \left[p - 2(\omega_1 H_{\langle 0\rangle} + \omega_2 \mathcal{O})\right] v \mathrm{d}\Omega.$$

Here, the mapping $\mathbf{e} \to L(\mathbf{e})$ from $\mathscr{U}_{ad}(\Omega)$ into $V^*(\Omega)$ is continuous.

We define the friction functional $\Phi([\mathcal{O},\mathscr{F}],v) : V(\Omega) \to \mathbb{R}^+_\infty$ by the formula

$$\Phi([\mathcal{O},\mathscr{F}],v) := \int_{\partial\Omega_{\mathsf{CONTACT}}} \mathscr{F}|v|\mathrm{d}s + I_{\mathscr{K}(\mathcal{O},\Omega)}(v), \tag{2.11}$$

where

$$\mathscr{F} \in \mathscr{U}^{\mathscr{F}}_{ad}(\partial\Omega_{\mathsf{CONTACT}}) = \{\mathscr{F} \in C^{(0),1}(\overline{\partial\Omega}_{\mathsf{CONTACT}}) : 0 \leqslant \mathscr{F}(s) \leqslant \mathscr{F}_{\mathrm{MAX}}, \\ |\mathrm{d}\mathscr{F}|\mathrm{d}s| \leqslant \text{constant}_{\langle a\rangle},\ \text{a.e. in } \partial\Omega_{\mathsf{CONTACT}}\},$$

with given positive constants $[\mathscr{F}_{\mathrm{MAX}}, \text{constant}_{\langle a\rangle}]$ and $I_{\mathscr{K}(\mathcal{O},\Omega)}(\cdot)$ is the indicator function of the set $\mathscr{K}(\mathcal{O},\Omega)$.

In what follows, we set $\mathbf{e} = [\mathcal{O},\mathscr{F}]$ and define

$$\mathscr{U}_{ad}(\Omega) = \mathscr{U}^{\mathcal{O}}_{ad}(\Omega) \times \mathscr{U}^{\mathscr{F}}_{ad}(\partial\Omega_{\mathsf{CONTACT}}).$$

Further, we introduce the function $\mathscr{S} \in C(\bar\Omega)$, to be given, describing a lower unilateral obstacle. The obstacle function $\mathscr{S} : \bar\Omega \to \mathbb{R}$ fulfils the condition:

(H0) $\quad \mathrm{MAX}_{[\mathrm{x},\mathrm{y}]\in\partial\Omega}\, \mathscr{S}(\mathrm{x},\mathrm{y}) + (\mathcal{O}_{\mathrm{MAX}} + \mathrm{H}_{\langle 0\rangle}) < 0.$

Let us use the virtual displacement principle to establish a variational formulation of the problem. To this end we introduce the set:

$$\mathscr{K}(\mathcal{O},\Omega) = \{v \in V(\Omega) : v(x,y) \geqslant \mathscr{S}(x,y) + (\mathcal{O}(x,y) + H_{\langle 0\rangle}) \text{ for } [x,y] \in \Omega\}.$$

On the basis of the virtual displacement principle, we introduce the following STATE PROBLEM: Given any $\mathbf{e} \in \mathscr{U}_{ad}(\Omega)$, find $u(\mathbf{e}) \in \mathscr{K}(\mathcal{O},\Omega)$ such that

$$\langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), v - u(\mathbf{e})\rangle_{V(\Omega)} + \Phi(\mathbf{e},v) - \Phi(\mathbf{e},u(\mathbf{e})) \geqslant \langle \mathrm{L}(\mathbf{e}), v - u(\mathbf{e})\rangle_{V(\Omega)}, \tag{2.12}$$

holds for all $v \in \mathscr{K}(\mathcal{O},\Omega)$.

Later on, we shall prove that the variational inequality has a unique solution for any $e \in \mathscr{U}_{ad}(\Omega)$.

This is a mathematical model of an elastic orthotropic plate in the state of static equilibrium, interacting with an obstacle (the obstacle shape is defined by the equation $z = \mathscr{S}(x, y), [x, y] \in \Omega$). Furtermore, due to (2.9), from (2.12) (here inserting $v = u(e) + \theta$, $\theta \in \mathscr{D}(\Omega)$ and $\theta \geqslant 0$), we immediately deduce that

$$\mu([e, u(e)], \Omega) = \mathsf{Q}(\mathcal{O})u(e) - (p - 2\omega_1 H_{\langle 0\rangle} - 2\omega_2 \mathcal{O}),$$

is a positive distribution on $\Omega$, and consequently, a nonnegative Radon measure in $\Omega$. This measure describes the work of interaction forces between the plate and the obstacle.

Let a specific weight $\omega_i \in \mathrm{L}_\infty(\Omega)$ be given, $\omega_i > 0$, $i = 1, 2$. Thus, the weight of the orthotropic (three-layered) plate is determined by

$$\mathscr{L}_{\text{WEIGHT}}(\mathbf{e}) = 2\int_\Omega [\omega_1 H_{\langle 0\rangle} + \omega_2 \mathcal{O}]\mathrm{d}\Omega.$$

Moreover, the following constraints will be considered (the Norris strength criterion)

$$\text{(A1)} \quad \mathscr{S}_{\mathsf{Q}}(\mathbf{e}, \mathbf{M}(\mathbf{e})) \leqslant 0, \quad \mathsf{Q} = 1, 2, ..., N_{\mathsf{Q}}, \quad N_{\mathsf{Q}}(< +\infty),$$

where

$$\mathscr{S}_{\mathsf{Q}}(\mathbf{e}, \mathbf{M}(\mathbf{e})) = 9/4\,\text{MEAS}\,\Omega^*_{\mathsf{Q}} \int_{\Omega^*_{\mathsf{Q}}} \frac{1}{(H_{\langle 0\rangle} + \mathcal{O})^4} \times [M^2_{xx}(\mathbf{e}) + M^2_{yy}(\mathbf{e}) + (\sigma_R/\tau_R)^2 M^2_{xy}(\mathbf{e})]\mathrm{d}\Omega - \sigma^2_R,$$

$\Omega^*_{\mathcal{O}} \subset \bar{\Omega}$ are given subdomains, $\sigma_R$, $\tau_R$ are given positive constants and $\mathbf{M}(\mathbf{e})$ are the bending moment and torque, derived by the relation (2.3) from the solution $u(\mathbf{e})$ of (2.12).

Let us introduce the set of statically admissible control variables:

$$\mathscr{G}_{ad}(\Omega) = \{\mathcal{O} \in \mathscr{U}_{ad}(\Omega) : \sum_{\mathsf{Q}=1}^{N_{\mathsf{Q}}} ([\mathscr{S}_{\langle \mathsf{Q}\rangle}(\mathbf{e}, \mathbf{M}(\mathbf{e}))]^+) = 0\}.$$

where $[a]^+ = \max\{0, a\}$ denotes the positive part of $a$.

Here, we assume

$$\mathscr{G}_{ad}(\Omega) \neq \emptyset. \tag{2.13}$$

Now, our main task is to solve the Optimal Control Problem:

$$(\mathscr{P}) \quad \mathbf{e}_{\langle \mathsf{Q}\rangle} = \operatorname*{ArgMin}_{\mathbf{e} \in \mathscr{G}_{ad}(\Omega)} \mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}).$$

In the following, we remove the constraints (A1) by means of a penalty method. To this end we introduce a penalized cost functional

$$\boldsymbol{\mathscr{L}}_{\langle\varepsilon\rangle,\mathsf{WEIGHT}}(\mathbf{e},\mathbf{M}(\mathbf{e})) = \mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e})+(1/\varepsilon)\sum_{\mathsf{Q}=1}^{N_{\mathsf{Q}}}\big([\mathscr{S}_{\langle\mathsf{Q}\rangle}(\mathbf{e},\mathbf{M}(\mathbf{e}))]^{+}\big),\ \varepsilon>0$$

and a penalized optimal control problem

$$(\mathscr{P}_{\langle\varepsilon\rangle})\quad \mathbf{e}_{\langle\varepsilon\rangle} = \operatorname*{ArgMin}_{\mathbf{e}\in\mathscr{U}_{ad}(\Omega)} \boldsymbol{\mathscr{L}}_{\langle\varepsilon\rangle,\mathsf{WEIGHT}}(\mathbf{e},\mathbf{M}(\mathbf{e})). \tag{2.14}$$

## 3. Existence of a solution to the optimal control problem

We shall consider a class of abstract optimal control problems and prove their solvability. Then, we shall apply the general result to our optimal control problem $(\mathscr{P})$.

Let $U(\Omega)$ be a Banach space of controls, $U_{ad}(\Omega)$ is a subset of admissible controls. We assume that $U_{ad}(\Omega)$ is compact in $U(\Omega)$. Let reflexive Banach space $V(\Omega)$ be endowed with a norm $||.||_{V(\Omega)}$ and let $V^*(\Omega)$ be its dual with a norm $||.||_{V^*(\Omega)}$, the duality pairing between $V(\Omega)$ and $V^*(\Omega)$ being denoted by $\langle .,.\rangle_{V(\Omega)}$.

Definition 3.1 *We say that a sequence $\{K_n(\Omega)\}_{n\in N}$ of convex subsets of $V(\Omega)$ converges to a set $K(\Omega)$, i.e. $K(\Omega) = \underset{n\to\infty}{Lim} K_n(\Omega)$ (convergence in the sense of Mosco) if the following two conditions are satisfied:*

$$\begin{cases} 1^\circ \textit{ For any } v\in K(\Omega) \textit{ a sequence } \{v_n\}_{n\in N} \textit{ exists, such that } v_n\in K_n(\Omega) \\ \quad\textit{and } \lim\limits_{n\to\infty} v_n = v \textit{ in } V(\Omega). \\ 2^\circ \textit{ If } v_n\in K_n(\Omega) \textit{ and } v_n\to v \textit{ weakly in } V(\Omega)\textit{, then } v\in K(\Omega). \end{cases}$$

Definition 3.2 *Let $\mathscr{W}_n : V(\Omega)\to[0,\infty]$, $n=1,2,\dots$ be a sequence of functionals. We say that*

$$\mathscr{W} = \underset{n\to\infty}{Lim}\,\mathscr{W}_n,$$

*if the following conditions hold:*

$$\begin{cases} 1^\circ \textit{ For each } V(\Omega) \textit{ there exists a sequence } \{v_n\}_{n\in N} \textit{ such that } v_n\in\ V(\Omega), \\ \lim\limits_{n\to\infty} v_n = v \textit{ in } V(\Omega), \limsup\limits_{n\to\infty}\mathscr{W}_n(v_n)\leqslant\mathscr{W}(v). \\ 2^\circ \textit{ For each sequence } \{\mathscr{W}_{n_k}\}_{k\in N} \textit{ and each sequence } \{v_k\}_{k\in N},\ v_k\in\ V(\Omega), \\ \quad\textit{weakly convergent to } v\in\ V(\Omega)\textit{, the inequality} \\ \mathscr{W}(v)\leqslant\liminf\limits_{n_k\to\infty}\mathscr{W}_{n_k}(v_{n_k}) \\ \textit{holds.} \end{cases}$$

In view of Definition 3.2, $\mathscr{W} = \operatorname*{Lim}_{n\to\infty} \mathscr{W}_n$ implies that for each $v \in V(\Omega)$ there exists a sequence $\{v_n\}_{n\in N}$ such that $v_n \to v$ strongly in $V(\Omega)$ and $\lim_{n\to\infty} \mathscr{W}_n(v_n) = \mathscr{W}(v)$.

Let us consider a system $\{\boldsymbol{\mathcal{K}}(e_n,\Omega)\}_{n\in N}$, $e_n \in U_{ad}(\Omega)$, of closed convex subsets, $\boldsymbol{\mathcal{K}}(e_n,\Omega) \subset V(\Omega)$ and a family $\{\mathrm{A}(e_n)\}_{n\in N}$ of operators $\mathrm{A}(e_n) : V(\Omega) \to V^*(\Omega)$, satisfying the following assumptions

$$
\text{(H1)}\begin{cases}
1^\circ \quad \bigcap_{e\in U_{ad}(\Omega)} \boldsymbol{\mathcal{K}}(e,\Omega) \neq \emptyset,\\
2^\circ \quad e_n \to e \text{ strongly in } U(\Omega),\ e_n \in U_{ad}(\Omega) \Rightarrow \boldsymbol{\mathcal{K}}(e,\Omega) = \operatorname*{Lim}_{n\to\infty} K(e_n,\Omega),\\
3^\circ \quad \text{There exist constants: } 0 < \alpha_{\mathcal{A}} < M_{\mathcal{A}} \text{ independent of } e \in U_{ad}(\Omega)\\
\qquad \text{and such that}\\
\qquad \begin{cases} \alpha_{\mathcal{A}}||v-z||^2_{V(\Omega)} \leqslant \langle \mathrm{A}(e)v - \mathrm{A}(e)z, v-z\rangle_{V(\Omega)},\\ ||\mathrm{A}(e)v - \mathrm{A}(e)z||_{V^*(\Omega)} \leqslant M_{\mathcal{A}}||v-z||_{V(\Omega)}, \end{cases}\\
4^\circ \quad e_n \to e \text{ strongly in } U(\Omega),\ e_n \in U_{ad}(\Omega) \Rightarrow \mathrm{A}(e_n)v \to \mathrm{A}(e)v\\
\qquad \text{strongly in } V^*(\Omega) \text{ holds for all } v \in V(\Omega).
\end{cases}
$$

Moreover, we consider a system of functionals $\{\Phi(e,.)\}$, $e \in U_{ad}(\Omega)$, $\Phi(e,.) : V(\Omega) \to [0,\infty]$, lower semicontinuous and convex on $V(\Omega)$ and such that

$$
\text{(H2)}\begin{cases}
1^\circ \quad \begin{cases} e_n \in U_{ad}(\Omega),\ e_n \to e \text{ strongly in } U(\Omega) \Rightarrow \Phi(e,.) = \operatorname*{Lim}_{n\to\infty} \Phi(e_n,.),\\ \operatorname{dom} \Phi(e,.) \equiv \{v \in V(\Omega) : \Phi(e,v) < +\infty\} = \boldsymbol{\mathcal{K}}(e,\Omega)\\ \qquad \text{for all } e \in U_{ad}(\Omega). \end{cases}\\
\text{Additionally we suppose that there is a bounded sequence } \{a_n\}_{n\in N} \text{ with}\\
2^\circ \quad \begin{cases} a_n \in \boldsymbol{\mathcal{K}}(e_n,\Omega) \text{ and } \limsup_{n\to\infty} \Phi(e_n,a_n) < +\infty \text{ for each sequence}\\ \{e_n\}_{n\in N},\ e_n \in U_{ad}(\Omega) \text{ such that } e_n \to e \text{ strongly in } U(\Omega). \end{cases}
\end{cases}
$$

Finally, let a functional $f \in V^*(\Omega)$ and a continuous mapping $B : U(\Omega) \to V^*(\Omega)$ be given.

For any $e \in U_{ad}(\Omega)$ let us consider the following state variational inequality: Find $u(e) \in \boldsymbol{\mathcal{K}}(e,\Omega)$ such that

$$
\begin{cases} \langle \mathrm{A}(e)u(e), v-u(e)\rangle_{V(\Omega)} + \Phi(e,v) - \Phi(e,u(e)) \geqslant \langle f + Be, v-u(e)\rangle_{V(\Omega)}\\ \text{for all } v \in \boldsymbol{\mathcal{K}}(e,\Omega). \end{cases} \tag{3.1}
$$

Here, we note that there exists a unique solution $u(e) \in \boldsymbol{\mathcal{K}}(e,\Omega)$ for any $e \in U_{ad}(\Omega)$. In fact, we may employ the general theory of variational inequalities (see Cea, 1971; Duvaut, Lions, 1972).

Next, let a functional $\mathscr{L} : U(\Omega) \times V(\Omega) \to R$ be given such that

$$
\begin{cases} e_n \to e \text{ strongly in } U(\Omega),\ e_n \in U_{ad}(\Omega) \text{ and } v_n \to v \text{ weakly in } V(\Omega) \Rightarrow\\ \liminf_{n\to\infty} \mathscr{L}(e_n,v_n) \geqslant \mathscr{L}(e,v). \end{cases} \tag{3.2}
$$

Let us introduce a functional $J : U_{ad}(\Omega) \to \mathbb{R}$ by the formula $J(e) = \mathscr{L}(e, u(e))$, $u(e)$ is the solution of the state problem (3.1). Here we shall solve the optimization problem

$$(\mathscr{B}) \quad e_{\langle * \rangle} = \operatorname*{ArgMin}_{e \in U_{ad}(\Omega)} J(e).$$

THEOREM 3.1 *Let the data of the state problem (3.1) satisfy the assumptions (H2). Let $e_n \in U_{ad}(\Omega)$, $e_n \to e_{\langle * \rangle}$ strongly in $U(\Omega)$. Then one has: $u(e_n) \to u(e_*)$ strongly in $V(\Omega)$.*

*Proof.* Let us consider the state inequality (3.1) for any $e_n$, $n = 1, 2, \ldots$. Next, set $v = a_n$ (see ((H1),2°), adding the term $\langle \mathrm{A}(e_n)a_n, u(e_n) - a_n \rangle_{V(\Omega)}$ to both sides, we derive the inequality

$$\begin{cases} \langle \mathrm{A}(e_n)u(e_n) - \mathrm{A}(e_n)a_n, u(e_n) - a_n \rangle_{V(\Omega)} + \Phi(e_n, u(e_n)) \\ \leqslant \langle f + Be_n, u(e_n) - a_n \rangle_{V(\Omega)} + \langle \mathrm{A}(e_n)a_n, a_n - u(e_n) \rangle_{V(\Omega)} + \Phi(e_n, a_n). \end{cases}$$

Hence in view of ((H1),3°,4°), (H2) and the continuity of $B$, we deduce: $||u(e_n)||_{V(\Omega)} \leqslant$ constant for all $n$.

Thus there exists a subsequence $\{u(e_{n_k})\}_{k \in N} \subset \{u(e_n)\}_{n \in N}$ and element $u_{\langle * \rangle} \in V(\Omega)$, such that

$$u(e_{n_k}) \to u_{\langle * \rangle} \text{ weakly in } V(\Omega). \tag{3.3}$$

The assumption ((H1),2°) implies that: $u_{\langle * \rangle} \in \boldsymbol{\mathcal{K}}(e_{\langle * \rangle}\Omega)$. From here, taking into account (H2), we have

$$\Phi(e_{\langle * \rangle}, u_{\langle * \rangle}) < +\infty \tag{3.4}$$

By virtue of Definition 3.2 we can find a sequence $\{\theta_k\}_{k \in N}$, such that $\theta_k \in \boldsymbol{\mathcal{K}}(e_k, \Omega)$ and

$$\begin{cases} \theta_k \to u_{\langle * \rangle} \text{ strongly in } V(\Omega), \\ \lim_{k \to \infty} \Phi(e_{n_k}, \theta_k) = \Phi(e_{\langle * \rangle}, u_{\langle * \rangle}). \end{cases} \tag{3.5}$$

Here $\theta_k \in \boldsymbol{\mathcal{K}}(e_n, \Omega)$ follows from (H2) and (3.4), (3.5).

Further, we consider again the inequality (3.1) for $e = e_{n_k}$, inserting $v := \theta_k$, and adding term $\langle \mathrm{A}(e_{n_k})\theta_k, u(e_{n_k}) - \theta_k \rangle_{V(\Omega)}$ to both sides of (3.1). We obtain

$$\begin{aligned} &\limsup_{k \to \infty} \langle \mathrm{A}(e_{n_k})u(e_{n_k}) - \mathrm{A}(e_{n_k})\theta_k, u(e_{n_k}) - \theta_k \rangle_{V(\Omega)} \\ &\leqslant \limsup_{k \to \infty} \langle \mathrm{A}(e_{n_k})\theta_k, \theta_k - u(e_{n_k}) \rangle_{V(\Omega)} \\ &\qquad + \limsup_{k \to \infty} \langle f + Be_{n_k}, u(e_{n_k}) - \theta_k \rangle_{V(\Omega)} \\ &\qquad + \limsup_{k \to \infty} \Phi(e_{n_k}, \theta_k) - \liminf_{k \to \infty} \Phi(e_{n_k}, u(e_{n_k}) \leqslant 0. \end{aligned} \tag{3.6}$$

The last inequality follows from the weak convergence of $\{u(e_{n_k})\}_{k\in N}$, (3.5), the continuity of $B$ and the following assertion

$$\begin{aligned} & e_k \to e \text{ strongly in } U(\Omega),\ e_k \in U_{ad}(\Omega) \text{ and } v_k \to v \text{ strongly in } V(\Omega) \\ & \Rightarrow ||\mathrm{A}(e_k)v_k - \mathrm{A}(e)v||_{V^*(\Omega)} \\ & \qquad \leqslant M_{\mathcal{A}}||v_k - v||_{V(\Omega)} + ||\mathrm{A}(e_k)v - \mathrm{A}(e)v||_{V^*(\Omega)} \to 0, \end{aligned} \tag{3.7}$$

which is a consequence of ((H1),3°,4°).

Moreover, due to the uniform monotonicity of $[\mathrm{A}(e_{n_k})]$ ((H1), 3°) and in view of (3.6), we may write

$$\lim_{k\to\infty} ||u(e_{n_k}) - \theta_k||_{V(\Omega)} = 0. \tag{3.8}$$

Thus, taking into account (3.8) and (3.5), we have

$$u(e_{n_k}) \to u_{\langle *\rangle} \text{ strongly in } V(\Omega). \tag{3.9}$$

Then the relations (3.7) and (3.9) give

$$\mathrm{A}(e_{n_k})u(e_{n_k}) \to \mathrm{A}(e_{\langle *\rangle})u_{\langle *\rangle} \text{ strongly in } V^*(\Omega). \tag{3.10}$$

Further (due to (H2) and Definition 3.2) for any $v \in \mathcal{K}(e,\Omega)$ there exists a sequence $\{\vartheta_k\}_{k\in N}$ such that $\vartheta_k \in \mathcal{K}(e_{n_k},\Omega)$, $\vartheta_k \to v$ strongly in $V(\Omega)$ and we may write

$$\Phi(e_{n_k},\vartheta_k) \to \Phi(e_{\langle *\rangle}, v). \tag{3.10$_*$}$$

Hence, passing to the $\limsup\limits_{k\to\infty}$ on both sides of the inequality:

$$\begin{aligned} & \langle \mathrm{A}(e_{n_k})u(e_{n_k}), u(e_{n_k}) - \vartheta_k\rangle_{V(\Omega)} - \langle f + Be_{n_k}, u(e_{n_k}) - \vartheta_k\rangle_{V(\Omega)} \\ & \leqslant \Phi(e_{n_k},\vartheta_k) - \Phi(e_{n_k}, u(e_{n_k})), \end{aligned}$$

we arrive at (by virtue of (3.9), (3.10), $(3.10)_*$ and (H2))

$$\langle \mathrm{A}(e_{\langle *\rangle})u_{\langle *\rangle}, u_{\langle *\rangle} - v\rangle_{V(\Omega)} - \langle f + Be_{\langle *\rangle}, u_{\langle *\rangle} - v\rangle_{V(\Omega)} \leqslant \Phi(e_{\langle *\rangle}, v) - \Phi(e_{\langle *\rangle}, u_{\langle *\rangle}).$$

Then, from the uniqueness of $u(e_{\langle *\rangle})$, we deduce $u_{\langle *\rangle} = u(e_{\langle *\rangle})$. Hence, the whole sequence $\{u(e_{n_k})\}_{n\in N}$ converges weakly to $u(e_{\langle *\rangle})$ in $V(\Omega)$. ■

THEOREM 3.2 *Let the data of the state problem (3.1) satisfy the assumptions (H1). Let the functional $\mathscr{L}$ satisfy the condition (3.2). Then there exists at least one solution of the OPTIMAL CONTROL PROBLEM ($\mathscr{B}$).*

*Proof.* Since the set $U_{ad}(\Omega)$ is compact in $U(\Omega)$, there exists a sequence $\{e_n\}_{n\in N}$, such that $e_n \in U_{ad}(\Omega)$, $e_n \to e_{\langle *\rangle}$ strongly in $U(\Omega)$, $e_{\langle *\rangle} \in U_{ad}(\Omega)$, $J(e_n) \to \operatorname*{Inf}\limits_{e\in U_{ad}(\Omega)} J(e)$. Then, (3.2) and Theorem 3.1 imply that

$$\mathscr{L}(e_{\langle *\rangle}, u(e_{\langle *\rangle})) \leqslant \liminf_{n\to\infty} \mathscr{L}(e_n, u(e_n)) = \operatorname*{Inf}_{e\in U_{ad}(\Omega)} \mathscr{L}(e, u(e)).$$

As a consequence, $e_{\langle * \rangle}$ is a solution to the problem $(\mathscr{B})$. ∎

We now consider the family of optimization problems $(\mathscr{P}_{\langle \varepsilon_n \rangle})$ which depend on $\varepsilon_n > 0$. Here we apply a penalty method for the existence of an optimal solution $(\mathscr{P})$.

Lemma 3.1 *For any $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$ the set $\mathscr{K}(\mathcal{O}, \Omega)$ defined in (2.9), is a nonempty closed and convex subset of $V(\Omega)$ and $\mathcal{O}_n \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$, $\mathcal{O}_n \to \mathcal{O}$ strongly in $\mathscr{U}^{\mathcal{O}}(\Omega) \Rightarrow \mathscr{K}(\mathcal{O}, \Omega) = \underset{n\to\infty}{Lim}\, \mathscr{K}(\mathcal{O}_n, \Omega)$.*

*Proof.* The condition (H0) ensures that the set $\mathscr{K}(\mathcal{O}, \Omega)$ is nonempty for any $\mathcal{O} \in U_{ad}^{\mathcal{O}}(\Omega) v = 0 \in \cap \mathscr{K}(\mathcal{O}, \Omega)$ for all $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$. Let $v_n \to v$ strongly in $V(\Omega)$, where $v_n \in \mathscr{K}(\mathcal{O}, \Omega)$. Next, due to the embedding theorem for the space $H^2(\Omega)$ we get: $\lim_{n\to\infty} v_n(x, y) = v(x, y)$ for every point $(x, y) \in \Omega$. Here, as $v_n(x, y) \geqslant \mathscr{S} + (\mathcal{O} + H_{\langle 0 \rangle})$ for all $(x, y) \in \Omega$, we obtain $v(x, y) \geqslant \mathscr{S} + (\mathcal{O} + H_{\langle 0 \rangle})$ in $\Omega$ and hence $v \in \mathscr{K}(\mathcal{O}, \Omega)$ as claimed.

$$\begin{cases} \text{For any } v \in \mathscr{K}(\mathcal{O}, \Omega) \text{ there exists a sequence } \{v_n\}_{n\in N}, \text{ such that:} \\ v_n \in V(\Omega),\ v_n \in \mathscr{K}(\mathcal{O}, \Omega) \text{ for } n \text{ sufficiently great and } v_n \to v \\ \text{strongly in } V(\Omega),\ \text{as } n \to \infty. \end{cases} \tag{3.11}$$

Indeed, let us define: $\boldsymbol{\mathcal{O}} = v - (\mathscr{S} + \mathcal{O} + H_{\langle 0 \rangle})$ so that $\boldsymbol{\mathcal{O}} \in C(\bar{\Omega})$, $\boldsymbol{\mathcal{O}} \geqslant 0$ in $\bar{\Omega}$ and

$$\vartheta_n = (\mathcal{O}_n - \mathcal{O}) - \boldsymbol{\mathcal{O}} = \mathcal{O}_n - v + (\mathscr{S} + H_{\langle 0 \rangle}) \in C(\bar{\Omega}),$$
$$\Pi_n = \{[x, y] \in \Omega : \vartheta_n(x, y) \geqslant \mathcal{C}/2\},$$

where

$$\mathcal{C} = \max_{[x,y]\in\partial\Omega} \mathscr{S}(x, y) + (\mathcal{O}_{\text{MAX}} + H_{\langle 0 \rangle}) < 0$$

due to the assumption (H0).

Next, there exists an open set $\Pi \subset \overline{\Pi} \subset \Omega$ such that

$$\Pi_n \subset \Pi \text{ for any } n. \tag{3.12}$$

To see this, we realize that: $\vartheta_n = \mathscr{S} + (\mathcal{O}_n + H_{\langle 0 \rangle}) \leqslant \mathcal{C}$ on the boundary $\partial\Omega$. Hence, the continuity of $\vartheta_n(x, y)$ and the constraints $|\partial \mathcal{O}_n / \partial x| \leqslant \text{constant}_{\langle 1 \rangle}$ and $|\partial \mathcal{O}_n / \partial y| \leqslant \text{constant}_{\langle 2 \rangle}$ imply that $\bigcup_{n=1}^{\infty} \Pi_n \subset\subset \Omega$ and (3.12) follows. Obviously, there exists a function $\boldsymbol{\mathcal{N}} \in C^\infty(\bar{\Omega})$ such that $\boldsymbol{\mathcal{N}}(x, y) = 1$ for $[x, y] \in \Pi$ and $\boldsymbol{\mathcal{N}}(x, y) = 0$, $\partial \boldsymbol{\mathcal{N}}(x, y)/\partial n = 0$ for $[x, y] \in \partial\Omega_{\text{DISPLACEMENT}}$, $0 < \boldsymbol{\mathcal{N}}(x, y) \leqslant 1$ for $[x, y] \in \Omega$. Let us set: $v_n = v + ||\mathcal{O}_n - \mathcal{O}||_{L_\infty(\Omega)} \boldsymbol{\mathcal{N}}$. Then, $v_n \in V(\Omega)$ and

$$||v - v_n||_{V(\Omega)} = ||\mathcal{O}_n - \mathcal{O}||_{L\infty(\Omega)} (||\boldsymbol{\mathcal{N}}||_{V(\Omega)}) \to 0 \text{ as } n \to \infty.$$

On the other hand, we may show that there exists $n_{\langle \circ \rangle} > 0$ such that

$$n > n_{\langle \circ \rangle} \Rightarrow v_n \geqslant \mathscr{S} + (\mathcal{O}_n + H_{\langle 0 \rangle}) \text{ in } \bar{\Omega} \Rightarrow v_n \in \mathscr{K}(\mathcal{O}_n, \Omega). \tag{3.13}$$

Indeed, let

$$\begin{cases} 1^\circ & [x,y] \in \Pi. \text{ Then one has} \\ & v_n = v + ||\mathcal{O}_n - \mathcal{O}||_{L_\infty(\Omega)} \geqslant v + (\mathcal{O}_n - \mathcal{O}) \geqslant \mathscr{S} + (\mathcal{O}_n + H_{\langle 0 \rangle}). \\ 2^\circ & \text{Let } [x,y] \in \bar{\Omega} \setminus \Pi. \text{ Then we have} \\ & v_n \geqslant \mathscr{S} + (\mathcal{O} + H_{\langle 0 \rangle}) + \boldsymbol{\mathcal{O}} + |\mathcal{O}_n - \mathcal{O}|\boldsymbol{\mathcal{N}}. \end{cases} \tag{3.14}$$

Taking into account that $[x,y] \notin \Pi$, $[x,y] \notin \Pi_n$ for any $n$ and $\vartheta_n \leqslant \mathcal{C}/2$ one has: $(\mathcal{O}_n - \mathcal{O}) - \boldsymbol{\mathcal{O}} \leqslant (\mathcal{C}/2)$, $-\mathcal{C}(\boldsymbol{\mathcal{N}}/2) + (1 - \boldsymbol{\mathcal{N}})\boldsymbol{\mathcal{O}} \leqslant \boldsymbol{\mathcal{O}} + |\mathcal{O}_n - \mathcal{O}|\boldsymbol{\mathcal{N}}$.

Hence (inserting into (3.14, 2°)), we obtain

$$v_n \geqslant \mathscr{S} + (\mathcal{O} + H_{\langle 0 \rangle}) + \mathcal{A} \text{ where } \mathcal{A} = -\boldsymbol{\mathcal{N}}(\mathcal{C}/2) + (1 - \boldsymbol{\mathcal{N}})\boldsymbol{\mathcal{O}}.$$

The function $\mathcal{A}$ is continuous and attains a positive minimum in the compact set $\bar{\Omega}\backslash\Pi$ : $\boldsymbol{\mathcal{M}} = \mathcal{A}([x_\circ, y_\circ]) = \min_{\bar{\Omega}\backslash\Pi} \mathcal{A} > 0$. Notice that, if $\boldsymbol{\mathcal{N}}(x_\circ, y_\circ) = 0$, then $[x_\circ, y_\circ] \in \partial\Omega_{\text{DISPLACEMENT}}$ and we have

$$\mathcal{A}(x_\circ, y_\circ]) = \boldsymbol{\mathcal{O}}([x_\circ, y_\circ]) = -[\mathscr{S}(x_\circ, y_\circ) + (\mathcal{O}(x_\circ, y_\circ) + H_{\langle 0 \rangle})] \geqslant -\mathcal{C} > 0.$$

Next, taking into account: $\boldsymbol{\mathcal{N}}(x_\circ, y_\circ) > 0$ one has $\mathcal{A}(x_\circ, y_\circ]) \geqslant -\mathcal{C}\boldsymbol{\mathcal{N}}(x_\circ, y_\circ)/2 > 0$. On the other hand there exists $n_\circ(\boldsymbol{\mathcal{M}})$ such that: $n \geqslant n_\circ(\boldsymbol{\mathcal{M}}) \Rightarrow ||\mathcal{O}_n - \mathcal{O}||_{L_\infty(\Omega)} \leqslant \boldsymbol{\mathcal{M}}$. Hence, then we have: $\mathcal{A}([x,y]) \geqslant \mathcal{A}[x_\circ, y_\circ]) \geqslant ||\mathcal{O}_n - \mathcal{O}||_{L_\infty(\Omega)} \geqslant [\mathcal{O}_n(x,y) - \mathcal{O}(x,y)]$, so that: $v_n(x,y) \geqslant \mathscr{S}(x,y) + [\mathcal{O}_n(x,y) + H_{\langle 0 \rangle}]$ for $n > n_\circ(\boldsymbol{\mathcal{M}}) \Rightarrow v_n \in \mathscr{K}(\mathcal{O}_n, \Omega)$. This means that the condition 1° in the Definition 3.1 is verified. Next we verify the condition 2°. As $v_n \in \mathscr{K}(\mathcal{O}_n, \Omega)$, $\mathcal{O}_n \to \mathcal{O}$ strongly in $\mathscr{U}^{\mathcal{O}}(\Omega)$ and $v_n \to v$ weakly in $V(\Omega)$, then $v_n \to v$ and $\mathcal{O}_n \to \mathcal{O}$ strongly in $C(\bar{\Omega})$ and the inequality for the limit remains valid.

The form of $\mathscr{K}(\mathcal{O}, \Omega)$ follows directly from its definition. Since: $\mathscr{S} + (\mathcal{O}_{\text{MAX}} + H_{\langle 0 \rangle}) \leqslant 0$ on $\Omega$ and due to the assumption (H0), the zero function belongs to $\mathscr{K}(\mathcal{O}, \Omega)$ for any $\mathcal{O} \in \mathscr{U}^{\mathcal{O}}_{ad}(\Omega)$. As a consequence ((H1),1°,2°) are satisfied. ■

The subspace $\boldsymbol{\mathcal{R}}(\Omega) := \{v \in V(\Omega) : \langle \mathscr{A}(\mathcal{O})v, v \rangle_{V(\Omega)} = 0\}$ is the set of rigid body motion of the plate. Let $P_V(\Omega)$ be the subspace of all possible (virtual) rigid body displacement of the middle plane, i.e., $P_V(\Omega) := \{v \in V(\Omega) : (\partial^2 v/\partial x^2)^2 = 0,\ (\partial^2 v/\partial y^2)^2 = 0,\ (\partial^2 v/\partial x \partial y)^2 = 0\}$.

Lemma 3.2 *Let* $v \in H^2(\Omega)$ *and* $(\partial^2 v/\partial x^2)^2 = 0$, $(\partial^2 v/\partial y^2)^2 = 0$, $(\partial^2 v/\partial x \partial y)^2 = 0$. *Then* $P_V(\Omega) = \{0\}$, *i.e.* $P_V(\Omega)$ *reduces to the zero element.*

*Proof.* The regularization of the displacement $v$ gives an element $v^{\langle h\rangle} \in \mathscr{C}(\bar{\Omega})$ for which

$$\begin{cases} \partial^2 v^{\langle h\rangle}/\partial x^2 = [\partial^2 v/\partial x^2]^{\langle h\rangle} = 0, \\ \partial^2 v^{\langle h\rangle}/\partial y^2 = [\partial^2 v/\partial y^2]^{\langle h\rangle} = 0, \\ \partial^2 v^{\langle h\rangle}/\partial x\partial y = [\partial^2 v/\partial x\partial y]^{\langle h\rangle} = 0, \end{cases} \tag{3.15}$$

holds for every domain $\Omega_{\langle *\rangle}$ such that $\bar{\Omega}_{\langle *\rangle} \subset \Omega$, provided $h$ is sufficiently small ($h < \operatorname{dis}(\bar{\Omega}_{\langle *\rangle}, \partial\Omega)$). Then from the conditions (3.15) we conclude that $v^{\langle h\rangle}$ is a linear polynomial. Since $v^{\langle h_n\rangle}$ converges to $v$ in $L_2(\Omega)$ as $h_n \to 0$ and finite-dimensional subspaces are closed in $L_2(\Omega)$, we conclude that $v^{\langle h_n\rangle}$ is a linear polynomial in every interior subdomain $\Omega_{\langle *\rangle}$, $\bar{\Omega}_{\langle *\rangle} \subset \Omega$ and thus throughout in $\Omega$. Thus, the homogeneous boundary condition of $\partial\Omega_{\text{DISPLACEMENT}}$, however, yields $v = 0$. ■

Lemma 3.3 *The system* $\{\Phi(\boldsymbol{e}, .)\}$, $\boldsymbol{e} \in \mathscr{U}_{ad}(\Omega)$ *of functionals defined by (2.11) satisfies the assumptions (H2).*

*Proof.* Since the integral is continuous on $V(\Omega)$ and the indicatrix lower semicontinuous, their sum is lower semicontinuous on $V(\Omega)$ for any $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$ and $\mathscr{F} \in \mathscr{U}_{ad}^{\mathscr{F}}(\Omega)$. The convexity is immediate. For any $\mathbf{e} \in \mathscr{U}_{ad}(\Omega)$, $v \in \mathscr{K}(\mathcal{O}, \Omega)$, the integral is finite and the indicatrix vanishes.

Let us verify the assumption (H2). In the following we set

$$\Phi(\mathbf{e}, v) = \Phi_{\langle\circ\rangle}\mathbf{e}, v) + \Phi_{\langle\bar{\circ}\rangle}(\mathbf{e}, v)$$

where

$$\Phi_{\langle\circ\rangle}(\mathbf{e}, v) = \int_{\partial\Omega_{\text{CONTACT}}} \mathscr{F}|v|\mathrm{d}S, \quad \Phi_{\langle\bar{\circ}\rangle}(\mathbf{e}, v) = I_{\mathscr{K}(\mathcal{O},\Omega)}(v).$$

From this we conclude that

$$\begin{aligned} |\Phi(\mathbf{e}_n, v_n) - \Phi(\mathbf{e}, v)| \leqslant &|\Phi_{\langle\circ\rangle}(\mathbf{e}_n, v_n) - \Phi_{\langle\circ\rangle}(\mathbf{e}, v)| \\ &+ |\Phi_{\langle\bar{\circ}\rangle}(\mathbf{e}_n, v_n) - \Phi_{\langle\bar{\circ}\rangle}(\mathbf{e}, v)| = |\boldsymbol{\mathcal{Z}}_{\langle\circ\rangle}| + |\boldsymbol{\mathcal{Z}}_{\langle\bar{\circ}\rangle}|. \end{aligned}$$

Next we shall verify the conditions (1°) and (2°) of Definition 3.2. First let $v \in \mathscr{K}(\mathcal{O}, \Omega)$ . Then by virtue of Lemma 3.1 there exists a sequence $\{v_n\}_{n\in N}$ such that $v_n \in \mathscr{K}(\mathcal{O}_n, \Omega)$, and $v_n \to v$ strongly in $V(\Omega)$. Then we have

$$\begin{aligned} |\boldsymbol{\mathcal{Z}}_{\langle\circ\rangle}| &\leqslant |\Phi_{\langle\circ\rangle}(\mathbf{e}_n, v_n) - \Phi_{\langle\circ\rangle}(\mathbf{e}, v_n)| + |\Phi_{\langle\circ\rangle}(\mathbf{e}, v_n) - \Phi_{\langle\circ\rangle}(\mathbf{e}, v)| \\ &\leqslant \int_{\partial\Omega_{\text{CONTACT}}} (|\mathscr{F}_n - \mathscr{F}|\,|v_n| + \mathscr{F}|v_n - v|)\mathrm{d}S \\ &\leqslant \text{constant}\,(||\mathscr{F}_n - \mathscr{F}||_{L_\infty(\partial\Omega)}||v_n||_{H^2(\Omega)} + \mathscr{F}_{\text{MAX}}||v_n - v||_{H^2(\Omega)}) \to 0. \\ \boldsymbol{\mathcal{Z}}_{\langle\bar{\circ}\rangle} &= I_{\mathscr{K}(\mathcal{O}_n,\Omega)}(v_n) - I_{\mathscr{K}(\mathcal{O},\Omega)}(v) = 0. \end{aligned}$$

Hence, we may write

$$\lim_{n\to\infty} \Phi(\mathbf{e}_n, v_n) = \Phi(\mathbf{e}, v). \tag{3.16}$$

On the other hand, let $v \notin \mathscr{K}(\mathcal{O}, \Omega)$. Here we set $v_n = v$ for all $n = 1, 2, \ldots$. From this we deduce that

$$\limsup_{n\to\infty} \Phi(\mathbf{e}_n, v_n) \leqslant \limsup_{n\to\infty} \int_{\partial\Omega_{\mathrm{CONTACT}}} \mathscr{F}_n |v| \mathrm{d}S + \limsup_{n\to\infty} I_{\mathscr{K}(\mathcal{O}_n,\Omega)}(v)$$

$$\leqslant \int_{\partial\Omega_{\mathrm{CONTACT}}} \mathscr{F}|v| \mathrm{d}S + \infty = \Phi_{\langle\mathrm{O}\rangle}(\mathbf{e}, v) + \Phi_{\langle\bar{\mathrm{O}}\rangle}(\mathbf{e}, v) = \Phi(\mathbf{e}, v), \tag{3.17}$$

since $I_{\mathscr{K}(\mathcal{O},\Omega)}(v) = +\infty$. Then we would have (due to (3.16) and (3.17))

$$\limsup_{n\to\infty} \Phi(\mathbf{e}_n, v_n) \leqslant \Phi(\mathbf{e}, v) \quad \text{for any } v \in V(\Omega).$$

As a consequence, condition ((H2),1°) is satisfied.

Next let $v_n \to v$ weakly in $V(\Omega)$. From this follows

$$\liminf_{n\to\infty} \Phi(\mathbf{e}_n, v_n) \geqslant \liminf_{n\to\infty} \Phi_{\langle\mathrm{O}\rangle}(\mathbf{e}_n, v_n) + \liminf_{n\to\infty} \Phi_{\langle\bar{\mathrm{O}}\rangle}(\mathbf{e}_n, v_n),$$

so that in view of the compactness of the trace mapping $H^1(\Omega) \to L(\partial\Omega)$, we may write

$$|\Phi_{\langle\mathrm{O}\rangle}(\mathbf{e}_n, v_n) - \Phi_{\langle\mathrm{O}\rangle}(\mathbf{e}, v)| \leqslant \int_{\partial\Omega_{\mathrm{CONTACT}}} (|\mathscr{F}_n - \mathscr{F}|\,|v_n| + \mathscr{F}|v_n - v|) \mathrm{d}S \to 0.$$

Hence one has

$$\lim_{n\to\infty} \Phi_{\langle\mathrm{O}\rangle}(\mathbf{e}_n, v_n) = \Phi_{\langle\mathrm{O}\rangle}(\mathbf{e}, v).$$

Further, one has

$$\liminf_{n\to\infty} \Phi_{\langle\bar{\mathrm{O}}\rangle}(\mathbf{e}_n, v_n) = \liminf_{n\to\infty} I_{\mathscr{K}(\mathcal{O}_n,\Omega)}(v_n) = \Pi,$$

where $\Pi$ is either $+\infty$ or zero. As $\Pi = +\infty$, then obviously

$$\Pi \geqslant I_{\mathscr{K}(\mathcal{O},\Omega)}(v). \tag{3.18}$$

If $\Pi = 0$, there exists a subsequence $\{v_k\}_{k\in N} \subset \{v_n\}_{n\in N}$ such that: $v_k \in \mathscr{K}(\mathcal{O}_k, \Omega)$ for all $k \to \infty$. Then by virtue of Lemma 3.1 the weak limit $v$ belongs to $\mathscr{K}(\mathcal{O}, \Omega)$ so that $I_{\mathscr{K}(\mathcal{O},\Omega)}(v) = 0$ and (3.18) holds again. As a consequence, condition (H2,2°) is fulfilled, as well. Hence we conclude that

$$\Phi(e, .) = \operatorname*{Lim}_{n\to\infty} \Phi(\mathbf{e}_n, .)$$

holds, provided $\mathbf{e}_n \to \mathbf{e}$ strongly in $\mathscr{U}(\Omega)$.

Finally, we can set $a_n = 0$, $n = 1, 2, \ldots$, since $0 \in \mathscr{K}(\mathcal{O}, \Omega)$ for all $\mathcal{O} \in \mathscr{U}^{\mathcal{O}}_{ad}(\Omega)$ (due to (H0)). Then one has: $\Phi(\mathbf{e}_n, a_n) = 0$ for all $n$. ■

Lemma 3.4 *The family of operators* $\{\mathscr{A}(\mathcal{O}_n)\}_{n\in N}$, $\mathcal{O}_n \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$, *defined by (2.10), satisfies the assumption ((H1),* $3^\circ$, $4^\circ$*).*

*Proof.* It is readily seen that

$$\begin{cases} \langle \mathscr{A}(\mathcal{O})v, v\rangle_{V(\Omega)} \geqslant \alpha_{\mathscr{A}} \|v\|_{V(\Omega)}^2, \\ |\langle \mathscr{A}(\mathcal{O})v, z\rangle_{V(\Omega)}| \leqslant M_{\mathscr{A}} \|v\|_{V(\Omega)} \|z\|_{V(\Omega)}, \end{cases} \tag{3.19}$$

for any $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$, for any $[v, z] \in V(\Omega)$, with the positive constants $[\alpha_{\mathscr{A}}, M_{\mathscr{A}}]$, independent of $[\mathcal{O}, v]$.

Indeed, due to the Sylvester criterion and the assumption ((2.2),$2^\circ$) we deduce the quadratic form

$$\frac{E_1}{1-\mu_{12}\mu_{21}}\xi_1^2 + \frac{2\mu_1 E_2}{1-\mu_{12}\mu_{21}}\xi_1\xi_2 + \frac{E_2}{1-\mu_{12}\mu_{21}}\xi_2^2,$$

for all $[\xi_1, \xi_1] \in \mathbb{R}$ to be a positive definite. Hence, we have

$$a(\mathcal{O}, v, v) \geqslant \alpha_\circ \int_\Omega [(\partial^2 v/\partial x^2)^2 + (\partial^2 v/\partial y^2)^2 + (\partial^2 v/\partial x\partial y)^2]\mathrm{d}\Omega,$$

where $v \in V(\Omega)$, $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$, $\alpha_\circ = \text{constant} > 0$.

On the other hand, the definition of $\boldsymbol{\mathcal{R}}(\Omega)$, the present estimate for $a(\theta, v, v)$ and Lemma 3.2 imply that $\boldsymbol{\mathcal{R}}(\Omega) = \{0\}$.

Then, by Corollary 1.6.1 (Litvinov, 2000) the formula

$$\left\{ \int_\Omega [(\partial^2 v/\partial x^2)^2 + (\partial^2 v/\partial y^2)^2 + (\partial^2 v/\partial x\partial y)^2]\mathrm{d}\Omega \right\}^{1/2}$$

defines a norm in $V(\Omega)$, which is equivalent to the original one, i.e., to the norm of $H^2(\Omega)$.

Further, we may write:

$$\begin{aligned} |\langle \mathscr{A}(\mathcal{O})v, w\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})z, w\rangle_{V(\Omega)}| = \Big| \int_\Omega \big[(D_{11}(\mathcal{O})(\partial^2(v-z)/\partial x^2)(\partial^2 w/\partial x^2) \\ +D_{22}(\mathcal{O})(\partial^2(v-z)/\partial y^2)(\partial^2 w/\partial y^2) + D_{12}(\mathcal{O})((\partial^2 v(v-z)/\partial x^2)(\partial^2 w/\partial y^2) \\ +(\partial^2(v-z)/\partial y^2)(\partial^2 w/\partial x^2)) + 2D_{33}(\mathcal{O})(\partial^2(v-z)/\partial x\partial y)(\partial^2 w/\partial x\partial y)\big]\mathrm{d}\Omega\Big| \end{aligned}$$

$$
\begin{aligned}
&\leqslant D_{11}(\mathcal{O}_{\mathrm{MAX}})\int\limits_\Omega |(\partial^2(v-z)/\partial x^2)|\,|\partial^2 w/\partial x^2|\mathrm{d}\Omega\\
&\quad +D_{22}(\mathcal{O}_{\mathrm{MAX}})\int\limits_\Omega |(\partial^2(v-z)/\partial y^2)|\,|\partial^2 w/\partial y^2|\mathrm{d}\Omega\\
&\quad +D_{12}(\mathcal{O}_{\mathrm{MAX}})\int\limits_\Omega (|(\partial^2(v-z)/\partial x^2)|\,|\partial^2 w/\partial y^2|\\
&\qquad\qquad +|(\partial^2(v-z)/\partial y^2)|\,|\partial^2 w/\partial x^2|)\mathrm{d}\Omega\\
&\quad +2D_{33}(\mathcal{O}_{\mathrm{MAX}})\int\limits_\Omega |(\partial^2(v-z)\partial x\partial y)|\,|\partial^2 w/\partial x\partial y|\mathrm{d}\Omega\\
&\leqslant \max\,[D_{11}(\mathcal{O}_{\mathrm{MAX}}), D_{22}(\mathcal{O}_{\mathrm{MAX}}), D_{12}(\mathcal{O}_{\mathrm{MAX}}), 2D_{33}(\mathcal{O}_{\mathrm{MAX}})]||w-z||_{V(\Omega)}||w||_{V(\Omega)}\\
&\leqslant \text{constant}\,[\mathcal{O}^3_{\mathrm{MAX}}+\mathcal{O}^2_{\mathrm{MAX}}+\mathcal{O}_{\mathrm{MAX}}]||v-z||_{V(\Omega)}||w||_{V(\Omega)}.
\end{aligned}
$$

As a consequence, the assumption ((H1),3°) is satisfied.

Next, in order to to verify ((H1),4°), we write

$$
\begin{aligned}
&|\langle \mathscr{A}(\mathcal{O}_n)v-\mathscr{A}(\mathcal{O})v, w\rangle_{V(\Omega)}| = \Big|\int\limits_\Omega \big[(D_{11}(\mathcal{O}_n)-D_{11}(\mathcal{O}))(\partial^2 v/\partial x^2)(\partial^2 w/\partial x^2)\\
&\quad +(D_{22}(\mathcal{O}_n)-D_{22}(\mathcal{O}))(\partial^2 v/\partial y^2)(\partial^2 w/\partial y^2)\\
&\quad +(D_{12}(\mathcal{O}_n)-D_{12}(\mathcal{O}))((\partial^2 v/\partial x^2)(\partial^2 w/\partial y^2)+(\partial^2 v/\partial y^2)(\partial^2 w/\partial x^2))\\
&\quad +2(D_{33}(\mathcal{O}_n)-D_{33}(\mathcal{O}))(\partial^2 v/\partial x\partial y)(\partial^2 w/\partial x\partial y)\big]\mathrm{d}\Omega\Big|\\
&\leqslant \text{constant}\,\big(||\mathcal{O}^3_n-\mathcal{O}^3||_{L_\infty(\Omega)}+||\mathcal{O}^2_n-\mathcal{O}^2||_{L_\infty(\Omega)}\\
&\qquad\qquad +||\mathcal{O}_n-\mathcal{O}||_{L_\infty(\Omega)}\big)\,||v||_{V(\Omega)}||w||_{V(\Omega)}.
\end{aligned}
$$

Then one has

$$
\begin{aligned}
&||\mathscr{A}(\mathcal{O}_n)v-\mathscr{A}(\mathcal{O})v||_{V^*(\Omega)}\\
&\leqslant \text{constant}\,(||\mathcal{O}^3_n-\mathcal{O}^3||_{L_\infty(\Omega)}+||\mathcal{O}^2_n-\mathcal{O}^2||_{L_\infty(\Omega)}\\
&\qquad +||\mathcal{O}_n-\mathcal{O}||_{L_\infty(\Omega)})||v||_{V(\Omega)}\to 0,
\end{aligned}
\tag{3.20}
$$

as $\mathcal{O}_n\to\mathcal{O}$ strongly in $\mathscr{U}^{\mathcal{O}}(\Omega)$.

We may introduce the state variational inequality (2.12) for $u(\mathcal{O}_n)\in\mathscr{K}(\mathcal{O}_n,\Omega)$

$$
\begin{aligned}
&\langle \mathscr{A}(\mathcal{O}_n)u(\mathbf{e}_n), v-u(\mathbf{e}_n)\rangle_{V(\Omega)}+\Phi(\mathbf{e}_n,v)-\Phi(\mathbf{e}_n,u(\mathbf{e}_n))\\
&\geqslant \langle \mathrm{L}(\mathbf{e}_n), v-u(\mathbf{e}_n)\rangle_{V(\Omega)}
\end{aligned}
\tag{3.21}
$$

for all $v\in\mathscr{K}(\mathcal{O}_n,\Omega)$.

Consequently, by virtue of Lemma 3.1, we may write (inserting the sequence $\{a_n\}_{n\in N}$ (with $v=a_n$ due to Lemma 3.3 into the variational inequality for $\mathbf{e}_n\in\mathscr{U}_{ad}(\Omega)$)

$$\langle \mathscr{A}(\mathcal{O}_n)u(\mathbf{e}_n), a_n - u(\mathbf{e}_n)\rangle_{V(\Omega)} + \Phi(\mathbf{e}_n, a_n) - \Phi(\mathbf{e}_n, u(\mathbf{e}_n)) \geqslant \langle L(\mathbf{e}_n), a_n - u(\mathbf{e}_n)\rangle_{V(\Omega)} \tag{3.22}$$

for $n > n_{\circ}$.

Hence in view of ((3.19,1°) and Lemma 3.3, we deduce that (we can set $a_n = 0$, $n = 1, 2, \ldots$, since the zero function belongs to $\mathscr{K}(\mathcal{O}, \Omega)$ for any $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$)

$$\alpha_{\mathscr{A}}||u(\mathbf{e}_n)||^2_{V(\Omega)} + \Phi(\mathbf{e}_n, u(\mathbf{e}_n)) \leqslant \langle L(\mathbf{e}_n), v - u(\mathbf{e}_n)\rangle_{V(\Omega)} \leqslant \text{constant}||u(\mathbf{e}_n)||_{V(\Omega)}$$

and

$$||u(\mathbf{e}_n)||_{V(\Omega)} \leqslant \text{constant, for any } n.$$

This means that there exists $u_{\diamondsuit} \in V(\Omega)$ and a subsequence $\{u(\mathbf{e}_{n_k})\}_{k \in N} \subset \{u(\mathbf{e}_n)\}_{n \in N}$, such that

$$u(\mathbf{e}_{n_k}) \to u_{\diamondsuit} \text{ weakly in } V(\Omega). \tag{3.23}$$

The functional $v \to \langle \mathscr{A}(\mathcal{O})v, v\rangle_{V(\Omega)}$ is weakly lower semicontinuous on $V(\Omega)$ for any $\mathcal{O} \in \mathscr{U}_{ad}^{\mathcal{O}}(\Omega)$. Consequently,

$$\liminf_{n\to\infty} \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} \geqslant \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, u_{\diamondsuit}\rangle_{V(\Omega)}, \text{ since } \mathbf{e} \in \mathscr{U}_{ad}(\Omega).$$

Moreover, we have

$$\begin{aligned} &|\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)}| \\ &\leqslant \text{constant}\left[\, ||\mathcal{O}^3_{n_k} - \mathcal{O}^3||_{L_\infty(\Omega)} + ||\mathcal{O}^2_{n_k} - \mathcal{O}^2||_{L_\infty(\Omega)} \right. \\ &\left. + ||\mathcal{O}_{n_k} - \mathcal{O}||_{L_\infty(\Omega)}\right] ||u(\mathbf{e}_{n_k})||^2_{V(\Omega)} \to 0, \end{aligned} \tag{3.24}$$

as $\mathbf{e}_{n_k} \to \mathbf{e}$ strongly in $\mathscr{U}(\Omega)$.

From the above argument, we conclude that

$$\begin{aligned} &\liminf_{k\to\infty} \langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} = \liminf_{k\to\infty} (\langle \mathscr{A}(\mathcal{O})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} \\ &\qquad + [\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)}]) \\ &\geqslant \liminf_{k\to\infty} \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} \geqslant \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, u_{\diamondsuit}\rangle_{V(\Omega)}. \end{aligned} \tag{3.25}$$

Further, we may write (using the decomposition:

$$\begin{aligned} &\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), \mathcal{Q}\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, \mathcal{Q}\rangle_{V(\Omega)} \\ &= [\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), \mathcal{Q}\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}_{n_k}), \mathcal{Q}\rangle_{V(\Omega)}] \\ &\qquad + \langle \mathscr{A}(\mathcal{O})(u(\mathbf{e}_{n_k}) - u_{\diamondsuit}), \mathcal{Q}\rangle_{V(\Omega)} \end{aligned}$$

and the weak convergence of $\{u(\mathbf{e}_{n_k})\}_{k \in N}$)

$$\lim_{k\to\infty} \langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), \mathcal{Q}\rangle_{V(\Omega)} = \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, \mathcal{Q}\rangle_{V(\Omega)} \text{ for any } \mathcal{Q} \in V(\Omega). \tag{3.26}$$

Taking into account (3.11), we obtain

$$|\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), v_k - v\rangle_{V(\Omega)}| \leqslant \text{constant}||u(\mathbf{e}_{n_k})||_{V(\Omega)}||v_k - v||_{V(\Omega)} \to 0, \text{ as } k \to \infty. \tag{3.27}$$

Then due to (3.26) and (3.27), we arrive at

$$\begin{aligned}&|\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), v_k\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}| \\ &\leqslant |\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), v_k - v\rangle_{V(\Omega)}| \\ &\qquad + |\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), v\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}| \to 0.\end{aligned} \tag{3.28}$$

Furthermore, weak convergence of $\{u(\mathbf{e}_{n_k})\}_{k\in N}$ and (3.11) yield that

$$\langle \mathrm{L}(\mathbf{e}_{n_k}), v_k - u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} \to \langle \mathrm{L}(\mathbf{e}), v - u_\diamondsuit\rangle_{V(\Omega)}, \tag{3.29}$$

when $\mathbf{e}_{n_k} \to \mathbf{e}$ strongly in $\mathscr{U}(\Omega)$.

On the other hand, from the inequality (3.21), we deduce that

$$\begin{aligned}&\langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} + \Phi(\mathbf{e}_{n_k}, u(\mathbf{e}_{n_k})) + \langle \mathrm{L}(\mathbf{e}_{n_k}), v_k - u(\mathbf{e}_{n_k})\rangle_{V(\Omega)} \\ &\leqslant \langle \mathscr{A}(\mathcal{O}_{n_k})u(\mathbf{e}_{n_k}), v_k\rangle_{V(\Omega)} + \Phi(\mathbf{e}_{n_k}, v_k).\end{aligned} \tag{3.30}$$

Passing here in (3.30) to the limit inferior on both sides with $k \to \infty$, using (3.25), (3.28), (3.29) and Lemma 3.3, we obtain

$$\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)} + \Phi(\mathbf{e}, u_\diamondsuit) + \langle \mathrm{L}(\mathbf{e}), v - u_\diamondsuit\rangle_{V(\Omega)} \leqslant \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)} + \Phi(\mathbf{e}, v).$$

Consequently, $u_\diamondsuit$ satisfies the inequality (2.12). Since the solution $u(\mathbf{e})$ of (2.12) is unique, $u_\diamondsuit = u(\mathbf{e})$ follows and the whole sequence $\{u(\mathbf{e}_n)\}_{n\in N}$ converges to $u(\mathbf{e})$ weakly in $V(\Omega)$.

Finally, it remains to verify the strong convergence. By virtue of (3.30), (3.28) and (3.29), we can write

$$\begin{aligned}&\limsup_{n\to\infty} \langle \mathscr{A}(\mathcal{O}_n)u(\mathbf{e}_n), u(\mathbf{e}_n)\rangle_{V(\Omega)} \\ &\leqslant \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), v\rangle_{V(\Omega)} + \Phi(\mathbf{e}, v) - \Phi(\mathbf{e}, u(\mathbf{e})) + \langle \mathrm{L}(\mathbf{e}), u(\mathbf{e}) - v\rangle_{V(\Omega)},\end{aligned} \tag{3.31}$$

for any $v \in \mathscr{K}(\mathcal{O}, \Omega)$.

Hence (we put $v := u(\mathbf{e})$ in (3.31)) due to (3.25), we get

$$\begin{aligned}&\langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)} \leqslant \liminf_{n\to\infty} \langle \mathscr{A}(\mathcal{O}_n)u(\mathbf{e}_n), u(\mathbf{e}_n)\rangle_{V(\Omega)} \\ &\leqslant \limsup_{n\to\infty} \langle \mathscr{A}(\mathcal{O}_n)u(\mathbf{e}_n), u(\mathbf{e}_n)\rangle_{V(\Omega)} \leqslant \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)}.\end{aligned}$$

This means that

$$\lim_{n\to\infty} \langle \mathscr{A}(\mathcal{O}_n)u(\mathbf{e}_n), u(\mathbf{e}_n)\rangle_{V(\Omega)} = \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)}. \tag{3.32}$$

Taking into account (3.24) and (3.32), we arrive at

$$\lim_{n\to\infty}\langle\mathscr{A}(\mathcal{O})u(\mathbf{e}_n),u(\mathbf{e}_n)\rangle_{V(\Omega)}=\langle\mathscr{A}(\mathcal{O})u(\mathbf{e}),u(\mathbf{e})\rangle_{V(\Omega)}. \tag{3.33}$$

Further, we equip the space $V(\Omega)$ with the scalar product

$$\langle\mathscr{A}(\mathcal{O})u(\mathbf{e}),v\rangle_{V(\Omega)}=(u(\mathbf{e}),v)_{\mathscr{A}}.$$

Hence (3.33) implies that the associated norms $||u(\mathbf{e}_n)||_{\mathscr{A}}$ tend to $||u(\mathbf{e})||_{\mathscr{A}}$. Since the norms $||\cdot||_{\mathscr{A}}$ and $||\cdot||_{V(\Omega)}$ are equivalent, we are led to the strong convergence

$$||u(\mathbf{e}_n)-u(\mathbf{e})||_{V(\Omega)}\to 0. \tag{3.34}$$

■

Lemma 3.5 *Let* $\mathbf{e}_n\to\mathbf{e}$ *strongly in* $\mathscr{U}(\Omega)$ *as* $n\to\infty$, $\mathbf{e}\in\mathscr{U}_{ad}(\Omega)$. *Then one has* $\mathbf{M}(\mathbf{e}_n)\to\mathbf{M}(\mathbf{e})$ *strongly in* $[\mathrm{L}_2(\Omega)]^3$.

*Proof.* We may write

$$\begin{aligned}&||M_{xx}(\mathbf{e}_n)-M_{xx}(\mathbf{e})||_{L_2(\Omega)}\\&\leqslant||(D_{11}(\mathcal{O}_n)-D_{11}(\mathcal{O}))\partial^2u(\mathbf{e}_n)/\partial x^2+(D_{12}(\mathcal{O}_n)-D_{12}(\mathcal{O}))\partial^2u(\mathbf{e}_n)/\partial y^2||_{L_2(\Omega)}\\&\quad+||D_{11}(\mathcal{O})(\partial^2u(\mathbf{e}_n)/\partial x^2-\partial^2u(\mathbf{e})/\partial x^2)\\&\quad+D_{12}(\mathcal{O})(\partial^2u(\mathbf{e}_n)/\partial y^2-\partial^2u(\mathbf{e})/\partial y^2)||_{L_2(\Omega)}\\&=M_{\langle 1n\rangle}+M_{\langle 2n\rangle}.\end{aligned}$$

Next, in view of (3.34) we have: $M_{\langle 1n\rangle}\to 0$, $M_{\langle 2n\rangle}\to 0$ as $n\to\infty$. Equally, we obtain

$$||M_{yy}(\mathbf{e}_n)-M_{yy}(\mathbf{e})||_{L_2(\Omega)}\to 0,\quad ||M_{xy}(\mathbf{e}_n)-M_{xy}(\mathbf{e})||_{L_2(\Omega)}\to 0.$$

■

Lemma 3.6 *Let* $\mathbf{e}_n\to\mathbf{e}$ *strongly in* $\mathscr{U}(\Omega)$ *as* $n\to\infty$, $\mathbf{e}\in\mathscr{U}_{ad}(\Omega)$. *Then for any* $\mathsf{Q}=1,2,\ldots,N_{\mathsf{Q}}$

$$([\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_n,M(\mathbf{e}_n))]^+)\to([\mathscr{S}_{\mathsf{Q}}(\mathbf{e},M(\mathbf{e})]^+).$$

*Proof.* Due to the estimate: $|a^+-b^+|\leqslant|a-b|$, we may write:

$$\begin{aligned}&\big|([\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_n,\mathbf{M}(\mathbf{e}_n))]^+)-([\mathscr{S}_{\mathsf{Q}}(\mathbf{e},\mathbf{M}(\mathbf{e}))]^+)\big|\leqslant\big|\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_n,\mathbf{M}(\mathbf{e}_n))-\mathscr{S}_{\mathsf{Q}}(\mathbf{e},\mathbf{M}(\mathbf{e}))\big|\\&\leqslant(36/\operatorname{meas}\Omega^*_{\mathsf{Q}}\int_{\Omega^*_{\mathsf{Q}}}\big|(1/(H_{\langle 0\rangle}+\mathcal{O}_n)^4)(M^2_{xx}(\mathbf{e}_n)+M^2_{yy}(\mathbf{e}_n)+(\sigma_0/\tau_0)^2M^2_{xy}(\mathbf{e}_n))\\&\quad-(1/(H_{\langle 0\rangle}+\mathcal{O})^4)(M^2_{xx}(\mathbf{e})+M^2_{yy}(\mathbf{e})+(\sigma_0/\tau_0)^2M^2_{xy}(\mathbf{e}))\big|\mathrm{d}\Omega\end{aligned}$$

$$\leqslant \text{constant} \int\limits_{\Omega^*_{\mathrm{Q}}} \big|(1/(H_{\langle 0\rangle}+\mathcal{O}_n)^4)([M^2_{xx}(\mathbf{e}_n)-M^2_{xx}(\mathbf{e})]+[M^2_{yy}(\mathbf{e}_n)-M^2_{yy}(\mathbf{e})]$$
$$+(\sigma_0/\tau_0)^2[M^2_{xy}(\mathbf{e}_n)-M^2_{xy}(\mathbf{e})])+(1/(H_{\langle 0\rangle}+\mathcal{O}_n)^4)$$
$$-(1/(H_{\langle 0\rangle}+\mathcal{O})^4)[M^2_{xx}(\mathbf{e})+M^2_{yy}(\mathbf{e})+(\sigma_0/\tau_0)^2 M^2_{xy}(\mathbf{e})])\big|\mathrm{d}\Omega$$
$$\leqslant \text{constant } (1/(H_{\langle 0\rangle}+\mathcal{O}_{\mathrm{MIN}})^4)\Big\{\int\limits_{\Omega^*_{\mathrm{Q}}}\big(|M_{xx}(\mathbf{e}_n)+M_{xx}(\mathbf{e})||M_{xx}(\mathbf{e}_n)-M_{xx}(\mathbf{e})|$$
$$+|M_{yy}(\mathbf{e}_n)+M_{yy}(\mathbf{e})||M_{yy}(\mathbf{e}_n)-M_{yy}(\mathbf{e})|+(\sigma_0/\tau_0)^2|M_{xy}(\mathbf{e}_n)+M_{xy}(\mathbf{e})|$$
$$\times|M_{xy}(\mathbf{e}_n)-M_{xy}(\mathbf{e})|\big)\mathrm{d}\Omega\Big\}+\text{constant } (\mathbf{e})||(1/(H_{\langle 0\rangle}+\mathcal{O}_n)^4)$$
$$-(1/(H_{\langle 0\rangle}+\mathcal{O})^4)||_{L_\infty(\Omega)}\to 0,$$

as $n\to\infty$, due to Lemma 3.5. ■

LEMMA 3.7 *The penalized optimal control problem* $(\mathscr{P}_{\langle\varepsilon_n\rangle})$ *has a solution for any* $\varepsilon_n>0$.

*Proof.* Note that the functionals $\mathscr{L}(\mathbf{e})$ and $([\mathscr{S}_{\mathrm{Q}}(\mathbf{e},\mathbf{M}(\mathbf{e}))]^+)$ are continuous in $\mathscr{U}_{ad}(\Omega)$ and $\mathscr{U}_{ad}(\Omega)$ is compact in $\mathscr{U}(\Omega)$ . Hence, there exists a minimizer $\mathbf{e}_{\langle\varepsilon_n\rangle}$ of $\mathscr{L}_{\langle\varepsilon_n\rangle}(\mathbf{e},\mathbf{M}(\mathbf{e}))$ in $\mathscr{U}_{ad}(\Omega)$. ■

THEOREM 3.3 *Let the condition (2.13) be satisfied. Let* $\{\varepsilon_n\}_{n\in N}$, $\varepsilon_n\to 0_+$ *be a sequence and* $\{\mathbf{e}_{\langle\varepsilon_n\rangle}\}_{n\in N}$ *a sequence of solutions of the penalized optimal control problems* $(\mathscr{P}_{\langle\varepsilon_n\rangle})$, $\{\mathbf{M}(\mathbf{e}_{\langle\varepsilon_n\rangle})\}_{n\in N}$ *the sequence of corresponding moment fields.*

*Then there exists a subsequence* $\{\varepsilon_{n_k}\}_{k\in N}\subset\{\varepsilon_n\}_{n\in N}$ *and an element* $e_{\langle\bigcirc\rangle}\in\mathscr{G}_{ad}(\Omega)$ *such that*

$$\begin{cases}\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}\to\mathbf{e}_{\langle\bigcirc\rangle} \text{ strongly in } \mathscr{U}(\Omega),\\ \mathbf{M}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle})\to\mathbf{M}(\mathbf{e}_{\langle\bigcirc\rangle}) \text{ strongly in } [L_2(\Omega)]^3,\end{cases}\tag{3.35}$$

*where* $e_{\langle\bigcirc\rangle}$ *is a solution of the optimal control problem* $(\mathscr{P})$.

*Proof.* There exists a subsequence $\{\varepsilon_{n_k}\}_{k\in N}\subset\{\varepsilon_n\}_{n\in N}$ (here, $\mathscr{U}_{ad}(\Omega)$ is compact in $\mathscr{U}(\Omega)$) such that ((3.35),1°) holds with $\mathbf{e}_{\langle\bigcirc\rangle}\in\mathscr{U}_{ad}(\Omega)$. In view of Lemma 3.5, we obtain ((3.35),2° ).

Further, the definition yields

$$\mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle})+(1/\varepsilon_{n_k})\sum_{\mathrm{Q}=1}^{N_{\mathrm{Q}}}([\mathscr{S}_{\mathrm{Q}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle},\mathbf{M}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}))]^+)$$
$$\leqslant\mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e})+(1/\varepsilon_{n_k})\sum_{\mathrm{Q}=1}^{N_{\mathrm{Q}}}([\mathscr{S}_{\mathrm{Q}}(\mathbf{e},\mathbf{M}(\mathbf{e}))]^+),\tag{3.36}$$

holds for any $\mathbf{e}\in\mathscr{U}_{ad}(\Omega)$.

On the other hand, for an arbitrary element $\mathbf{e}$ from $\mathscr{G}_{ad}(\Omega)$, we have

$$\begin{cases} \varepsilon_{n_k}\mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}) + \sum\limits_{\mathsf{Q}=1}^{N_{\mathsf{Q}}}([\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}, \mathbf{M}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}))]^{+}) \leqslant \varepsilon_{n_k}\mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}), \\ 0 \leqslant \sum\limits_{\mathsf{Q}=1}^{N_{\mathsf{Q}}}([\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}, \mathbf{M}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}))]^{+}) \leqslant \varepsilon_{n_k}\mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}). \end{cases}$$

Hence, passing to the limit with $\varepsilon_{n_k} \to 0$ and by virtue of Lemma 3.6, we arrive at

$$\sum_{\mathsf{Q}=1}^{N_{\mathsf{Q}}}([\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_{\langle\circ\rangle}, \mathbf{M}(\mathbf{e}_{\langle\circ\rangle}))]^{+}) = 0.$$

This means that the element $\mathbf{e}_{\langle\circ\rangle} \in \mathscr{G}_{ad}(\Omega)$.

Then, on account of (3.36), we obtain

$$\begin{aligned} \mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}) \leqslant \mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}) + (1/\varepsilon_{n_k}) \sum_{\mathsf{Q}=1}^{N_{\mathsf{Q}}}([\mathscr{S}_{\mathsf{Q}}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}, \mathbf{M}(\mathbf{e}_{\langle\varepsilon_{n_k}\rangle}))]^{+}) \\ \leqslant \mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}) \end{aligned} \tag{3.37}$$

for any $\mathbf{e} \in \mathscr{G}_{ad}(\Omega)$.

We deduce form (3.37) the estimate (passing to the limit with $\varepsilon_{n_k} \to 0$ (or $k \to \infty$) and we use ((3.35),1°))

$$\mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}_{\langle\circ\rangle}) \leqslant \mathscr{L}_{\mathsf{WEIGHT}}(\mathbf{e}),$$

for any $\mathbf{e} \in \mathscr{G}_{ad}(\Omega)$. ■

Lemma 3.8 *For nonempty set $\mathscr{G}_{ad}(\Omega)$ there exists at least one solution of the optimal control problem ($\mathscr{P}$).*

*Proof.* The proof follows immediately from Lemma 3.7 and Theorem 3.3. ■

## 4. Penalty method-cost functional with norm

We now investigate the technique in which the functional to be minimized is augmented by a so-called penalty functional. The penalty functional is designed so that it grows in magnitude as the amount by which the constraint is violated grows, in other words, we pay a penalty in our optimization process for violating the constraint, and the more we violate it, the larger the penalty paid. We remark that the penalty method is much more general than the Lagrange multiplier method (neither convexity for the functional or kinematically admissible set need be assumed).

For any $\mathbf{e} \in \mathscr{U}_{ad}(\Omega)$ consider the following variational inequality:

$$\text{Find } u(\mathbf{e}) \in \mathscr{K}(\mathbf{e},\Omega) \text{ such that}$$
$$\boldsymbol{\mathscr{A}}(\mathbf{e})u(\mathbf{e}) - u(\mathbf{e})\rangle_{V(\Omega)} \geqslant \langle L(\mathbf{e}), v - u(\mathbf{e})\rangle_{V(\Omega)} \text{ for all } v \in \mathscr{K}(\mathbf{e},\Omega) \tag{4.1}$$

where $V(\Omega) = H_0^2(\Omega)$, $L(\mathbf{e}) \in L_2(\Omega)$, $\mathscr{S} \in H^2(\Omega)$ and $\mathbf{e} = [0,0]^T$.

Consider now the optimal control problem $(\mathscr{B})$:

$$\text{Find an element } \mathbf{e}_0 \text{ such that}$$
$$\mathscr{L}(\mathbf{e}_\circ) \leqslant \mathscr{L}(\mathbf{e}) \text{ for any } \mathbf{e} \in \mathscr{U}_{ad}(\Omega), \tag{4.2}$$

where $\mathscr{L}(\mathbf{e}) = ||u(\mathbf{e}) - z_{ad}||^2_{L_2(\Omega)}$, and the element $z_{ad} \in L_2(\Omega)$ is given.

Further, we consider a family of optimization $(\mathscr{B}_\varepsilon)$ problems depending on $\varepsilon > 0$ . For a fixed $\varepsilon > 0$, an approximate problem is of the form

$$\operatorname*{Inf}_{\mathbf{e}\in\mathscr{U}_{ad}(\Omega)} \mathscr{L}_\varepsilon(\mathbf{e}) = \operatorname*{Inf}_{\mathbf{e}\in\mathscr{U}_{ad}(\Omega)} [||u_\varepsilon(\mathbf{e}) - z_{ad}||^2_{L_2(\Omega)}]. \tag{4.3}$$

Here, $u_\varepsilon(e) \in V(\Omega)$ is a solution to the nonlinear equation (the penalized problem)

$$u_\varepsilon(\mathbf{e}) \in V(\Omega) : \boldsymbol{\mathscr{A}}(\mathbf{e})u_\varepsilon(e) + (1/\varepsilon)\boldsymbol{\mathscr{Q}}(\mathbf{e}, u_\varepsilon(\mathbf{e})) = L(\mathbf{e}), \tag{4.4}$$

where

$$\boldsymbol{\mathscr{Q}}(\mathbf{e}, u_\varepsilon(\mathbf{e}))(x,y) = \begin{cases} [u_\varepsilon(\mathbf{e})(x,y) - (\mathscr{S} + (\mathbf{e} + H_{\langle 0\rangle}))(x,y)], \\ \qquad \text{if } [u_\varepsilon(\mathbf{e})(x,y) - (\mathscr{S} + (\mathbf{e} + H_{\langle 0\rangle}))(x,y)] \leqslant 0, \\ 0, \text{ if } [u_\varepsilon(\mathbf{e})(x,y) - (\mathscr{S} + (\mathbf{e} + H_{\langle 0\rangle}))(x,y)] > 0. \end{cases}$$

For any $\varepsilon > 0$ problem (4.3) is solvable. Next, we introduce the notations:

$$\mathcal{O} = \inf_{\mathbf{e}\in\mathscr{U}_{ad}(\Omega)} \mathscr{L}(\mathbf{e}) \text{ and } \mathscr{O}_{\langle\varepsilon\rangle} = \inf_{\mathbf{e}\in\mathscr{U}_{ad}(\Omega)} \in \mathscr{L}_\varepsilon(\mathbf{e}). \tag{4.5}$$

Since the penalty operator $\boldsymbol{\mathscr{Q}}(\mathbf{e},.)$ is a Lipschitz continuous, monotone operator in $L_2(\Omega)$, the penalized problem (4.4) is uniquely solvable. Moreover, $u_\varepsilon(\mathbf{e}) \in V(\Omega) \cap H^4(\Omega)$ holds.

Indeed, the penalizator $\boldsymbol{\mathscr{Q}}(\mathbf{e},.) : V(\Omega) \to V^*(\Omega)$ is of the form

$$\langle \boldsymbol{\mathscr{Q}}(\mathbf{e}, v), z\rangle_{V(\Omega)} = -\int_\Omega ([v - (\mathscr{S} + (\mathbf{e} + H_{\langle 0\rangle}))]^-) z \mathrm{d}\Omega, \tag{4.6}$$

with $([a]^-) = \inf(a, 0)$.

The linear form: $z \to \int_\Omega ([v - (\mathscr{S} + (\mathbf{e} + H_{\langle 0\rangle}))]^-) z \mathrm{d}\Omega$ is continuous on $V(\Omega)$ and defines the functional $\boldsymbol{\mathscr{Q}}(\mathbf{e}, v) \in V^*(\Omega)$.

Next, in view of the inequality $(([a+c]^-)-([b+c]^-))d \leqslant |a-b|\,|d|$, for arbitrary real numbers, we can write

$$\begin{aligned}
&\langle \mathcal{Q}(\mathbf{e},v)-\mathcal{Q}(\mathbf{e},z),w\rangle_{V(\Omega)}\\
&=\int_\Omega \{([v-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^-)-([z-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^-)\}w\mathrm{d}\Omega\\
&\leqslant \int_\Omega |v-z|\,|w|\mathrm{d}\Omega \leqslant \text{constant } ||v-z||_{V(\Omega)}||w||_{V(\Omega)}. \qquad (4.7)
\end{aligned}$$

Hence, from it follows the Lipschitz continuity of penalizator $\mathcal{Q}(\mathbf{e},v)$. Moreover, we have

$$\begin{aligned}
&\langle \mathcal{Q}(\mathbf{e},v)-\mathcal{Q}(\mathbf{e},z),v-z\rangle_{V(\Omega)}\\
&=-\int_\Omega \big\{([v-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^-)-([z-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^-)\big\}[v-z]\mathrm{d}\Omega\\
&=-\int_\Omega \{([v-(\mathscr{S}+H_{\langle 0\rangle})]^-)-([z-(\mathscr{S}+H_{\langle 0\rangle})]^-)\}\\
&\quad\times\{[v-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]-[z-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]\}\mathrm{d}\Omega\\
&\geqslant \int_\Omega \{[v-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^- -[z-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^-\}^2\mathrm{d}\Omega\\
&\geqslant 0, \qquad (4.8)
\end{aligned}$$

where we have used the relations:

$$\begin{aligned}
&-(([a]^- - ([b]^-))(a-b) = -(([a]^-)-([b]^-))\\
&\qquad\qquad\times[(([a]^+)-([a]^-))-(([b]^+)-([b]^-))]\\
&=-(([a]^-)-([b]^-))(([a]^+)-([b]^+))+(([a]^-)-([b]^-))^2\\
&\geqslant (([a]^-)-([b]^-))^2,
\end{aligned}$$

for any $[a,b]\in\mathbb{R}$.

Note that $\mathcal{Q}(\mathbf{e},v)=0 \Leftrightarrow ([v-(\mathscr{S}+(\mathbf{e}+H_{\langle 0\rangle}))]^-)=0 \Rightarrow v\in\mathscr{K}(\mathbf{e},\Omega)$.

Here the hemicontinuity is a consequence of Lipschitz continuity of the operator $\mathcal{Q}(\mathbf{e},.)$, i.e. for $[v,z,w]\in V(\Omega)$ the function $\lambda\to\langle\mathcal{Q}(\mathbf{e},v+\lambda z),w\rangle_{V(\Omega)}$ is continuous on $\mathbb{R}$.

Hence, due to the theory of monotone operators (Lions, 1969) we obtain a unique solution $u_\varepsilon(\mathbf{e})$ of the penalized equation (4.4).

Lemma 4.1 *Let $\{u_{\varepsilon_n}(\mathbf{e})\}_{n\in N}$ be a sequence of solutions of the penalized problems (4.4) for fixed $\mathbf{e}\in\mathscr{U}_{ad}(\Omega)$. Then there exists a subsequence $\{\varepsilon_{n_k}\}\subset\{\varepsilon_n\}_{n\in N}$ such that $u_{\varepsilon_{n_k}}(\mathbf{e})\to u(\mathbf{e})$ weakly in $V(\Omega)$, where $u(\mathbf{e})$ is a solution of the state problem (4.1).*

*Proof.* The operator $\mathcal{A}(\mathbf{e}) + (1/\varepsilon)\mathcal{Q}(\mathbf{e},.) : V(\Omega) \to V^*(\Omega)$ is bounded, hemicontinuous, monotone and coercive and due to Theorem 2.2.1 (Lions, 1969) for every $\mathbf{e} \in \mathcal{U}_{ad}(\Omega)$ there exists a solution $u_\varepsilon(\mathbf{e}) \in V(\Omega)$ of the penalized equation (4.4).

Let $v_{\mathcal{O}} \in \mathcal{K}(\mathbf{e},\Omega)$ be an arbitrary element. Then, by inserting $v = [u_{\varepsilon_n}(\mathbf{e}) - v_{\mathcal{O}}]$ in the equation

$$\langle \mathcal{A}(\mathbf{e})u_\varepsilon(\mathbf{e}), v\rangle_{V(\Omega)} + (1/\varepsilon)\langle \mathcal{Q}(\mathbf{e}), u_\varepsilon(\mathbf{e}), v\rangle_{V(\Omega)} = \langle L(\mathbf{e}), v\rangle_{V(\Omega)} \tag{4.9}$$

and since $\mathcal{Q}(\mathbf{e}, v_{\mathcal{O}}) = 0$, we arrive at

$$\begin{aligned}&\langle \mathcal{A}(\mathbf{e})u_{\varepsilon_n}(\mathbf{e}), u_{\varepsilon_n}(\mathbf{e}) - v_{\mathcal{O}}\rangle_{V(\Omega)}\\ &\qquad +(1/\varepsilon_n)\langle \mathcal{Q}(\mathbf{e}, u_\varepsilon(\mathbf{e})) - \mathcal{Q}(\mathbf{e}, v_{\mathcal{O}}), u_\varepsilon(\mathbf{e}) - v_{\mathcal{O}}\rangle_{V(\Omega)}\\ &= \langle L(\mathbf{e}), u_\varepsilon(\mathbf{e}) - v_{\mathcal{O}}\rangle_{V(\Omega)}.\end{aligned}$$

Then, using the monotonicity of $\mathcal{Q}(\mathbf{e},.)$, we have

$$\langle \mathcal{A}(e)u_{\varepsilon_n}(\mathbf{e}), u_{\varepsilon(n)}(\mathbf{e}) - v_{\mathcal{O}}\rangle_{V(\Omega)} \leqslant \langle L(\mathbf{e}), u_\varepsilon(\mathbf{e}) - v_{\mathcal{O}}\rangle_{V(\Omega)}.$$

Hence, the estimate (3.19) yields

$$\begin{aligned}\alpha_{\mathcal{A}}||u_{\varepsilon_n}(\mathbf{e})||^2_{V(\Omega)} \leqslant \text{constant } (&||u_{\varepsilon_n}(\mathbf{e})||_{V(\Omega)}||v_{\mathcal{O}}||_{V(\Omega)}\\ &+||u_{\varepsilon_n}(\mathbf{e})||_{V(\Omega)} + ||v_{\mathcal{O}}||_{V(\Omega)}).\end{aligned}$$

Thus, we obtain the estimate

$$u_{\varepsilon_n}(\mathbf{e}) \leqslant \text{constant (does not depend on } \varepsilon_n). \tag{4.10}$$

This means that there exist the element $u_\diamond$ and the subsequence $\{u_{\varepsilon n_k}(\mathbf{e})\}_{k\in N}$ such that

$$u_{\varepsilon n_k}(\mathbf{e}) \to u_\diamond \text{ weakly in } V(\Omega). \tag{4.11}$$

Further the equality (4.4) and the estimate (4.10) imply

$$\begin{aligned}||\mathcal{Q}(e, u_{\varepsilon n_k}(\mathbf{e}))||_{V^*(\Omega)} &= \sup_{v\neq 0}[-\langle \mathcal{A}(\mathbf{e})u_{\varepsilon n_k}(\mathbf{e}), v\rangle_{V(\Omega)}\\ &\quad +\langle L(\mathbf{e}), v\rangle_{V(\Omega)}]/||v||_{V(\Omega)} = 0(\varepsilon).\end{aligned} \tag{4.12}$$

In view of (4.10) and (4.12) we may write

$$\lim_{k\to\infty} \langle \mathcal{Q}(\mathbf{e}, u_{\varepsilon n_k}(\mathbf{e})), u_{\varepsilon n_k})(\mathbf{e}) - v\rangle_{V(\Omega)} = 0.$$

Next, we take into account the monotonicity of $\mathcal{Q}(\mathbf{e},.)$ and the weak convergence (4.11). Hence, we arrive at

$$\langle \mathcal{Q}(\mathbf{e}, v), u_\diamond - v\rangle_{V(\Omega)} \geqslant 0 \text{ for all } v \in V(\Omega). \tag{4.13}$$

Let $v = u_\Diamond + \lambda z$, $\lambda > 0$, $z \in V(\Omega)$. Then from the inequality (4.13), we deduce that

$$\langle \boldsymbol{\mathcal{Q}}(\mathbf{e}, u_\Diamond + \lambda z), z\rangle_{V(\Omega)} \geqslant 0 \text{ for any } \lambda > 0. \tag{4.14}$$

By using the hemicontinuity of $\boldsymbol{\mathcal{Q}}(\mathbf{e}, .)$ we get after $\lambda \to 0$

$$\langle \boldsymbol{\mathcal{Q}}(\mathbf{e}, u_\Diamond), z\rangle_{V(\Omega)} \geqslant 0 \text{ for all } z \in V(\Omega), \tag{4.15}$$

which implies $\boldsymbol{\mathcal{Q}}(\mathbf{e}, u_\Diamond) = 0$ and $u_\Diamond \in \mathscr{K}(\mathbf{e}, \Omega)$.

Now take in (4.9) $[v - u_{\varepsilon_{n_k}}(\mathbf{e})]$ instead of $v \in \mathscr{K}(\mathbf{e}, \Omega)$, so that we can write

$$\begin{aligned}&\langle \boldsymbol{\mathcal{A}}(\mathbf{e}) u_{\varepsilon_{n_k}}(\mathbf{e}), v - u_{\varepsilon_{n_k}}(\mathbf{e})\rangle_{V(\Omega)}\\ &\qquad -(1/\varepsilon_{n_k})\langle \boldsymbol{\mathcal{Q}}(\mathbf{e}, v) - \boldsymbol{\mathcal{Q}}(\mathbf{e}, u_{\varepsilon_{n_k}}(\mathbf{e})), v - u_{\varepsilon_{n_k}}(\mathbf{e})\rangle_{V(\Omega)}\\ &= \langle L(\mathbf{e}), v - u_{\varepsilon_{n_k}}(\mathbf{e})\rangle_{V(\Omega)}.\end{aligned}$$

Hence, one has

$$\langle \boldsymbol{\mathcal{A}}(\mathbf{e}) u_{\varepsilon_{n_k}}(\mathbf{e}), v - u_{\varepsilon_{n_k}}(\mathbf{e})\rangle_{V(\Omega)} \geqslant \langle L(\mathbf{e}), v - u_{\varepsilon_{n_k}}(\mathbf{e})\rangle_{V(\Omega)}. \tag{4.16}$$

We note that $w \to \langle \boldsymbol{\mathcal{A}}(\mathbf{e}) w, w\rangle_{V(\Omega)}$ is a lower weakly semicontinuous functional on $V(\Omega)$ and due to (4.11) we obtain

$$\langle \boldsymbol{\mathcal{A}}(\mathbf{e}) u_\Diamond, u_\Diamond\rangle_{V(\Omega)} \leqslant \liminf_{k\to\infty} \langle \boldsymbol{\mathcal{A}}(\mathbf{e}) u_{\varepsilon_{n_k}}(\mathbf{e}), u_{\varepsilon_{n_k}}(\mathbf{e})\rangle_{V(\Omega)}. \tag{4.17}$$

Taking into account (4.17) the state inequality (4.1) follows immediately from (4.16). Hence we have $u(\mathbf{e}) = u_\Diamond$. This concludes the proof. ■

Lemma 4.2 *For any $\varepsilon > 0$ there exists a solution $\langle \mathbf{e}_\varepsilon\rangle$ of the optimal control problem $(\mathscr{P}_\varepsilon)$. If $\{\varepsilon_n\}_{n\in N}$ is a sequence with the property $\varepsilon_n > 0$, $\varepsilon_n \to 0_+$, then there exists its subsequence $\{\varepsilon_{n_k}\}_{k\in N}$ such that*

$$\begin{cases} \mathbf{e}_{\varepsilon_{n_k}} \to \mathbf{e}_{\langle *\rangle} \text{ strongly in } \mathscr{U}(\Omega),\\ u_{\varepsilon_{n_k}}(\mathbf{e}_{\varepsilon_{n_k}}) \to u(\mathbf{e}_{\langle *\rangle}) \text{ strongly in } V(\Omega),\\ \mathscr{O}_{\varepsilon_{n_k}} \to \mathscr{O}, \end{cases} \tag{4.18}$$

*where $u_{\varepsilon_{n_k}}(\mathbf{e}_{\varepsilon_{n_k}}) \in V(\Omega)$ fulfils the penalized equation (4.4) and $u(\mathbf{e}_{\langle *\rangle}) \in \mathscr{K}(\mathbf{e}_{\langle *\rangle}, \Omega)$ solves the state variational inequality (4.1) with $\mathbf{e} = \mathbf{e}_{\langle *\rangle}$.*

*Proof.* Let $\{\mathbf{e}_{\langle n\rangle}\}_{n\in N} \subset \mathscr{U}_{ad}(\Omega)$ be a minimizing sequence for the functional $\mathscr{L}_\varepsilon(\,.\,)$. The set $\mathscr{U}_{ad}(\Omega)$ is compact in the Banach space $\mathscr{U}(\Omega)$ and hence it contains a subsequence $\{\mathbf{e}_{\varepsilon\langle n_k\rangle}\}_{k\in N}$ such that

$$\mathbf{e}_{\varepsilon\langle n_k\rangle} \to \mathbf{e}_{\langle\varepsilon\rangle} \quad (\in \mathscr{U}_{ad}(\Omega)) \text{ strongly in } \mathscr{U}(\Omega). \tag{4.19}$$

Let $u(\mathbf{e}_{\varepsilon\langle n\rangle})$ be the corresponding sequence of solutions to the penalized problem

$$\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon\langle n\rangle})u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle}) + (1/\varepsilon\langle n\rangle\boldsymbol{\mathcal{Q}}(\mathbf{e}_{\varepsilon\langle n\rangle}, u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle})) = L(\mathbf{e}_{\varepsilon\langle n\rangle}). \tag{4.20}$$

Making use of the inequality

$$\begin{aligned}&\langle\boldsymbol{\mathcal{Q}}(\mathbf{e}_{\varepsilon\langle n\rangle}, u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle})), u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle}) - (\mathscr{S} + (\mathbf{e}_{\varepsilon\langle n\rangle} + H_{\langle 0\rangle}))\rangle_{L_2(\Omega)}\\ &= \int\limits_\Omega \boldsymbol{\mathcal{Q}}(\mathbf{e}_{\varepsilon\langle n\rangle}, u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle}))(u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle}) - \mathscr{S} - \mathbf{e}_{\varepsilon\langle n\rangle} - H_{\langle 0\rangle})\mathrm{d}\Omega \geqslant 0,\end{aligned}$$

and the uniform coercivity of the family of operators $\{\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon\langle n\rangle})\}_{n\in N}$ we obtain the boundedness of the sequence $\{u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle})\}_{n\in N}$ in the space $V(\Omega)$. Hence, there exists the sequence $\{u_\varepsilon(\mathbf{e}_{\varepsilon\langle n_k\rangle})\}_{k\in N}$ such that

$$\begin{cases} u_\varepsilon(\mathbf{e}_{\varepsilon\langle n_k\rangle}) \to u_{\varepsilon\langle *\rangle} \text{ weakly in } V(\Omega),\\ \text{or}\\ u_\varepsilon(\mathbf{e}_{\varepsilon\langle n_k\rangle}) \to u_{\varepsilon\langle *\rangle} \text{ strongly in } L_2(\Omega). \end{cases} \tag{4.21}$$

Further, we introduce the penalty functional by the relation

$$\begin{aligned}&\boldsymbol{\mathcal{Q}}(\mathbf{e}, z) = -\left([z - \mathscr{S} - \mathbf{e} - H_{\langle 0\rangle}]^-\right)\\ &= \frac{1}{2}(z - \mathscr{S} - \mathbf{e} - H_{\langle 0\rangle} - |z - \mathscr{S} - \mathbf{e} - H_{\langle 0\rangle}|).\end{aligned}$$

Hence, we conclude that it is continuous as a function $\boldsymbol{\mathcal{Q}}(.,.) : \mathscr{U}(\Omega) \times L_2(\Omega) \to L_2(\Omega)$. On the other hand we have

$$\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon\langle n\rangle})u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle}) \to \boldsymbol{\mathcal{A}}(\mathbf{e}_\varepsilon)u_{\varepsilon\langle *\rangle} \text{ weakly in } V^*(\Omega).$$

Thus, from (4.20), we conclude that

$$\boldsymbol{\mathcal{A}}(\mathbf{e}_\varepsilon)u_{\varepsilon\langle *\rangle} + (1/\varepsilon)\boldsymbol{\mathcal{Q}}(\mathbf{e}_\varepsilon, u_{\varepsilon\langle *\rangle}) = L(\mathbf{e}_\varepsilon).$$

So, we have $u_{\varepsilon\langle *\rangle} = u_\varepsilon(\mathbf{e}_\varepsilon)$. Simultaneously, one has $\boldsymbol{\mathcal{A}}(\mathbf{e}_\varepsilon)u_\varepsilon(\mathbf{e}_\varepsilon) \in L_2(\Omega)$ and

$$\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon\langle n\rangle})u_\varepsilon(\mathbf{e}_{\varepsilon\langle n\rangle}) \to \boldsymbol{\mathcal{A}}(\mathbf{e}_\varepsilon)u_\varepsilon(\mathbf{e}_\varepsilon) \text{ strongly in } L_2(\Omega). \tag{4.22}$$

We note that (4.22) is a consequence of the weak convergence of the same sequence in the dual space $V^*(\Omega)$, its boundedness in $L_2(\Omega)$ and the density of the space $V(\Omega)$ in $L_2(\Omega)$.

Then, we can write

$$\lim_{k\to\infty} \mathscr{L}_\varepsilon(\mathbf{e}_{\varepsilon\langle n_k\rangle}, u_\varepsilon(\mathbf{e}_{\varepsilon\langle n_k\rangle})) = \mathscr{L}_\varepsilon(\mathbf{e}_\varepsilon, u_\varepsilon(\mathbf{e}_\varepsilon))$$

and $\mathbf{e}_\varepsilon \in \mathscr{U}_{ad}(\Omega)$ is a solution of the optimal control problem $(\mathscr{B}_\varepsilon)$ for fixed $\varepsilon > 0$.

Let $\varepsilon_n > 0$, $n = 1, 2, \ldots$ fulfil $\varepsilon_n \to 0$. The sequence $\{\mathbf{e}_{\varepsilon_n}\}_{n\in N}$ of penalized optimal thickness is bounded in $\mathscr{U}_{ad}(\Omega)$ and contains a subsequence $\{\mathbf{e}_{\varepsilon_{n_k}}\}_{k\in N}$ fulfilling the convergence ((4.18),1°). Consequently, one has

$$\mathbf{e}_{\varepsilon_{n_k}} \to \mathbf{e}_{\langle * \rangle} \text{ weakly in } V(\Omega). \tag{4.23}$$

Next, we consider the variational equality

$$\begin{aligned}
&\langle \boldsymbol{\mathscr{A}}(\mathbf{e}_{\varepsilon n_k}) u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k}), u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k}) - ([\mathscr{S} + (\mathbf{e}_{\varepsilon n_k} + H_{\langle 0 \rangle})]^+)\rangle_{V(\Omega)} \\
&\qquad -(1/\varepsilon_{n_k})\langle([u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k}) - (\mathscr{S} + (\mathbf{e}_{\varepsilon n_k} + H_{\langle 0 \rangle})]^-), u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k}) \\
&\qquad -([\mathscr{S} + \mathbf{e}_{\varepsilon n_k} + H_{\langle 0 \rangle}]^+)\rangle_{L_2(\Omega)} \\
&= \langle L(\mathbf{e}_{\varepsilon n_k}), u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k}) - ([\mathscr{S} + (\mathbf{e}_{\varepsilon n_k} + H_{\langle 0 \rangle})]^+)\rangle_{V(\Omega)}.
\end{aligned} \tag{4.24}$$

Here, the penalizing number of (4.24) is non-negative and the sequence $\{u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k})\}_{k\in N}$ of solutions to the penalized problem (4.20) is bounded in $V(\Omega)$. Therefore, we can choose from the sequence $\{u_{\varepsilon n_k}(\mathbf{e}_{\varepsilon n_k})\}_{k\in N}$ a subsequence $\{u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}})\}_{\Pi\in N}$ such that

$$\begin{cases} u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) \to u_{\langle * \rangle} \text{ weakly in } V(\Omega), \\ u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) \to u_{\langle * \rangle} \text{ strongly in } L_2(\Omega). \end{cases} \tag{4.25}$$

On the other hand from the boundedness of the sequence $\{u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}})\}_{\Pi\in N}$ in the space $V(\Omega)$ we deduce

$$||\boldsymbol{\mathcal{Q}}(\mathbf{e}_{\varepsilon_{n_\Pi}}, u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}))||_{V^*(\Omega)} \leqslant M\varepsilon_{n_\Pi}, \tag{4.26}$$

where $M > 0$ and $\Pi = 1, 2, \ldots$ .

Taking into account (4.26) and ((4.25),2°, ((4.18),1°) we get

$$\boldsymbol{\mathcal{Q}}(\mathbf{e}_{\langle * \rangle}, u_{\langle * \rangle}) = ([u_{\langle * \rangle} - (\mathscr{S} + \mathbf{e}_{\langle * \rangle} + H_{\langle 0 \rangle}]^-) = 0.$$

Hence, one has

$$u_{\langle * \rangle} \in \mathscr{K}(\mathbf{e}_{\langle * \rangle}, \Omega). \tag{4.27}$$

Let $u(\mathbf{e}_{\langle * \rangle}) \in \mathscr{K}(\mathbf{e}_{\langle * \rangle}, \Omega)$ be a solution of the variational inequality (4.1). Then one has

$$\langle \boldsymbol{\mathscr{A}}(\mathbf{e}_{\langle * \rangle}) u(\mathbf{e}_{\langle * \rangle}), u_{\langle * \rangle} - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)} \geqslant \langle L(\mathbf{e}_{\langle * \rangle}), u_{\langle * \rangle} - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)}. \tag{4.28}$$

Further, we have simultaneously the identity

$$
\begin{aligned}
&\langle \boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon_{n_\Pi}})u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle}) + \mathbf{e}_{\varepsilon_{n_\Pi}} - \mathbf{e}_{\langle * \rangle}\rangle_{V(\Omega)} \\
&\qquad +(1/\varepsilon_{n\Pi})\langle([u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - \mathscr{S} - \mathbf{e}_{\varepsilon_{n_\Pi}} - H_{\langle 0\rangle}]^-), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) \\
&\qquad -\mathscr{S} - \mathbf{e}_{\varepsilon_{n_\Pi}} - H_{\langle 0\rangle}\rangle_{L_2(\Omega)} \\
&\qquad +(1/\varepsilon_n)\langle([u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - \mathscr{S} - \mathbf{e}_{\varepsilon_{n_\Pi}} - H_{\langle 0\rangle}]^-), u(\mathbf{e}_{\langle * \rangle}) \\
&\qquad -\mathscr{S} - \mathbf{e}_{\langle * \rangle} - H_{\langle 0\rangle}\rangle_{L_2(\Omega)} \\
&= \langle L(\mathbf{e}_{\varepsilon_{n_\Pi}}), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle}) + \mathbf{e}_{\varepsilon_{n_\Pi}} - \mathbf{e}_{\langle * \rangle}\rangle_{V(\Omega)}.
\end{aligned} \tag{4.29}
$$

From here and from the relation (4.28), we get the inequality

$$
\begin{aligned}
&\langle \boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon_{n_\Pi}})[u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})], u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)} \\
&\quad +\langle \boldsymbol{\mathcal{A}}(\mathbf{e}_{\langle * \rangle})u(\mathbf{e}_{\langle * \rangle}), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)} \\
&\quad +\langle [\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - \boldsymbol{\mathcal{A}}(\mathbf{e}_{\langle * \rangle})]u(\mathbf{e}_{\langle * \rangle}), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)} \\
&\quad +\langle [\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon_{n_\Pi}})u(\mathbf{e}_{\langle * \rangle}), \mathbf{e}_{\varepsilon_{n_\Pi}} - \mathbf{e}_{\langle * \rangle}\rangle_{V(\Omega)} \\
&\quad +\langle \boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon_{n_\Pi}})[u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})], \mathbf{e}_{\varepsilon_{n_\Pi}} - \mathbf{e}_{\langle * \rangle}\rangle_{V(\Omega)} \\
&\leqslant \langle L(\mathbf{e}_{\varepsilon_{n_\Pi}}), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)} \\
&\quad +\langle L(\mathbf{e}_{\langle * \rangle}), u_{\varepsilon_{n_\Pi}}(\mathbf{e}_{\varepsilon_{n_\Pi}}) - u(\mathbf{e}_{\langle * \rangle})\rangle_{V(\Omega)} + \langle L(\mathbf{e}_{\varepsilon_{n_\Pi}}), \mathbf{e}_{\varepsilon_{n_\Pi}} - \mathbf{e}_{\langle * \rangle}\rangle_{V(\Omega)}.
\end{aligned} \tag{4.30}
$$

Next, taking into account that the members with coefficients $(1/\varepsilon_{n_\Pi})$ in (4.29) are non-negative, we can omit them. Moreover, the uniform coercivity of the family of operators $\{\boldsymbol{\mathcal{A}}(\mathbf{e}_{\varepsilon_n})\}_{n\in N}$, the relations ((4.18),1°, (3.24), (4.23), ((4.25),1°) and the continuity properties of the operators $\boldsymbol{\mathcal{A}}(\,.\,) : \mathscr{U}(\Omega) \to L(V(\Omega), V^*(\Omega))$, imply: $u_{\langle * \rangle} = u(\mathbf{e}_{\langle * \rangle})$ and the convergence ((4.18),2°). Thus, by (4.25) we have

$$
\begin{cases}
\liminf\limits_{\varepsilon_n \to 0} \mathscr{O}_{\varepsilon_n} \geqslant \mathscr{L}(\mathbf{e}_{\langle * \rangle}) \\
\text{and} \\
\mathscr{L}_{\varepsilon_n}(\mathbf{e}) \to \mathscr{L}(\mathbf{e}), \ \text{for any } \mathbf{e} \in \mathscr{U}_{ad}(\Omega).
\end{cases} \tag{4.31}
$$

Here $u_{\varepsilon_n}(\mathbf{e})$ is a solution to equation (4.4) and $u(\mathbf{e})$ denotes a solution to (4.1). Therefore one has

$$\mathscr{O}_{\varepsilon_n} \leqslant \mathscr{L}_{\varepsilon_n}(\hat{\mathbf{e}}_{\langle * \rangle}) \to \mathscr{L}(\hat{\mathbf{e}}_{\langle * \rangle}), \tag{4.32}$$

where $\hat{\mathbf{e}}_{\langle * \rangle}$ is a solution to (4.1) and (4.2). Consequently

$$\limsup_{\varepsilon_n \to 0} \mathscr{O}_{\varepsilon_n} \leqslant \mathscr{L}(\hat{\mathbf{e}}_{\langle * \rangle}). \tag{4.33}$$

Then, from ((4.31), 1°) and (4.33) it follows that $e_{\langle * \rangle}$ is a solution to (4.1), (4.2) and

$$\lim_{\varepsilon_n \to 0} \mathscr{O}_{\varepsilon_n} = \mathscr{L}(\mathbf{e}_{\langle * \rangle}) = \mathscr{O}.$$

This concludes the proof. ■

## 5. Approximate optimal control. The numerical solution by the finite element method

We shall propose approximate solutions of the optimization problem for an elastic three-layered plate by the finite element method. We restrict ourselves to particular domains (namely we suppose that $\Omega$ is a parallelogram) and introduce the convex set given by formula $\mathscr{K}(\Omega) = \{v \in V(\Omega) : \mathscr{M}_0 v(\partial\Omega_{\mathsf{CONTACT}}) \geqslant 0\}$, respectively. Here $\mathscr{K}(\Omega)$ is the closure of $K(\Omega) = \{v \in C^\infty(\bar\Omega),\ \mathscr{M}_0 v = 0,\ \mathscr{M}_1 v = 0$ on $\partial\Omega_{\mathsf{DISPLACEMENT}}$ and $\mathscr{M}_0 v \geqslant 0$ on $\partial\Omega_{\mathsf{CONTACT}}\}$ in the space $H^2(\Omega)$.

Let $\mathscr{T}_h$ denote a uniform partition of $\Omega$ into a finite number of small (open) parallelograms $\mathcal{H}_{\langle i\rangle}$ by means of two systems of equidistant straight lines parallel to the sides of $\Omega$. Then, we may write $\bar\Omega = \bigcup_{i=1}^{N(h)} \bar{\mathcal{H}}_{\langle i\rangle}, \mathcal{H}_{\langle i\rangle} \cap \mathcal{H}_{\langle j\rangle} = \emptyset$ for $i \neq j$ and denote $h = \operatorname{diam} \mathcal{H}_{\langle i\rangle}$. Assume that $\mathscr{T}_h$ is consistent with the partition of the boundary $\partial\Omega = \partial\Omega_{\mathsf{CONTACT}} \cup \partial\Omega_{\mathsf{DISPLACEMENT}}$, i.e. the number of points $\partial\bar\Omega_{\mathsf{DISPLACEMENT}} \cap \partial\bar\Omega_{\mathsf{CONTACT}}$ is finite and every point of this kind coincides with a node of $\mathscr{T}_h$. Thus, we may write: $\partial\Omega_{\mathsf{CONTACT}} = \sum_{j=1}^{N(h)} \overline{A^*_{\langle j-1\rangle h} A^*_{\langle j\rangle h}}$. We introduce the spaces $\mathcal{Q}_{\langle k\rangle}(\mathcal{H})$ of bilinear ($k = 1$) or bicubic ($k = 3$) polynomials defined on the parallelogram $\mathcal{H}$. If $\mathcal{H}$ is not rectangular, the spaces $\mathcal{Q}_{\langle k\rangle}(\mathcal{H})$ are defined via the affine mapping

$$[x_1, x_2] = \mathcal{V}(y_1, y_2) : x_1 = y_1 + y_2 \cos\alpha, x_2 = y_2 \sin\alpha, \tag{5.1}$$

which maps a rectangle $\mathcal{H}_{\langle *\rangle}$ onto $\mathcal{H}$. We set

$$v \in \mathcal{Q}_{\langle k\rangle}(\mathcal{H}) \Leftrightarrow v \circ \mathcal{V} = \hat v \in \mathcal{Q}_{\langle k\rangle}(\mathcal{H}_{\langle *\rangle}).$$

Let $\Sigma_h$ be the set of all vertices $A_{\boldsymbol{z}}$, $1 \leqslant \boldsymbol{z} \leqslant M(h)$ (nodes of $\mathscr{T}_h$) of parallelograms. Let $V_h(\Omega)$ be a finite-dimensional subspace of $V(\Omega)$ defined by

$$V_h(\Omega) = \{v \in V(\Omega) : v|_{\mathcal{H}_{\boldsymbol{o}}} \in \mathcal{Q}_3(\mathcal{H}_{\boldsymbol{o}}),\ 1 \leqslant \boldsymbol{o} \leqslant N(h)\},$$

i.e. $V_h(\Omega)$ contains those functions, which are continuous and continuously differentiable in $\bar\Omega$ and piecewise bicubic in each $\mathcal{H}_{\boldsymbol{o}}$. Then, $\mathscr{K}_h(\Omega)$ is defined be following way:

$$\mathscr{K}_h(\Omega) = \{v \in V_h(\Omega) : 0 \leqslant v(A_{\langle j\rangle h}),\ 1 \leqslant j \leqslant N(h)\}.$$

Let us take notations:

$$\mathbf{e}_h = [\mathcal{O}_h, \mathscr{F}_h]^T \in \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega) \times \mathscr{U}^{\mathscr{F}}_{ad\langle h\rangle}(\partial\Omega_{\mathsf{CONTACT}})\ (= \mathscr{U}_{ad\langle h\rangle}(\Omega)),$$

where

$$\mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega) = \mathscr{U}^{\mathcal{O}}_{ad}(\Omega) \cap \boldsymbol{z}_{\langle h\rangle}(\Omega)$$

and

$$\mathscr{U}^{\mathscr{F}}_{ad\langle h\rangle}(\partial\Omega_{\mathsf{CONTACT}}) = \mathscr{U}^{\mathscr{F}}_{ad}(\partial\Omega_{\mathsf{CONTACT}}) \cap \boldsymbol{\mathcal{Q}}_{\langle h\rangle,\,\mathsf{CONTACT}}(\Omega)$$

where

$$\boldsymbol{\mathcal{Q}}_{\langle h\rangle,\,\mathsf{CONTACT}}(\Omega) = \boldsymbol{\mathcal{Q}}_{\langle h\rangle}(\Omega)|_{\partial\Omega_{\mathsf{CONTACT}}}$$
$$\boldsymbol{\mathcal{Q}}_{\langle h\rangle}(\Omega) = \{v_h \in C(\bar\Omega)\colon v_h|_{\mathcal{H}_{\boldsymbol{\mathcal{O}}}} \in \mathcal{Q}_1(\mathcal{H}_{\boldsymbol{\mathcal{O}}}) \text{ for all parallelograms } \mathcal{H}_{\boldsymbol{\mathcal{O}}} \in \mathscr{T}_h\}.$$

In what follows, we shall consider any families $\{\mathscr{T}_{h_n}\}_{n\in N}$, $h_n \to 0^+$ of partitions, which refine the original partition $\mathscr{T}_{h_{\langle *\rangle}}$. We say that a family $\{\mathscr{T}_{h_n}\}_{n\in N}$ is regular if there exists a positive constant such that $(h_n/\rho) \leqslant \mathcal{C}$ for any $\mathcal{H}_{\boldsymbol{\mathcal{O}}} \in \bigcup\limits_{h_n} \mathscr{T}_{h_n}$ and $\Sigma_{h_1} \subset \Sigma_{h_2}$ if $h_1 > h_2$, where $\rho$ denotes the diameter of the maximal circle contained in $\mathcal{H}_{\boldsymbol{\mathcal{O}}}$.

Here the approximate friction functional has the form

$$\Phi_h(\mathbf{e}_h, v_h) = \sum_{\mathscr{O}\subset\partial\Omega_{\mathsf{CONTACT}}} \int_{\mathscr{O}} \mathscr{F}_h |v_h(\Gamma_{\mathscr{O}})| \mathrm{d}s + I_{\mathscr{K}_h(\Omega)}(v_h), \tag{5.2}$$

where $\mathscr{O}$ denotes the edge of rectangle $\mathcal{H} \in \mathscr{T}_h$ adjacent to $\partial\Omega_{\mathsf{CONTACT}}$ and $\Gamma_{\mathscr{O}}$ is the midpoint of $\mathscr{O}$. Now we may define the following APPROXIMATE STATE PROBLEM:

Given any $\mathbf{e}_h \in \mathscr{U}_{ad\langle h\rangle}(\Omega)$, find $u_h(\mathbf{e}_h) \in \mathscr{K}_h(\mathcal{O}_h, \Omega)$ such that

$$\begin{aligned}&\langle \mathscr{A}_h(\mathcal{O}_h)u_h(\mathbf{e}_h), v_h - u_h(\mathbf{e}_h)\rangle_{V_h(\Omega)} + \Phi_h(\mathbf{e}_h, v_h) - \Phi_h(\mathbf{e}_h, u_h(\mathbf{e}_h))\\ &\geqslant \langle L(\mathbf{e}_h), v_h - u_h(\mathbf{e}_h))\rangle_{V(\Omega)},\end{aligned} \tag{5.3}$$

holds for all $v_h \in \mathscr{K}_h(\Omega)$.

Observe that we shall employ some simple formulas of numerical integration: Thus instead of $\langle \mathscr{A}(\mathcal{O}_h)u_h(\mathbf{e}_h), v_h\rangle_{V(\Omega)}$, we introduce the form

$$\begin{aligned}\langle \mathscr{A}(\mathcal{O}_h)u_h(\mathbf{e}_h), v_h\rangle_{V(\Omega)} = \sum_{j=1}^{N(h)} \{&D_{11}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) \int_{\mathcal{H}_j} (\partial^2 u_h(\mathbf{e}_h)/\partial x^2)(\partial^2 v_h/\partial x^2)\mathrm{d}\Omega\\ &+D_{22}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) \int_{\mathcal{H}_j} (\partial^2 u_h(\mathbf{e}_h)/\partial y^2)(\partial^2 v_h/\partial y^2)\mathrm{d}\Omega\\ &+D_{12}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) \int_{\mathcal{H}_j} ((\partial^2 u_h(\mathbf{e}_h)/\partial x^2)(\partial^2 v_h/\partial y^2)\\ &\qquad+(\partial^2 u_h(\mathbf{e}_h)/\partial y^2)(\partial^2 v_h/\partial x^2))\mathrm{d}\Omega\\ &+2D_{13}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) \int_{\mathcal{H}_j} (\partial^2 u_h(\mathbf{e}_h)/\partial x\partial y)(\partial^2 v_h/\partial x\partial y)\mathrm{d}\Omega\},\end{aligned} \tag{5.4}$$

where $\mathcal{O}_h \in \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega)$, $[u_h(\mathbf{e}_h), v_h] \in V_h(\Omega)$ and $\Gamma_{\langle j\rangle}$ is the centroid of $\mathcal{H}_{\langle j\rangle}$.

Finally, let us define the penalized cost functional:

$$\mathscr{L}_{\langle\varepsilon\rangle\text{WEIGHT}\langle h\rangle} = 2\int\limits_{\Omega} [\omega_1 H_{\langle 0\rangle} + \omega_2 \mathcal{O}_h]\mathrm{d}\Omega + (1/\varepsilon)\sum_{\mathcal{O}=1}^{N_{\mathcal{O}}}([\mathscr{S}_{\mathcal{O}}(\mathbf{e}_h, \mathbf{M}_h(\mathbf{e}_h))]^+) = 0.$$

Here the approximate optimal control problem consists in finding a function $\mathbf{e}_{\varepsilon\langle h\rangle,\text{WEIGHT}}$ of the approximate optimal control problem such that

$$(\mathscr{P}_{\varepsilon\langle h\rangle}) \quad \mathbf{e}_{\varepsilon\langle h\rangle,\text{WEIGHT}} = \operatorname*{ArgMin}_{\mathbf{e}_h\in\mathscr{U}_{ad\langle h\rangle}(\Omega)} \{\mathscr{L}_{\text{WEIGHT}}(\mathbf{e}_h) + (1/\varepsilon)\sum_{\mathcal{O}=1}^{N_{\mathcal{O}}}([\mathscr{S}_{\mathcal{O}}(\mathbf{e}_h, \mathbf{M}_h(\mathbf{e}_h))]^+)\}.$$

Further, we shall prove the solvability of the problem $(\mathscr{P}_{\varepsilon\langle h\rangle})$. To this end we first establish the following lemmas.

Lemma 5.1 *For any* $\mathcal{O}_h \in \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega)$, $u_h(\mathbf{e}_h) \in V_h(\Omega)$, $v \in H^2(\Omega)$ *there holds*

$$|\langle\mathscr{A}_h(\mathcal{O}_h)u_h(\mathbf{e}_h), v\rangle_{V_h(\Omega)} - \langle\mathscr{A}(\mathcal{O}_h)u_h(\mathbf{e}_h), v\rangle_{V(\Omega)}| \leqslant \text{constant}\, h||u_h(\mathbf{e}_h)||_{V(\Omega)}||v||_{V(\Omega)}. \tag{5.5}$$

*Proof.* We may write

$$\begin{aligned}
&|\langle\mathscr{A}_h(\mathcal{O}_h)u_h(\mathbf{e}_h), v\rangle_{V_h(\Omega)} - \langle\mathscr{A}(\mathcal{O}_h)u_h(\mathbf{e}_h), v\rangle_{V(\Omega)}| \\
&= \Big|\sum_{j=1}^{N(h)}\int\limits_{\mathcal{H}_j}\Big\{[D_{11}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{11}(\mathcal{O}_h)](\partial^2 u_h(\mathbf{e}_h)/\partial x^2)(\partial^2 v/\partial x^2) \\
&\quad +[D_{22}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{22}(\mathcal{O}_h)](\partial^2 u_h(\mathbf{e}_h)/\partial y^2)(\partial^2 v/\partial y^2) \\
&\quad +[D_{12}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{12}(\mathcal{O}_h)]((\partial^2 u_h(\mathbf{e}_h)/\partial y^2)(\partial^2 v/\partial x^2) \\
&\quad +(\partial^2 u_h(\mathbf{e}_h)/\partial x^2)(\partial^2 v/\partial y^2)) \\
&\quad +[D_{33}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{33}(\mathcal{O}_h)](\partial^2 u_h(\mathbf{e}_h)/\partial x\partial y)(\partial^2 v/\partial x\partial y)\Big\}\mathrm{d}\Omega\Big| \\
&\leqslant \sum_{j=1}^{N(h)}\int\limits_{\mathcal{H}_j}\Big\{|D_{11}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{11}(\mathcal{O}_h)||(\partial^2 u_h(\mathbf{e}_h)/\partial x^2)(\partial^2 v/\partial x^2)| \\
&\quad +|D_{22}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{22}(\mathcal{O}_h)||(\partial^2 u_h(\mathbf{e}_h)/\partial y^2)(\partial^2 v/\partial y^2)| \\
&\quad +|D_{12}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{12}(\mathcal{O}_h)||(\partial^2 u_h(\mathbf{e}_h)/\partial y^2)(\partial^2 v/\partial x^2) \\
&\quad +(\partial^2 u_h(e_h)/\partial x^2)(\partial^2 v/\partial y^2)| \\
&\quad +|D_{33}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{33}(\mathcal{O}_h)||(\partial^2 u_h(\mathbf{e}_h)/\partial x\partial y)(\partial^2 v/\partial x\partial y)|\Big\}\mathrm{d}\Omega,
\end{aligned}$$

$$\begin{aligned}
&|D_{11}(\mathcal{O}_h(\Gamma_{\langle j\rangle})) - D_{11}(\mathcal{O}_h(x,y))| \leqslant (3||\mathcal{O}_h^2\,\mathbf{grad}\,\mathcal{O}_h||_{C(\bar{\Omega})} \\
&\quad +4H_{\langle 0\rangle}||\mathcal{O}_h\,\mathbf{grad}\,\mathcal{O}_h||_{C(\bar{\Omega})} + H^2_{\langle 0\rangle}||\mathbf{grad}\,\mathcal{O}_h||_{C(\bar{\Omega})})(h/2) \leqslant \text{constant}\ h
\end{aligned}$$

and we obtain the same estimates for $[D_{22}(\mathcal{O}_h), D_{12}(\mathcal{O}_h), D_{33}(\mathcal{O}_h)]$. Thus, the estimate (5.5) follows by inserting and summing. ■

LEMMA 5.2 *For any* $\mathcal{O}_h \in \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega)$ *and* $[v, z] \in V(\Omega)$ *there holds*

$$\begin{cases} |\langle \mathscr{A}_h(\mathcal{O}_h)v, z\rangle_{V_h(\Omega)}| \leqslant \text{constant}_{\langle 1\rangle}||v||_{V(\Omega)}||z||_{V(\Omega)}, \\ \langle \mathscr{A}_h(\mathcal{O}_h)v, v\rangle_{V_h(\Omega)} \geqslant \text{constant}_{\langle 2\rangle}||v||^2_{V(\Omega)}, \end{cases} \tag{5.6}$$

*where* $[\text{constant}_{\langle 1\rangle},\ \text{constant}_{\langle 2\rangle}]$ *are independent of* $[h, \mathcal{O}_h, v, z]$.

*Proof.* The estimates (5.6) follow immediately from (5.4) and the bounds for $\mathcal{O}_h$. ■

LEMMA 5.3 *The set* $\mathscr{K}_h(\Omega)$ *is a closed and convex subset of* $V_h(\Omega)$ *and* $\mathscr{K}(\Omega) = \underset{n\to\infty}{Lim}\ \mathscr{K}_{h_n}(\Omega)$ *(convergence in the sense of Glowinski).*

*Proof.* Let $\{v_{h_n}\}_{n\in N}$, $v_{h_n} \in \mathscr{K}_{h_n}(\Omega)$ be the sequence such that $v_{h_n} \to v$ weakly in $V(\Omega)$ for $n \to \infty$. We can show that $v \in \mathscr{K}(\Omega)$. Indeed, we have $v \in V(\Omega)$ and

$$\lim_{n\to\infty} ||v_{h_n} - v||_{C(\bar{\Omega})} = 0. \tag{5.7}$$

Here, we take into account the fact that the embedding of $V(\Omega)$ into $C(\bar{\Omega})$ is compact (Adams, 1975). Assume that $\mathscr{M}_0 v < 0$ in an interval $\vartheta \subset \partial\Omega_{\text{CONTACT}}$. Then $\varepsilon > 0$ and a subinterval $\vartheta_{\langle *\rangle} \subset \vartheta$ exist such that $\mathscr{M}_0 v(x, y) \leqslant -\varepsilon$ for all $[x, y] \in \vartheta_{\langle *\rangle}$. For sufficiently small $h$ there exists always a node $\text{A}_{\langle i\rangle h} \in \vartheta_{\langle *\rangle}$, $\text{A}_{\langle i\rangle h} \in \mathscr{T}_h$ and we may write

$$\begin{aligned} &|v_h(\text{A}_{\langle i\rangle h}) - \mathscr{M}_0 v(\text{A}_{\langle i\rangle h})| = v_h(\text{A}_{\langle i\rangle h}) - \mathscr{M}_0 v(\text{A}_{\langle i\rangle h}) \\ &\geqslant -\mathscr{M}_0 v(\text{A}_{\langle i\rangle h}) \geqslant \varepsilon, \end{aligned}$$

which contradicts (5.7). Hence we have $\mathscr{M}_0 v \geqslant 0$ on $\partial\Omega_{\text{CONTACT}}$.

Next, we consider $v \in \mathscr{K}(\Omega)$. Then functions $v_{\boldsymbol{\mathcal{O}}} \in K(\Omega)$ exist such that

$$\lim_{\boldsymbol{\mathcal{O}}\to\infty} ||v_{\boldsymbol{\mathcal{O}}} - v||_{V(\Omega)} = 0. \tag{5.8}$$

Denote by $\theta_{h_n} = R_{h_n} v_{\boldsymbol{\mathcal{O}}}$ the $V_h(\Omega)$-interpolate of $v_{\boldsymbol{\mathcal{O}}}$ over the partition $\mathscr{T}_{h_n}$. Then $\theta_{h_n} \in \mathscr{K}_{h_n}(\Omega)$ holds, since the nodal parameters involve all the values $v_{\boldsymbol{\mathcal{O}}}(\text{A}_{\langle j\rangle})$. On the other hand $||R_{h_n} v_{\boldsymbol{\mathcal{O}}} - v_{\boldsymbol{\mathcal{O}}}||_{V(\Omega)} \leqslant \text{constant } h_n^2 ||v_{\boldsymbol{\mathcal{O}}}||_{H^4(\Omega)}$ holds for any regular family $\{\mathscr{T}_{h_n}\}_{n\in N}$ and therefore

$$\lim_{n\to\infty} ||\theta_{h_n} - v||_{V(\Omega)} = 0, \tag{5.9}$$

which concludes the proof. ■

Lemma 5.4 *The system of functionals* $\Phi_{h\langle n\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}), \mathbf{e}_{h\langle n\rangle} \in \mathscr{U}_{ad\langle h\rangle}(\Omega)$ *defined in (5.2), satisfies the assumptions (H2).*

*Proof.* Let us write

$$\Phi_h(\mathbf{e}_h, v_h) = \Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_h) + \Phi_{h\langle\mathsf{Q}\rangle}(\mathbf{e}_h, v_h),$$

where

$$\begin{cases}\Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_h) = \sum\limits_{\mathscr{O}\subset\partial\Omega_{\mathsf{CONTACT}}} \int\limits_{\mathscr{O}} \mathscr{F}_h|v_h(\Gamma_{\mathscr{O}})|\,\mathrm{d}S,\\ \Phi_{h\langle\mathsf{Q}\rangle}(\mathbf{e}_h, v_h) = I_{\mathscr{K}_h(\Omega)}(v_h).\end{cases} \tag{5.10}$$

Further, we consider a sequence $\{\mathbf{e}_{h\langle n\rangle}\}_{n\in N}$, $\mathbf{e}_{h\langle n\rangle} \to \mathbf{e}_h$, as $n\to\infty$, $\mathbf{e}_{h\langle n\rangle} \in \mathscr{U}_{ad\langle h\rangle}(\Omega)$.

Let $v_h \in \mathscr{K}_h(\Omega)$, then, due to Lemma 5.3, there exists a sequence $\{v_{h\langle n\rangle}\}_{n\in N}$, $v_{h\langle n\rangle} \in \mathscr{K}_h(\Omega)$, such that $v_{h\langle n\rangle} \to v_h$ as $n\to\infty$. Hence, we may write

$$|\Phi_h(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) - \Phi_h(\mathbf{e}_h, v_h)| \leqslant |\Lambda_{\langle\Pi\rangle n}| + |\Lambda_{\langle\mathsf{Q}\rangle n}|, \tag{5.11}$$

where

$$\begin{cases}|\Lambda_{\langle\Pi\rangle n}| &= |\Phi_{h\langle\Pi\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) - \Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_h)|\\ &\leqslant |\Phi_{h\langle\Pi\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) - \Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_{h\langle n\rangle})|\\ &\quad + |\Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_{h\langle n\rangle}) - \Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_h)|\\ &\leqslant \sum\limits_{\mathscr{O}}(|v_{h\langle n\rangle}(\Gamma_{\mathscr{O}})| \int\limits_{\mathscr{O}} |\mathscr{F}_{h\langle n\rangle} - \mathscr{F}_h|\,\mathrm{d}S\\ &\quad + |v_{h\langle n\rangle}(\Gamma_{\mathscr{O}}) - v_h(\Gamma_{\mathscr{O}})| \int\limits_{\mathscr{O}} \mathscr{F}_h\,\mathrm{d}S) \to 0 \text{ as } n\to\infty,\\ |\Lambda_{\langle\mathsf{Q}\rangle n}| &= |I_{\mathscr{K}_h(\Omega)}(v_{h\langle n\rangle}) - I_{\mathscr{K}_h(\Omega)}(v_h)| = 0 \text{ for all } n.\end{cases} \tag{5.12}$$

Thus, from (5.11) and (5.12) we conclude that

$$\lim_{n\to\infty} \Phi_h(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) = \Phi_h(\mathbf{e}_h, v_h). \tag{5.13}$$

On the other hand, let $v_h \notin \mathscr{K}_h(\Omega)$, setting $v_{h\langle n\rangle} = v_h$ for all $n = 1, 2, \ldots$, we obtain

$$\limsup_{n\to\infty} \Phi_h(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) \leqslant \limsup_{n\to\infty} \sum_{\mathscr{O}} \int_{\mathscr{O}} \mathscr{F}_{h\langle n\rangle}|v_{h\langle n\rangle}(\Gamma_{\mathscr{O}})|\,\mathrm{d}S + \infty$$

$$= \sum_{\mathscr{O}} \int_{\mathscr{O}} \mathscr{F}_h|v_h(\Gamma_{\mathscr{O}})|\,\mathrm{d}S + \Phi_{h\langle\mathsf{Q}\rangle}(\mathbf{e}_h, v_h) = \Phi_h(\mathbf{e}_h, v_h). \tag{5.14}$$

Now in view of (5.13) and (5.14), we obtain

$$\limsup_{n\to\infty} \Phi_h(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) \leqslant \Phi_h(\mathbf{e}_h, v_h).$$

Let $v_{h\langle n\rangle} \to v_h$ as $n \to \infty$. We may write

$$\liminf_{n\to\infty} \Phi_h(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) \geqslant \liminf_{n\to\infty} \Phi_{h\langle\Pi\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) + \liminf_{n\to\infty} \Phi_{h\langle\mathcal{Q}\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}).$$

Here, by the same arguments as in the case of $\Lambda_{\langle\Pi\rangle n}$, we obtain

$$\lim_{n\to\infty} \Phi_{h\langle\Pi\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) = \Phi_{h\langle\Pi\rangle}(\mathbf{e}_h, v_h).$$

Next, we may write

$$\lim_{n\to\infty} I_{\mathscr{K}_h(\Omega)}(v_{h\langle n\rangle}) = \mathcal{Q},$$

where $\mathcal{Q}$ is either $+\infty$ or zero. If $\mathcal{Q} = +\infty$ then obviously

$$\mathcal{Q} \geqslant I_{\mathscr{K}_h(\Omega)}(v_h). \tag{5.15}$$

If $\mathcal{Q} = 0$, then there exists a subsequence $\{v_{h\langle n_k\rangle}\}_{k\in N} \subset \{v_{h\langle n\rangle}\}_{n\in N}$ such that $v_{h\langle n_k\rangle} \in \mathscr{K}_h(\Omega)$ for all $k \to \infty$. The limit $v_h$ belongs to $\mathscr{K}_h(\Omega)$, so that $I_{\mathscr{K}_h(\Omega)}(v_h) = 0$ and (5.15) holds again.

As a consequence, $\liminf_{n\to\infty} \Phi_{h\langle\mathcal{Q}\rangle}(\mathbf{e}_{h\langle n\rangle}, v_{h\langle n\rangle}) \geqslant \Phi_{h\langle\mathcal{Q}\rangle}(\mathbf{e}_h, v_h)$ and the condition ((H2),1°) is fulfilled. To satisfy condition ((H2),2°), we can choose $a_{\langle n\rangle} = 0$ for all $n$, since $0 \in \mathscr{K}_h(\Omega)$. Then one has $\Phi_h(\mathbf{e}_{h\langle n\rangle}, a_{\langle n\rangle}) = 0$ for all $n$. ∎

LEMMA 5.5 *The approximate problem $(\mathscr{P}_{\varepsilon\langle h\rangle})$ has at least one solution for any fixed rectangulation $\mathscr{T}_{h_O}$ and any $\varepsilon > 0$.*

*Proof.* For fixed $\mathscr{T}_{h_{\mathcal{O}}}$ and for $\mathbf{e}_{h\langle n\rangle} \to \mathbf{e}_h$ strongly in $V(\Omega)$, $\mathbf{e}_{h\langle n\rangle} \in \mathscr{U}_{ad,h}(\Omega)$, $n = 1, 2, \ldots$, we may prove (paralleling the proof of Lemma 3.5) that

$$\mathbf{M}_{h\langle n\rangle}(\mathbf{e}_{h\langle n\rangle}) \to \mathbf{M}_h(\mathbf{e}_h) \text{ strongly in } [L_2(\Omega)]^3.$$

Then, taking into account this relation, we prove that the functions $([\mathscr{S}_{\mathcal{O}}(\mathbf{e}_{h\langle n\rangle}, \mathbf{M}(\mathbf{e}_{h\langle n\rangle}))]^+)$ are continuous in $\mathscr{U}_{ad,h}(\Omega)$ (see the proof of the analogous Lemma 3.6). Thus we have proved the following: the cost functional in $(\mathscr{P}_{\varepsilon\langle h\rangle})$ is continuous, as well. Obviously one has: $\mathbf{e}_h \in \mathscr{U}_{ad,h}(\Omega) \Leftrightarrow \{\mathbf{e}_h(\mathrm{A}_{\langle\mathcal{Z}\rangle h})\}_{\mathcal{Z}=1}^{M(h)} \in \mathscr{P}_h \subset \mathbb{R}^{M(h)}$, where $\mathrm{A}_{\langle\mathcal{Z}\rangle h}$ are the vertices of $\mathscr{T}_h$. But here the set $\mathscr{P}_h$ is compact in $\mathbb{R}^{M(h)}$, being bounded and closed. Hence the cost functional attains its minimum in $\mathscr{U}_{ad,h}(\Omega)$. ∎

LEMMA 5.6 *Assume that a sequence $\{\mathbf{e}_{h_n}\}_{n\in N}$, $\mathbf{e}_{h_n} \in \mathscr{U}_{ad\langle h\rangle}(\Omega) (= \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega) \times \mathscr{U}^{\mathscr{F}}_{ad\langle h\rangle}(\partial\Omega))$ converges to a function $\mathbf{e}([\mathcal{O}, \mathscr{F}])$ in $C(\bar{\Omega}) \times C(\partial\bar{\Omega}_{\text{CONTACT}})$ for $h_n \to 0_+$.*

*The following state variational inequality for $u_{h_n}(\mathbf{e}_{h_n}) \in \mathscr{K}_{h_n}(\Omega)$*

$$\begin{aligned} &\langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n}) u_{h_n}(\mathbf{e}_{h_n}), v_{h_n} - u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)} + \Phi_h(\mathbf{e}_{h_n}, v_{h_n}) - \Phi_h(\mathbf{e}_{h_n}, u_{h_n}(\mathbf{e}_{h_n})) \\ &\quad \geqslant \langle L(\mathbf{e}_{h_n}), v_{h_n} - u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)}, \end{aligned} \tag{5.16}$$

*for all* $v_{h_n} \in \mathscr{K}_{h_n}(\Omega)$ *has a unique solution* $u_{h_n}(\mathbf{e}_{h_n})$ *for any* $\mathbf{e}_{h_n} \in \mathscr{U}_{ad\langle h\rangle}(\Omega)$ *and for any* $h$.

*Let* $u(\mathbf{e}) \in \mathscr{K}(\Omega)$ *be the (unique) solution of the variational inequality*

$$\langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), v-u(\mathbf{e})\rangle_{V(\Omega)} + \Phi(\mathbf{e}, v) - \Phi(\mathbf{e}, u(\mathbf{e})) \geqslant \langle L(\mathbf{e}), v-u(\mathbf{e})\rangle_{V(\Omega)}, \quad (5.17)$$

*for all* $v \in \mathscr{K}(\Omega)$. *Then one has*

$$\lim_{n\to\infty} ||u_{h_n}(\mathbf{e}_{h_n}) - u(\mathbf{e})||_{V(\Omega)} = 0.$$

*Proof.* Take the zero function for $v_{h_n}$ in (5.16). Hence, we obtain

$$\langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)} + \Phi_{h_n}(\mathbf{e}_{h_n}, u_{h_n}(\mathbf{e}_{h_n}))$$
$$\leqslant \langle L(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)}.$$

Next, due to the estimate (5.6),2°) and in view of (2.11), we may write

$$\text{constant}_{\langle 2\rangle}||u_{h_n}(\mathbf{e}_{h_n})||^2_{V(\Omega)} \leqslant ||L(\mathbf{e}_{h_n})||_{V^*(\Omega)}||u_{h_n}(\mathbf{e}_{h_n})||_{V(\Omega)}.$$

Hence, we conclude that

$$||u_{h_n}(\mathbf{e}_{h_n})||_{V(\Omega)} \leqslant \text{constant}, \text{ for all } h < h_{\langle 0\rangle} \quad (5.18)$$

Thus a subsequence $\{u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\}_{k\in N}$ exists such that

$$u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}) \to u_{\diamondsuit} \text{ weakly in } V(\Omega). \quad (5.19)$$

Since the embedding $H^2(\Omega) \subset C(\bar{\Omega})$ is completely continuous, we have

$$u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}) \to u_{\diamondsuit} \text{ strongly in } C(\bar{\Omega}). \quad (5.20)$$

Moreover, by inserting: $v_{h_k} = \theta_{h_k}$ into inequality (5.16) we obtain

$$\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), \theta_{h_k} - u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V_h(\Omega)} + \Phi_{h_{n_k}}(\mathbf{e}_{h_{n_k}}, \theta_{h_k})$$
$$-\Phi_{h_{n_k}}(\mathbf{e}_{h_{n_k}}, u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})) \geqslant \langle L(\mathbf{e}_{h_{n_k}}), \theta_{h_k} - u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}.$$

Next we can show for $h_{n_k} \to 0_+$ (or $k \to \infty$)

$$\liminf_{k\to\infty}\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)} \geqslant \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, u_{\diamondsuit}\rangle_{V(\Omega)}. \quad (5.21)$$

In fact, since $\mathcal{O} \in \mathscr{U}^{\mathcal{O}}_{ad}(\Omega)$, $w \to \langle \mathscr{A}(\mathcal{O})w, w\rangle_{V(\Omega)}$ is a lower weakly semicontinuous functional on $V(\Omega)$ and from (5.19) we conclude that

$$\lim_{k\to\infty} \langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)} \geqslant \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, u_{\diamondsuit}\rangle_{V(\Omega)}.$$

Due to the relation

$$|\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}|$$
$$\leqslant \text{constant}[||\mathcal{O}^3_{h_{n_k}} - \mathcal{O}^3||_{L_\infty(\Omega)} + ||\mathcal{O}^2_{h_{n_k}} - \mathcal{O}^2||_{L_\infty(\Omega)}$$
$$+||\mathcal{O}_{h_{n_k}} - \mathcal{O}||_{L_\infty(\Omega)}]||u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})||^2_{V(\Omega)} \to 0,$$

we may write

$$\liminf_{k\to\infty}(\langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}$$
$$+[\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}$$
$$-\langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}]) \geqslant \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)},$$

which is the estimate (5.21).

Next from Lemma 5.1 and in view of (5.21), (5.18) we obtain

$$\liminf_{k\to\infty}(\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}$$
$$+[\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V_h(\Omega)}$$
$$-\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)}]) \geqslant \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)} \quad (5.22)$$

Further, for any $v \in V(\Omega)$, we have

$$\lim_{k\to\infty}\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), v\rangle_{V(\Omega)} = \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}. \quad (5.23)$$

In fact, we may write

$$|[\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), v\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}]|$$
$$\leqslant |[\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), v\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), v\rangle_{V(\Omega)}]|$$
$$+|\langle \mathscr{A}(\mathcal{O})(u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}) - u_\diamondsuit), v\rangle_{V(\Omega)}| \to 0$$

in view of boundedness of $\{u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\}_{k\in N}$, (5.19), (3.20).

Thus, due to (5.23) and by Lemma 5.1, we derive that

$$\lim_{k\to\infty}\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), v\rangle_{V_h(\Omega)} = \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}. \quad (5.24)$$

Moreover, taking into account Lemmas 5.2, 5.3, (5.18) and (5.9), we conclude that

$$|\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), \theta_{h_{n_k}} - v\rangle_{V_h(\Omega)}|$$
$$\leqslant \text{constant}_{\langle 1\rangle}||u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})||_{V(\Omega)}||\theta_{h_{n_k}} - v||_{V(\Omega)} \to 0. \quad (5.25)$$

Then combining (5.25) with (5.24), we arrive at

$$|\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), \theta_{h_{n_k}}\rangle_{V_h(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}|$$
$$\leqslant |\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), \theta_{h_k} - v\rangle_{V_h(\Omega)}|$$
$$+|\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), v\rangle_{V_h(\Omega)} - \langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)}| \to 0. \quad (5.26)$$

Consider the estimate

$$|\Phi_{h_n}(\mathbf{e}_{h_n}, v_{h_n}) - \Phi(\mathbf{e}_{h_n}, v_{h_n})| = |\sum_{\mathscr{O}\subset\partial\Omega_{\text{CONTACT}}} \int_{\mathscr{O}} \mathscr{F}_{h_n}(|v_{h_n}(\Gamma_{\mathscr{O}})| - |v_{h_n}|)\mathrm{d}S|$$

$$\leqslant \mathscr{F}_{\text{MAX}} \sum_{\mathscr{O}} \int_{\mathscr{O}} |v_{h_n}(\Gamma_{\mathscr{O}}) - v_h|\mathrm{d}S.$$

We may write

$$|v_{h_n}(\Gamma_{\mathscr{O}}) - v_{h_n}(s)| \leqslant (1/2)\,\text{MEAS}\ \mathscr{O}|\partial v_{h_n}/\partial s|$$
$$\leqslant (1/2)\,\text{MEAS}\ \mathscr{O}||\mathbf{grad}\, v_{h_n}||_{L_2(\mathcal{H}_{\mathscr{O}})}$$
$$\leqslant \text{constant}\ \ \text{MEAS}\ \mathscr{O}\rho_{\mathcal{H}}^{-1} h_{\mathcal{H}}(\text{MEAS}\ \mathcal{H}_{\mathscr{O}})^{-(1/4)}|v_{h_n}|_{H^1(\mathcal{H}_{\mathscr{O}})}$$
$$\leqslant \text{constant}_{\langle *\rangle} h_n^{3/2}|v_{h_n}|_{H^1(\mathcal{H}_{\mathscr{O}})},$$

using the estimate MEAS $\mathcal{H}_{\mathscr{O}} \leqslant \text{constant}_{\mathscr{O}} h_{\mathcal{H}}^2$.

Here, $\mathcal{H}_{\mathscr{O}}$ is the parallelogram adjacent to the edge $\mathscr{O}$ and $\rho_{\mathcal{H}}$ is the radius of the largest circle inscribed in $\mathcal{H}_{\mathscr{O}}$. Thus, we obtain

$$\sum_{\mathscr{O}} \int_{\mathscr{O}} |v_{h_n}(\Gamma_{\mathscr{O}}) - v_h|\mathrm{d}S \leqslant \text{constant}_{\langle *\rangle} h_n^{(3/2)} \sum_{\mathscr{O}} \text{MEAS}\ \mathscr{O}|v_{h_n}|_{H^1(\mathcal{H}_{\mathscr{O}})}$$

$$\leqslant \text{constant}_{\langle *\rangle} h_n^{(3/2)} \Big(\sum_{\mathscr{O}} \text{MEAS}\ \mathscr{O}\Big)^{(1/2)} \Big(\sum_{\mathscr{O}} |v_{h_n}|^2_{H^1(\mathcal{H}_{\mathscr{O}})}\Big)^{(1/2)}$$

$$\leqslant \text{constant}_{\langle *\rangle} h^{(3/2)}\ \text{MEAS}\ \partial\Omega_{\text{CONTACT}})^{(1/2)}|v_{h_n}|_{H^1(\Omega)} \to 0.$$

As a consequence

$$\Lambda_{\langle\Pi\rangle h} := |\Phi_{h_n}(\mathbf{e}_{h_n}, v_{h_n}) - \Phi(\mathbf{e}_{h_n}, v_{h_n})| \to 0. \tag{5.27}$$

Since $v \in \mathscr{K}(\Omega)$, we have

$$\Lambda_{\langle\text{Q}\rangle h} := |\Phi(\mathbf{e}_{h_n}, v_{h_n}) - \Phi(\mathbf{e}_{h_n}, v)| = |\int_{\partial\Omega_{\text{CONTACT}}} \mathscr{F}_{h_n}(|v_{h_n}| - |\mathscr{M}_0 v|)\mathrm{d}S$$

$$\leqslant \mathscr{F}_{\text{MAX}}(\text{MEAS}\ \partial\Omega_{\text{CONTACT}})^{(1/2)}||v_{h_n} - \mathscr{M}_0 v||_{L_2(\partial\Omega_{\text{CONTACT}})} \to 0. \tag{5.28}$$

Finally, we may write

$$\Lambda_{\langle\text{O}\rangle h} := |\Phi(\mathbf{e}_{h_n}, v) - \Phi(\mathbf{e}, v)| = |\int_{\partial\Omega_{\text{CONTACT}}} (\mathscr{F}_{h_n} - \mathscr{F})|\mathscr{M}_0 v|\mathrm{d}S|$$

$$\leqslant ||\mathscr{F}_{h_n} - \mathscr{F}||_{L_\infty(\partial\Omega_{\text{CONTACT}})} \int_{\partial\Omega_{\text{CONTACT}}} |\mathscr{M}_0 v|\mathrm{d}S \to 0. \tag{5.29}$$

Hence, by virtue of (5.27) to (5.29), we arrive at

$$|\Phi_{h_n}(\mathbf{e}_{h_n}, v_{h_n}) - \Phi(\mathbf{e}, v)| \leqslant \Lambda_{\langle\Pi\rangle h} + \Lambda_{\langle\mathbb{Q}\rangle h} + \Lambda_{\langle\bigcirc\rangle h} \to 0 \text{ as } h_n \to 0_+. \quad (5.30)$$

On the other hand, in a parallel way, we can deduce that

$$|\Phi_{h_n}(\mathbf{e}_{h_n}, u_{h_n}(\mathbf{e}_{h_n})) - \Phi(\mathbf{e}, u_\diamondsuit)| \to 0 \text{ as } h_n \to 0_+, \quad (5.31)$$

using the boundedness of $\{u_{h_n}(\mathbf{e}_{h_n})\}_{n\in N}$ and compactness of the trace operator $\mathscr{M}_0$.

Here, state inequality (5.16) can be rewritten as follows

$$\begin{aligned}&\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V_h(\Omega)} + \Phi_h(\mathbf{e}_{h_{n_k}}, u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}))\\ &\leqslant \langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), \theta_{h_k}\rangle_{V_h(\Omega)} + \Phi_h(\mathbf{e}_{h_{n_k}}, \theta_{h_k})\\ &\quad + \langle L(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}) - \theta_{h_k}\rangle_{V(\Omega)}. \end{aligned} \quad (5.32)$$

Hence, passing to the $\liminf_{k\to\infty}$ on both sides of the inequality, we deduce that the left-hand side is bounded below by

$$\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega)} + \Phi(\mathbf{e}, u_\diamondsuit), \text{ due to (5.22) and (5.31).} \quad (5.33)$$

At the same time, the right hand side has the following limit

$$\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega)} + \Phi(\mathbf{e}, v) + \langle L(\mathbf{e}), u_\diamondsuit - v\rangle_{V(\Omega)}$$

as follows from (5.26), (5.30) and the continuity of $L(\mathbf{e})$.

Thus, we arrive at

$$\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit - v\rangle_{V(\Omega)} + \Phi(\mathbf{e}, u_\diamondsuit) \leqslant \langle L(\mathbf{e}), u_\diamondsuit - v\rangle_{V(\Omega)} + \Phi(\mathbf{e}, v).$$

Now, since $v \in \mathscr{K}(\Omega)$ was arbitrary and the inequality (5.17) has a unique solution (see (3.19)), $u_\diamondsuit = u(\mathbf{e})$ and the whole sequence $\{u_{h_n}(\mathbf{e}_{h_n})\}_{n\in N}$ tends to $u(\mathbf{e})$ weakly in the space $V(\Omega)$.

Finally, we have to prove the strong convergence. Here, referring to the variational inequality (5.16) and passing to limes inferior or limes superior as $h_n \to 0_+$, we apply (5.22), (5.24), (5.30) and (5.31), and deduce that

$$\begin{aligned}&\langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)} \leqslant \liminf_{n\to\infty} \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)}\\ &\leqslant \limsup_{n\to\infty} \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)} \leqslant \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), v\rangle_{V(\Omega)}\\ &\quad + \Phi(\mathbf{e}, v) - \Phi(\mathbf{e}, u(\mathbf{e})) + \langle L(\mathbf{e}), u(\mathbf{e}) - v\rangle_{V(\Omega)}, \end{aligned} \quad (5.34)$$

for all $v \in \mathscr{K}(\Omega)$.

We may set $v := u(\mathbf{e})$ in (5.34) to obtain

$$\lim_{n\to\infty} \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)} = \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)}. \quad (5.35)$$

Next, we have

$$\begin{aligned}
&|\langle \mathscr{A}(\mathcal{O})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)} - \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)}| \\
&\leqslant |\langle \mathscr{A}(\mathcal{O})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)} - \langle \mathscr{A}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)}| \\
&\quad + |\langle \mathscr{A}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)} - \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V_h(\Omega)}| \\
&\leqslant ||\mathscr{A}(\mathcal{O})u_{h_n}(\mathbf{e}_{h_n}) - \mathscr{A}(\mathcal{O}_{h_n})u_{h_n}(\mathbf{e}_{h_n})||_{V^*(\Omega)}||u_{h_n}(\mathbf{e}_{h_n})||_{V(\Omega)} \\
&\quad + \text{constant } h_n||u_{h_n}(\mathbf{e}_{h_n})||_{V(\Omega)}||u_{h_n}(\mathbf{e}_{h_n})||_{L_2(\Omega)} \to 0, \qquad (5.36)
\end{aligned}$$

making use of (3.20), (5.18) and (5.5). Thus, from (5.35) and (5.36), we conclude that

$$\lim_{n\to\infty} \langle \mathscr{A}(\mathcal{O})u_{h_n}(\mathbf{e}_{h_n}), u_{h_n}(\mathbf{e}_{h_n})\rangle_{V(\Omega)} = \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)}. \qquad (5.37)$$

Next, from (5.37) and (2.10), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} a(\mathcal{O}, u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})) \\
&= \lim_{n\to\infty} \langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}), u_{h_{n_k}}(\mathbf{e}_{h_{n_k}})\rangle_{V(\Omega)} \\
&= \langle \mathscr{A}(\mathcal{O})u(\mathbf{e}), u(\mathbf{e})\rangle_{V(\Omega)} = a(\mathcal{O}, u(\mathbf{e}), u(\mathbf{e})). \qquad (5.38)
\end{aligned}$$

But, the bilinear form $a(\mathcal{O}, ., .)$ can be taken for a scalar product in $V(\Omega)$ (in view of ((3.19),1°). On the other hand, by virtue of (5.38) and the weak convergence of $\{u_{h_n}(\mathbf{e}_{h_n})\}_{n\in N}$, we conclude that

$$\lim_{n\to\infty} a(\mathcal{O}, (u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}) - u(\mathbf{e})), (u_{h_{n_k}}(\mathbf{e}_{h_{n_k}}) - u(\mathbf{e}))) = 0,$$

which in turn implies that $u_{h_n}(e_{h_n}) \to u(e)$ strongly in $V(\Omega)$. ■

LEMMA 5.7 *Let* $\{\mathbf{e}_{h_n}\}_{n\in N}$, $h_n \to 0^+$ *be a sequence of* $\mathbf{e}_{h_n} \in \mathscr{U}_{ad\langle h\rangle}(\Omega)$ *such that* $\mathbf{e}_{h_n} \to \mathbf{e}$ *strongly in* $\mathscr{U}(\Omega)$, *as* $h_n \to 0^+$.

*Then one has*

$$\mathbf{M}_{h_n}(\mathbf{e}_{h_n}) \to \mathbf{M}(\mathbf{e}) \textit{ strongly in } [L_2(\Omega)]^3. \qquad (5.39)$$

*Proof.* By using inclusion: $\mathscr{U}_{ad\langle h_n\rangle}(\Omega) \subset \mathscr{U}_{ad}(\Omega)$ and Lemma 3.5, we conclude that

$$||\mathbf{M}(\mathbf{e}_{h_n}) - \mathbf{M}(\mathbf{e})||_{[L_2(\Omega)]^3} \to 0.$$

Further, from Lemma 5.6, we see that

$$||\mathbf{M}_{h_n}(\mathbf{e}_{h_n}) - \mathbf{M}(\mathbf{e}_{h_n})||_{[L_2(\Omega)]^3} \to 0.$$

Then, in view of the triangle inequality, we obtain

$$\begin{aligned}
&||\mathbf{M}_{h_n}(\mathbf{e}_{h_n}) - \mathbf{M}(\mathbf{e})||_{[L_2(\Omega)]^3} \\
&\leqslant ||\mathbf{M}_{h_n}(\mathbf{e}_{h_n}) - \mathbf{M}(\mathbf{e}_{h_n})||_{[L_2(\Omega)]^3} + ||\mathbf{M}(\mathbf{e}_{h_n}) - \mathbf{M}(\mathbf{e})||_{[L_2(\Omega)]^3} \to 0
\end{aligned}$$

as $h_n \to 0^+$. ■

LEMMA 5.8 *We have*

$$\boldsymbol{\mathcal{L}}_{\langle\varepsilon\rangle,\text{WEIGHT}}(\mathbf{e}_{h_n}, \mathbf{M}_{h_n}(\mathbf{e}_{h_n})) \to \boldsymbol{\mathcal{L}}_{\langle\varepsilon\rangle,\text{WEIGHT}}(\mathbf{e}, \mathbf{M}(\mathbf{e})), \text{ as } h_n \to 0^+.$$

*Proof.* Proof is analogous to that of Lemma 3.6, being based on Lemma 5.7. ■

LEMMA 5.9 *For any* $\mathbf{e} \in \mathscr{U}_{ad}(\Omega)$ *there exists a sequence* $\{\mathbf{e}_{h_n}\}_{n\in N}$, $h_n \to 0^+$, *such that* $\mathbf{e}_{h_n} \in U_{ad\langle h_n\rangle}(\Omega)$ *and* $\mathbf{e}_{h_n} \to \mathbf{e}$ *strongly in* $\mathscr{U}(\Omega)$, *as* $h_n \to 0^+$.

*Proof.* Here, we introduce the parallelogram $\Omega$ and use the skew coordinates $([\xi,\eta])$ via the mapping (5.1). Let $\Omega = \mathscr{F}(\Omega_{\langle 0\rangle})$, $\Omega_{\langle 0\rangle} = (0, L_A) \times (0, L_B)$, $h_{\langle 1\rangle} = (L_A/m)$, $h_{\langle 2\rangle} = (L_B/n)$. Further, denote by $\mathcal{H}_{ij}$ the grid points with coordinates: $\xi = ih_{\langle 1\rangle}$, $\eta = jh_{\langle 2\rangle}$, $i = 0,1,2,\dots,m$, $j = 0,1,2,\dots,n$

$$\begin{cases} \mathcal{O}_{ij}^{\langle 0\rangle} = & [(i-1)h_{\langle 1\rangle}, ih_{\langle 1\rangle}] \times [(j-1)h_{\langle 2\rangle}, jh_{\langle 2\rangle}], \mathcal{O}_{ij} = \mathscr{F}(\mathcal{O}_{ij}^{\langle 0\rangle}), \\ \boldsymbol{\mathcal{O}}_{ij}^{\langle 0\rangle} = & ((i-(1/2))h_{\langle 1\rangle}, (i+(1/2))h_{\langle 1\rangle}) \times ((j-(1/2))h_{\langle 2\rangle}, \\ & (j+(1/2))h_{\langle 2\rangle}) \cap \Omega_{\langle 0\rangle}, \\ \boldsymbol{\mathcal{O}}_{ij} = & \mathscr{F}(\boldsymbol{\mathcal{O}}_{ij}^{\langle 0\rangle}). \end{cases}$$

From this, we have that $\boldsymbol{\mathcal{O}}_{ij}$ is a neighbourhood of the point $\mathscr{F}(\mathcal{H}_{ij})$. Here we set

$$\mathcal{O}_h(\mathscr{F}(\mathcal{H}_{ij})) = (\text{MES}\,\boldsymbol{\mathcal{O}}_{ij})^{-1} \int_{\boldsymbol{\mathcal{O}}_{ij}} \mathcal{O}(x,y)\mathrm{d}\Omega, \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant n. \tag{5.40}$$

Next, we interpolate the nodal values (5.40) by functions from $\mathcal{Q}_1(\mathcal{O}_{ij})$. Hence, we obtain $\mathcal{O}_h \in \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega)$. We may write

$$\int_{\mathcal{O}_{ij}} \mathcal{O}_h \mathrm{d}\Omega = (1/4)\,\text{MES}\,\mathcal{O}_{ij} \sum_{k=1}^{4} \mathcal{O}_h(\mathcal{H}_{ij}^{\langle k\rangle})$$

where $\mathcal{H}_{ij}^{\langle k\rangle}$ are vertices of the parallelogram $\mathcal{O}_{ij}$. Introduce the notation: $\mathscr{S}_{\langle ij\rangle}$ (the union of all parallelograms $\mathcal{O}_{ij}$ which are adjacent to the node $\mathscr{F}(\mathcal{H}_{ij})$), then we have

$$\begin{aligned} \int_{\Omega} \mathcal{O}_h \mathrm{d}\Omega &= \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{\mathcal{O}_{ij}} \mathcal{O}_h \mathrm{d}\Omega = \sum_{i=1}^{m}\sum_{j=1}^{n} (1/4)\,\text{MES}\,\mathcal{O}_{ij} \sum_{k=1}^{4} \mathcal{O}_h(\mathscr{F}(\mathcal{H}_{ij}^{\langle k\rangle})) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} \mathcal{O}_h(\mathscr{F}(\mathcal{H}_{ij}))(1/4)\,\text{MES}\,\mathscr{S}_{\langle ij\rangle} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} (\text{MES}\,\mathscr{S}_{\langle ij\rangle}/4\,\text{MES}\,\boldsymbol{\mathcal{O}}_{ij}) \int_{\boldsymbol{\mathcal{O}}_{ij}} \mathcal{O}\mathrm{d}\Omega \\ &= \int_{\Omega} \mathcal{O}\mathrm{d}\Omega, \text{ since } \text{MES}\,\mathscr{S}_{\langle ij\rangle} = 4\,\text{MES}\,\boldsymbol{\mathcal{O}}_{ij}, \quad \bigcup_{i,j} \bar{\boldsymbol{\mathcal{O}}}_{ij} = \Omega. \end{aligned}$$

Further, we introduce the functions $\tilde{\mathcal{O}} = \mathcal{O} \circ \mathscr{F}$, $\tilde{\mathcal{O}}_h = \mathcal{O}_h \circ \mathscr{F}$. Then, we may transform (5.40) into the formula

$$\tilde{\mathcal{O}}_h(\mathcal{H}_{ij}) = (1/\operatorname{MES}\boldsymbol{\mathcal{O}}_{ij}^{\langle 0\rangle}) \int_{\boldsymbol{\mathcal{O}}_{ij}^{\langle 0\rangle}} \tilde{\mathcal{O}}\,\mathrm{d}\xi\mathrm{d}\eta. \tag{5.41}$$

Next, take the system $[\xi, \eta]$ as a skew coordinate system, parallel with the edges of $\Omega$. From this identification follows that

$$\begin{cases} \partial\mathcal{O}/\partial\xi = \partial\tilde{\mathcal{O}}/\partial\xi, \quad \partial\mathcal{O}/\partial\eta = \partial\tilde{\mathcal{O}}/\partial\eta \\ \text{and} \\ \partial\mathcal{O}_h/\partial\xi = \partial\widetilde{\mathcal{O}}_h/\partial\xi, \quad \partial\mathcal{O}_h/\partial\eta = \partial\widetilde{\mathcal{O}}_h/\partial\eta, \end{cases}$$

for the corresponding points.

Let us extend $\widetilde{\mathcal{O}}$ onto a rectangle $(-h_{\langle 1\rangle}/2, L_x + h_{\langle 1\rangle}/2) \times (-h_{\langle 2\rangle}/2, L_y + h_{\langle 2\rangle}/2)$, so that the extension $\mathcal{O}_{\langle 0\rangle} = \widetilde{\mathcal{O}}$ in $\Omega_{\langle 0\rangle}$ and $\mathcal{O}_{\langle 0\rangle}$ is symmetric with respect to the sides, namely: $\mathcal{O}_{\langle 0\rangle}(L_x + s, \eta) = \mathcal{O}_{\langle 0\rangle}(L_x - s, \eta)$ for any $\eta \in (-h_{\langle 2\rangle}/2, L_y + h_{\langle 2\rangle}/2)$, for any $s \in (0, h_{\langle 1\rangle}/2)$ and similarly along the other sides of $\partial\Omega_{\langle 0\rangle}$. Taking this into account, we may write instead of (5.41)

$$\tilde{\mathcal{O}}_h(\mathcal{H}_{ij}) = (1/h_{\langle 1\rangle}h_{\langle 2\rangle}) \int_{S_{ij}^{\langle 0\rangle}} \mathcal{O}_{\langle 0\rangle}\mathrm{d}\xi\mathrm{d}\eta, \quad 0 \leqslant i \leqslant m, \quad 0 \leqslant j \leqslant n, \tag{5.42}$$

where $S_{ij}^{\langle 0\rangle}$ denotes the (complete) rectangle with the center $\mathcal{H}_{ij}$ and the lengths of sides $h_{\langle 1\rangle}$, $h_{\langle 2\rangle}$.

Further, we have

$$\begin{aligned} &(1/h_{\langle 1\rangle})|\widetilde{\mathcal{O}}_h(\mathcal{H}_{i+1,j}) - \widetilde{\mathcal{O}}_h(\mathcal{H}_{i,j})| = (1/h_{\langle 1\rangle}^2 h_{\langle 2\rangle})\Big| \int_{S_{i+1,j}^{\langle 0\rangle}} \mathcal{O}_{\langle 0\rangle}\mathrm{d}\xi\mathrm{d}\eta - \int_{S_{ij}^{\langle 0\rangle}} \mathcal{O}_{\langle 0\rangle}\mathrm{d}\xi\mathrm{d}\eta\Big| \\ &= (1/h_{\langle 1\rangle}h_{\langle 2\rangle})\Big| \int_{S_{ij}^{\langle 0\rangle}} (1/h_{\langle 1\rangle})[\mathcal{O}_{\langle 0\rangle}(\xi + h_{\langle 1\rangle}, \eta) - \mathcal{O}_{\langle 0\rangle}(\xi, \eta)]\mathrm{d}\xi\mathrm{d}\eta\Big| \\ &\leqslant (1/h_{\langle 1\rangle}h_{\langle 2\rangle})\,\mathrm{constant}_{\langle\xi\rangle}\,\operatorname{MES} S_{ij}^{\langle 0\rangle} = \mathrm{constant}_{\langle\xi\rangle}, \end{aligned} \tag{5.43}$$

where we use the fact that $|\partial\mathcal{O}_{\langle 0\rangle}/\partial\xi| \leqslant \mathrm{constant}_{\langle\xi\rangle}$ holds almost everywhere. It follows from $\widetilde{\mathcal{O}}_h \in \mathcal{Q}_1(\mathcal{O}_{ij}^{\langle 0\rangle})$ in $\mathcal{O}_{ij}^{\langle 0\rangle}$ that the derivative $\partial\widetilde{\mathcal{O}}_h/\partial\xi$ attains its maximum at the boundary $\partial\mathcal{O}_{ij}^{\langle 0\rangle}$. Then, in view of (5.43), we get the estimate $|\partial\mathcal{O}_h/\partial\xi| \leqslant \mathrm{constant}_{\langle\xi\rangle}$ for any $[x, y] \in \Omega$.

Moreover, the upper bound $\mathrm{constant}_{\langle\eta\rangle}$ for $|\partial\mathcal{O}_h/\partial n|$ can be derived in a parallel way. Here we note that the maximum of $\widetilde{\mathcal{O}}_h$ in $\mathcal{O}_{ij}^{\langle 0\rangle}$ is attained at some vertex of $\mathcal{O}^{\langle 0\rangle}_{ij}$. Then, in view of (5.41), we easily verify that: $\mathcal{O}_{\text{MIN}} \leqslant \mathcal{O}_h(x,y) \leqslant \mathcal{O}_{\text{MAX}}$ for any $[x,y] \in \bar{\Omega}$. Hence, we have proven that $\mathcal{O}_h \in \mathscr{U}^{\mathcal{O}}_{ad\langle h\rangle}(\Omega)$.

Notice that in order to get a convergence of $\{\mathcal{O}_{h_n}\}_{n\in N}$, we consider an arbitrary point $[x,y] \in \bar{\Omega}$ and we may write (for $[\xi,\eta] = \mathscr{F}^{-1}(x,y) \in \mathcal{O}^{\langle 0\rangle}_{ij}$)

$$|\mathcal{O}_{h_n}(x,y) - \mathcal{O}(x,y)| \leqslant \Big|\sum_{k=1}^{4} \widetilde{\mathcal{O}}_{h_n}(\mathcal{H}^{\langle k\rangle}_{i,j})\omega_k(\xi,\eta) - \sum_{k=1}^{4} \mathcal{O}(\xi,\eta)\omega_k(\xi,\eta)\Big|$$

where $\omega_k$ are the shape functions of $\mathcal{Q}_1(\mathcal{O}^{\langle 0\rangle}_{ij})$ (here one has $\omega_k(\mathcal{H}^{\langle m\rangle}_{ij}) = \delta_{km}$ at the vertices).

Hence in view of (5.41), we obtain

$$\begin{aligned}
&|\mathcal{O}_{h_n}(x,y) - \mathcal{O}(x,y)| \leqslant \sum_{k=1}^{4} \big|\widetilde{\mathcal{O}}_{h_n}(\mathcal{H}^{\langle k\rangle}_{ij}) - \widetilde{\mathcal{O}}(\xi,\eta)|\omega_k(\xi,\eta) \\
&= \sum_{k-1}^{4} |(1/h_{\langle 1\rangle}h_{\langle 2\rangle}) \int_{S^{\langle 0\rangle k}_{ij}} \mathcal{O}_{\langle 0\rangle}(s_1,s_2)\mathrm{d}s_1\mathrm{d}s_2 \\
&\qquad -(1/h_{\langle 1\rangle}h_{\langle 2\rangle}) \int_{S^{\langle 0\rangle k}_{ij}} \widetilde{\mathcal{O}}(\xi,\eta)\mathrm{d}s_1\mathrm{d}s_2\big|\omega_k(\xi,\eta) \\
&\leqslant \sum_{k-1}^{4} (1/h_{\langle 1\rangle}h_{\langle 2\rangle}) \int_{S^{\langle 0\rangle k}_{ij}} |\mathcal{O}_{\langle 0\rangle}(s_1,s_2) - \widetilde{\mathcal{O}}(\xi,\eta)|\mathrm{d}s_1\mathrm{d}s_2,
\end{aligned} \tag{5.44}$$

where $S^{\langle 0\rangle k}_{ij}$ denotes the rectangle with the center at $\mathcal{H}^k_{ij}$ and $\text{MES}\, S^{\langle 0\rangle k}_{ij} = h_{\langle 1\rangle}h_{\langle 2\rangle}$.

On the other hand, we get

$$\begin{aligned}
&|\mathcal{O}_{\langle 0\rangle}(s_1,s_2) - \widetilde{\mathcal{O}}(\xi,\eta)| = |\mathcal{O}_{\langle 0\rangle}(s_1,s_2) - \mathcal{O}_{\langle 0\rangle}(\xi,\eta)| \\
&\leqslant |\mathcal{O}_{\langle 0\rangle}(s_1,s_2) - \mathcal{O}_{\langle 0\rangle}(\xi,s_2)| + |\mathcal{O}_{\langle 0\rangle}(\xi,s_2) - \mathcal{O}_{\langle 0\rangle}(\xi,\eta)| \\
&\leqslant (3/2)(h_{\langle 1\rangle}\text{constant}_{\langle\xi\rangle} + h_{\langle 2\rangle}\text{constant}_{\langle\eta\rangle}).
\end{aligned} \tag{5.45}$$

Then, due to (5.45) and (5.44), we have

$$|\mathcal{O}_{h_n}(x,y) - \mathcal{O}(x,y)| \leqslant 12h_n \max(\text{constant}_{\langle\xi\rangle}, \text{constant}_{\langle\eta\rangle}).$$

Let $\Pi_h\mathscr{F}$ denote the Lagrange linear interpolant of $\mathscr{F}$ over partition of $\partial\Omega_{\mathsf{CONTACT}}$, generated by $\mathscr{T}_h$. Since $\mathscr{F} \in W^1_\infty(\partial\Omega_{\mathsf{CONTACT}})$, interpolation theory (Ciarlet, 1978) yields

$$||\mathscr{F} - \Pi_{h_n}\mathscr{F}||_{L_\infty(\partial\Omega_{\mathsf{CONTACT}})} \leqslant \text{constant } h_n||\mathscr{F}||_{W^1_\infty(\partial\Omega_{\mathsf{CONTACT}})}.$$

Obviously, $0 \leqslant \Pi_{h_n}\mathscr{F} \leqslant \mathscr{F}_{\mathrm{MAX}}$ everywhere. For any straight-line segment $\overline{PQ} \in \mathcal{H}$, where $P$, $Q \in \partial\Omega_{\text{CONTACT}}$ and $\mathcal{H} \in \mathscr{T}_h$, we have

$$|\partial\Pi_{h_n}\mathscr{F}/\partial s| \leqslant (1/\operatorname{MEAS} \overline{PQ})|\mathscr{F}(Q) - \mathscr{F}(P)|$$
$$\leqslant (1/\operatorname{MEAS} \overline{PQ}) \int_P^Q |\partial\mathscr{F}/\partial s| \mathrm{d}S \leqslant \text{constant}_{\mathscr{F}}.$$

Now, $\mathbf{e}_h = [\mathcal{O}_h, \Pi_h\mathscr{F}]^T$ satisfies the conditions of the lemma. ■

THEOREM 5.1 *Let* $\{\mathbf{e}_{\varepsilon\langle h_n\rangle,\text{WEIGHT}}\}_{n\in N}$*,* $h_n \to 0^+$ *be a sequence of solutions of the approximate optimal control problem* $(\mathscr{P}_{\varepsilon\langle h_n\rangle})$*.*

*Then there exists a subsequence* $\{\mathbf{e}_{\varepsilon\langle h_{n_k}\rangle,\text{WEIGHT}}\}_{k\in N} \subset \{\mathbf{e}_{\varepsilon\langle h_n\rangle,\text{WEIGHT}}\}_{n\in N}$ *and an element* $\mathbf{e}_{\varepsilon\langle *\rangle,\text{WEIGHT}} \in \mathscr{U}_{ad}(\Omega)$ *such that*

$$\begin{cases} \mathbf{e}_{\varepsilon\langle h_{n_k}\rangle,\text{WEIGHT}} \to \mathbf{e}_{\varepsilon\langle *\rangle,\text{WEIGHT}} \text{ strongly in } \mathscr{U}(\Omega), \\ \mathbf{M}_{h_{n_k}}(\mathbf{e}_{\varepsilon\langle h_{n_k}\rangle,\text{WEIGHT}}) \to \mathbf{M}(\mathbf{e}_{\varepsilon,\text{WEIGHT}} \text{ strongly in } [L_2(\Omega)]^3, \end{cases} \tag{5.46}$$

*and* $\mathbf{e}_{\varepsilon,\text{WEIGHT}}$ *is a solution of the penalized optimal control problem* $(\mathscr{P}_\varepsilon)$*. Each uniformly convergent subsequence* $\{e_{\varepsilon\langle h_n\rangle,\text{WEIGHT}}\}_{n\in N}$ *tends to a solution of* $(\mathscr{P}_\varepsilon)$ *and ((5.46),2°) holds.*

*Proof.* Here we have $\mathscr{U}_{ad\langle h\rangle}(\Omega) \subset \mathscr{U}_{ad}(\Omega)$ and $\mathscr{U}_{ad}(\Omega)$ is compact in $\mathscr{U}(\Omega)$ ( $= C(\bar{\Omega}) \times C(\overline{\partial\Omega_{\text{CONTACT}}})$). Hence, there exists a subsequence of $\{e_{\varepsilon\langle h_n\rangle,\text{WEIGHT}}\}_{n\in N}$ such that ((5.46),1°) holds with $e_{\varepsilon,\text{WEIGHT}} \in \mathscr{U}_{ad}(\Omega)$. Then, from Lemma 5.7 we obtain ((5.46),2°). In the following, we prove that $e_{\langle\varepsilon\rangle,\text{WEIGHT}}$ is a solution of the problem $(\mathscr{P}_{\langle\varepsilon\rangle})$. Consider any $\mathbf{e} \in \mathscr{U}_{ad}(\Omega)$ and we apply Lemma 5.9 to obtain $\{\mathbf{e}_{h_k}\}_{k\in N}$, $\mathbf{e}_{h_k} \in \mathscr{U}_{ad\langle h_k\rangle}(\Omega)$, such that $\mathbf{e}_{h_k} \to e$ strongly in $\mathscr{U}(\Omega)$.

Now, the definition $(\mathscr{P}_{\langle\varepsilon\rangle})$ implies that

$$\begin{aligned} &\boldsymbol{\mathscr{L}}_{\langle\varepsilon\rangle,\text{WEIGHT}_{\langle h\rangle}}(\mathbf{e}_{\varepsilon_{\langle h_k\rangle,\text{WEIGHT}}}, \mathbf{M}_{h_{n_k}}(\mathbf{e}_{\varepsilon_{\langle h_{n_k}\rangle},\text{WEIGHT}})) \\ &\leqslant \boldsymbol{\mathscr{L}}_{\langle\varepsilon\rangle,\text{WEIGHT}_{\langle h\rangle}}(\mathbf{e}_{h_k}, \mathbf{M}_{h_k}(\mathbf{e}_{h_k})) \end{aligned} \tag{5.47}$$

for any $h_{n_k}$.

Taking into account (5.47) and passing to the limit with $h_{n_k} \to 0_+$ and applying Lemma 5.8 to both sides, we arrive at

$$\boldsymbol{\mathscr{L}}_{\langle\varepsilon\rangle,\text{WEIGHT}}(\mathbf{e}_\varepsilon, \mathbf{M}(\mathbf{e}_\varepsilon)) \to \boldsymbol{\mathscr{L}}_{\langle\varepsilon\rangle,\text{WEIGHT}}(\mathbf{e}, \mathbf{M}(\mathbf{e})).$$

Thus, the element $\mathbf{e}_{\varepsilon,\text{WEIGHT}}$ is a solution to the problem $(\mathscr{P}_{\langle\varepsilon\rangle})$, what completes the proof. ■

## 6. Application to a circular plate with annular opening

We consider the case of an elastic circular plate with annular openings (the midplane of the plate occupy a given bounded domain $\Omega = \{[x, y] \in \mathbb{R}^2, R_A < x^2 + y^2 < R_B\}$) subjected to the axially symmetric loads (concentrated load at the curve (circle) $\mathscr{O}_{\langle a\rangle}$ and $\mathscr{O}_{\langle b\rangle}$). Let the boundary $\partial\Omega$ be decomposed as follows: $\partial\Omega = \overline{\partial\Omega}_{\mathsf{DISPLACEMENT}} \cup \overline{\partial\Omega}_{\mathsf{CONTACT}}$, where on $\partial\Omega_{\mathsf{DISPLACEMENT}}$ (with radius $R_A$) homogeneous conditions are prescribed, whereas on $\partial\Omega_{\mathsf{CONTACT}}$ (with radius $R_B$) the plate is unilaterally supported and is subject to a contact with friction. The transversal displacements (deflection of axisymmetric plate) $v$ belong to the space (on $\partial\Omega_{\mathsf{DISPLACEMENT}}$ a homogeneous kinematic conditions are prescribed)

$$V(\Omega) := \{v \in H^2(\Omega) : \mathscr{M}_0 v = 0,\ \mathscr{M}_1 v = 0 \text{ on } \partial\Omega_{\mathsf{DISPLACEMENT}}\}.$$

The investigated plate is circular, therefore we introduce the polar coordinates by the transformation: $x = r\cos\theta$, $y = r\sin\theta$. Here we introduce unilateral constraint imposed upon the deflection $v$ on $\partial\Omega_{\mathsf{CONTACT}}$ (the plate is subjected to a contact with friction)

$$\mathscr{K}(\Omega) := \{v \in V(\Omega) : \mathscr{M}_0 v \geqslant 0 \text{ on } \partial\Omega_{\mathsf{CONTACT}}\}$$

or

$$\mathscr{K}(\Omega_{\langle AB\rangle}) := \{v \in V(\Omega_{\langle AB\rangle}) : \mathscr{M}_0 v \geqslant 0 \text{ at the point } r = R_A\},$$

where

$$V(\Omega_{\langle AB\rangle}) := \{v \in H^2(\Omega_{\langle AB\rangle}) : \mathscr{M}_0 v = 0, \mathscr{M}_1 v = 0 \text{ at the point } r = R_B\}.$$

Due to the axisymmetric load and axisymmetric boundary conditions on $\partial\Omega$ the state function depends only on the radius $r$. We deduce the following relations for the bending moments $[M_{rr}(\mathcal{O}, r), M_{\theta\theta}(\mathcal{O}, r)]$ and for the torque $M_{r\theta}(\mathcal{O}, r)$

$$\begin{cases} M_{rr}(\mathcal{O}, r) = M_{rr}(\mathcal{O}) = D_{11}(\mathcal{O})(\mathrm{d}^2 v/\mathrm{d}r^2) + D_{12}(\mathcal{O})(1/r)(\mathrm{d}v/\mathrm{d}r), \\ M_{\theta\theta}(\mathcal{O}, r) = M_{\theta\theta}(\mathcal{O}) = D_{21}(\mathcal{O})(\mathrm{d}^2 v \mathrm{d}r^2) + D_{22}(\mathcal{O})(1/r)(\mathrm{d}v\mathrm{d}r), \\ M_{r\theta}(\mathcal{O}) = M_{\theta r}(\mathcal{O}) = 0. \end{cases}$$

from which we conclude the isotropic material to be a partial case of the orthotropic one

$$E_{11} = E_{22} = E, \quad \nu_{12} = \nu_{21} = \nu, \quad G = E/2(1+\nu).$$

Let us use the virtual displacement principle to establish a variational formulation of the problem. To this end we introduce a bilinear form

$$\langle \mathscr{A}(\mathcal{O})v, z\rangle_{V(\Omega_{\langle AB\rangle})} = a(\mathcal{O}, v, z)$$
$$= 2\pi \int_{\Omega_{\langle AB\rangle}} \langle \mathrm{d}^2 v/\mathrm{d}r^2, (1/r)\mathrm{d}v/\mathrm{d}r\rangle [\mathbb{K}(\mathcal{O})] \langle \mathrm{d}^2 z/\mathrm{d}r^2, (1/r)\mathrm{d}z/\mathrm{d}r\rangle^T r\mathrm{d}r \quad (6.1)$$

and the virtual work of external loads by the formula

$$\langle \boldsymbol{\mathcal{L}}, v\rangle_{V(\Omega)} = -\int_{\mathscr{O}_{R_1}} p_{\langle 1\rangle} v \mathrm{d}S + \int_{\mathscr{O}_{R_2}} p_{\langle 2\rangle} v \mathrm{d}S + 2\pi \int_{\Omega} r p(r) v \mathrm{d}r,$$

or

$$\langle \boldsymbol{\mathcal{L}}, v\rangle_{V(\Omega_{\langle AB\rangle})} = 2\pi[-p_{\langle 1\rangle} R_{\mathscr{O}_{R_1}} v(R_{\mathscr{O}_{R_1}}) + p_{\langle 2\rangle} R_{\mathscr{O}_{R_2}} v(R_{\mathscr{O}_{R_1}})] + 2\pi \int_{\Omega_{\langle AB\rangle}} r p(r) v \mathrm{d}r,$$

where

$$[\mathbb{K}(\mathcal{O})] = \begin{bmatrix} D_{11}(\mathcal{O}) & D_{12}(\mathcal{O}) \\ D_{21}(\mathcal{O}) & D_{22}(\mathcal{O}) \end{bmatrix}$$

and

$$[\mathbb{K}(\mathcal{O})] = (2E\mathcal{O}^3/3(1-\nu^2)) \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix},$$

for isotropic plate, $v$ is the Poisson ratio, and $2\mathcal{O}$ is the thickness of the plate.

Here we assume that the thickness $\mathcal{O}$ of the exterior layers (or the plate half-thickness) is not constant over the whole area $\Omega$. Now, let it be axisymmetric. Then we can assume that it is possible to express the thickness of the plate by a function $\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB\rangle})$ where

$$\begin{aligned} \mathscr{U}_{ad}(\Omega_{\langle AB\rangle}) = \big\{ & \mathcal{O} \in C^{(0),1}(\bar{\Omega}_{\langle AB\rangle}), |\mathrm{d}\mathcal{O}/\mathrm{d}r| \leqslant \text{ constant}_{\langle \mathcal{O}\rangle}, \\ & 0 < \mathcal{O}_{\min} \leqslant \mathcal{O}(r) \leqslant \mathcal{O}_{\max}, \\ & r \in \bar{\Omega}_{\langle AB\rangle}, \text{ where constant}_{\langle \mathcal{O}\rangle} \text{ is a positive number} \big\}. \end{aligned}$$

LEMMA 6.1 *The set* $\mathscr{K}(\Omega_{\langle AB\rangle}) \cap C^{\infty}(\bar{\Omega}_{\langle AB\rangle})$ *is dense in* $\mathscr{K}(\Omega_{\langle AB\rangle})$.

*Proof.* Consider a $v \in \mathscr{K}(\Omega)$. Taking into account the partition of unity such that $\{R_A\} \subset \operatorname{supp} \theta_1$, $\{R_B\} \subset \operatorname{supp} \theta_2$, $\operatorname{supp} \theta_1 \cap \{R_B\} = \emptyset$, $\operatorname{supp} \theta_2 \cap \{R_A\} = \emptyset$, we may write

$$v = \sum_{j=1}^{2} v\theta_j, v\theta_j = v_j$$

and solve the approximation of every $v_j$ separately.

If $v_2(R_B) > 0$, we extend the function by its tangent at $r = R_B$ for $r > R_B$ and regularize by the formula:

$$\mathscr{R}_\kappa f(r) = A\kappa^{-1} \int_{-\infty}^{\infty} \omega(r-\rho, \kappa) f(\rho) \mathrm{d}\rho,$$

$$\omega(a,\kappa) = \exp(a^2/(a^2-\kappa^2)), \text{ for } |a| \leqslant \kappa \quad \text{ or } \omega(a,\kappa) = 0, \text{ for } |a| > \kappa,$$

where $A$ and $\kappa$ are constants.

Since the extension $Ev_2$ is positive in a neighbourhood of the point $r = R_B$, its regularization $\mathscr{R}_\kappa Ev_2 > 0$ for $\kappa$ sufficiently small and

$$||\mathscr{R}_{\kappa_n} Ev_2 - v_2||_{H^2(\Omega_{\langle AB\rangle})} \to 0 \text{ for } \kappa_n \to 0. \tag{6.2}$$

If $v_2(R_B) = 0$, we extend $v_2$ in such way that $Ev_2$ is antisymmetric with respect to the point $r = R_B$, i.e. $Ev_2(r) = -v_2(2R_B - r)$ for $r > R_B$.

Hence we deduce that $\mathscr{R}_\kappa Ev_2(R_B) = 0$ and (6.2) holds.

Next, the function $v_1$ will be extended by zero for $r < R_A$, then shifted to the right and regularized (for $\kappa$ less than the shift). Here we obtain functions: $v_{1\kappa_n} \in C_0^\infty(\Omega_{\langle AB\rangle})$ such that

$$||v_{1_{\kappa_n}} - v_1||_{H^2(\Omega_{\langle AB\rangle})} \to 0 \text{ for } \kappa_n \to 0. \tag{6.3}$$

On the other hand, making use of (6.2) and (6.3) we have

$$\begin{aligned} &||(v_1 - v_{1\kappa_n}) + (v_2 - \mathscr{R}_{\kappa_n} Ev_2)||_{H^2(\Omega_{\langle AB\rangle})} \\ &\leqslant ||v_1 - v_{1\kappa_n}||_{H^2(\Omega_{\langle AB\rangle})} + ||v_2 - \mathscr{R}_{\kappa_n} Ev_2||_{H^2(\Omega_{\langle AB\rangle})} \to 0, \end{aligned}$$

which completes the proof. ■

Let us use the virtual displacement principle to establish a variational formulation of the problem (shape optimization of elastic axisymmetric plate). Then, the general statement of such a problem is as follows:

Given any $\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB\rangle})$ find $u(\mathcal{O}) \in \mathscr{K}(\Omega_{\langle AB\rangle})$ such that

$$\langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), v - u(\mathcal{O})\rangle_{V(\Omega_{\langle AB\rangle})} \geqslant \langle \boldsymbol{\mathscr{L}}, v - u(\mathcal{O})\rangle_{V(\Omega_{\langle AB\rangle})}, \tag{6.4}$$

holds for all $v \in \mathscr{K}(\Omega_{\langle AB\rangle})$.

Now, we define the cost functional (desired deflection)

$$\begin{cases} \mathscr{L}_{\text{DESIRED DEFLECTION}}(v) = 2\pi \displaystyle\int_{\Omega_{\langle AB\rangle}} [v - z_{ad}]^2 r\mathrm{d}r \\ \text{and} \\ \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}, v) = 2\pi \displaystyle\int_{\Omega_{\langle AB\rangle}} [\sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta}] r\mathrm{d}r, \\ = \left(\dfrac{9}{4\,\text{MEAS}\ \Omega_{\langle AB\rangle}}\right) 2\pi \displaystyle\int_{\Omega_{\langle AB\rangle}} (H_{\langle 0\rangle} + \mathcal{O})^4 [M_{rr}^2(\mathcal{O}) + M_{\theta\theta}^2(\mathcal{O}) \\ \qquad - M_{rr}(\mathcal{O})M_{\theta\theta}(\mathcal{O})] r\mathrm{d}r \\ \text{or} \\ \left(\dfrac{18\pi}{4\,\text{MEAS}\ \Omega_{\langle AB\rangle}}\right) \displaystyle\int_{\Omega_{\langle AB\rangle}} \{(\mathcal{O}(r))^2 [(\mathrm{d}^2 v/\mathrm{d}r^2)^2 + (\mathrm{d}v/\mathrm{d}r)^2 (1/r^2)] \\ \qquad \times (1 - \nu + \nu^2) + [(\mathrm{d}v/\mathrm{d}r)(\mathrm{d}^2 v/\mathrm{d}r^2](1/r)(-1 + 4\nu - \nu^2)\} r\mathrm{d}r \\ = \mathsf{Q} \displaystyle\int_{\Omega_{\langle AB\rangle}} r(\mathcal{O}(r))^2 \varphi([v, v]) dr. \end{cases} \tag{6.5}$$

The choice corresponds to the minimization of the expression (the so called von Mises equivalent stress) evaluated on the surface of the plate.

Now, we define the following optimal control problems

$$\begin{cases} \mathcal{O}_{\langle * \rangle, \text{DESIRED DEFLECTION}} = \underset{\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB \rangle})}{\operatorname{ArgMin}} \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}, u(\mathcal{O})) \\ \text{and} \\ \mathcal{O}_{\langle * \rangle, \text{INTENSITY OF SHEAR STRESS}} = \underset{\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB \rangle})}{\operatorname{ArgMin}} \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}, u(\mathcal{O})). \end{cases} \tag{6.6}$$

**Approximation of the problem**

Let $N(h)$ be an integer and $\mathscr{T}_h$ a partition of interval $[R_A, R_B]$ into $N(h)$ subintervals $\mathscr{O}_j = [\mathscr{R}_{j-1}, \mathscr{R}_j]$ of the length $h$, $j = 1, 2, \ldots, N(h)$, $\mathscr{R}_0 = R_A$, $\mathscr{R}_{N(h)} = R_B$. Here we shall consider only $h$ that lead to a uniform partition of the interval. Let $P_k(\mathscr{O})$ be the polynomials whose order is at most $k$. We define

$$\begin{cases} \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle}) := \{\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB \rangle}) : \mathcal{O}|_{\mathscr{O}_j} \in P_1(\mathscr{O}_j), \text{ for any } j\}, \\ V_h(\Omega_{\langle AB \rangle}) := \{v \in V(\Omega_{\langle AB \rangle}) : v|_{\mathscr{O}_j} \in P_3(\mathscr{O}_j), \text{ for any } j\}, \\ \mathscr{K}_h(\Omega_{\langle AB \rangle}) := \mathscr{K}(\Omega_{\langle AB \rangle}) \cap V_h(\Omega_{\langle AB \rangle}). \end{cases} \tag{6.7}$$

Approximate State Problem:

Given any $\mathcal{O}_h \in \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle})$, find $u_h(\mathcal{O}_h) \in \mathscr{K}_h(\Omega_{\langle AB \rangle})$ such that

$$\langle \mathscr{A}_h(\mathcal{O}_h) u_h(\mathcal{O}_h), v_h - u_h(\mathcal{O}_h) \rangle_{V_h(\Omega_{\langle AB \rangle})} \geqslant \langle \boldsymbol{\mathscr{L}}, v_h - u_h(\mathcal{O}_h) \rangle_{V(\Omega_{\langle AB \rangle})}, \tag{6.8}$$

holds for all $v_h \in \mathscr{K}_h(\Omega_{\langle AB \rangle})$.

Let the approximate Optimal Control Problem $(\mathscr{P}_h)$ be defined in the following way:

Find $\mathcal{O}_{\langle * \rangle h, \text{DESIRED DEFLECTION}} \in \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle})$
and $\mathcal{O}_{\langle * \rangle h, \text{INTENSITY OF SHEAR STRESS}} \in \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle})$ such that

$$\begin{cases} \mathcal{O}_{\langle * \rangle h, \text{DESIRED DEFLECTION}} = \underset{\mathcal{O}_h \in \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle})}{\operatorname{ArgMin}} \mathscr{L}_{\text{DESIRED DEFLECTION}}(u_h(\mathcal{O}_h)) \\ \mathcal{O}_{\langle * \rangle h, \text{INTENSITY OF SHEAR STRESS}} = \underset{\mathcal{O}_h \in \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle})}{\operatorname{ArgMin}} \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}\langle h \rangle}(\mathcal{O}_h, u_h(\mathcal{O}_h)). \end{cases} \tag{6.9}$$

Next, we introduce a numerical quadrate finite element approximation of (6.1). Taking into account (6.7), we have

$$\langle \mathscr{A}_h(\mathcal{O}_h) v_h, z_h \rangle_{V_h(\Omega_{\langle AB \rangle})} = \sum_{j=1}^{N(h)} 2\pi \int_{\mathscr{O}_j} (\langle \mathrm{d}^2 v_h/\mathrm{d}r^2, (1/r)\mathrm{d}v_h/\mathrm{d}r \rangle [\mathbb{K}(\mathcal{O}_h(a_j))]$$
$$\times \langle \mathrm{d}^2 z_h/\mathrm{d}r^2, (1/r)\mathrm{d}z_h/\mathrm{d}r \rangle^T) r \mathrm{d}r, \tag{6.10}$$

where $a_j = (1/2)(\mathscr{R}_{j-1} + \mathscr{R}_j)$, $\mathcal{O}_h \in \mathscr{U}_{ad\langle h \rangle}(\Omega_{\langle AB \rangle})$, $[v_h, z_h] \in V(\Omega_{\langle AB \rangle})$.

LEMMA 6.2 *For any* $\Theta_h \in \mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB\rangle})$ *and* $v_h, z_h \in V_h(\Omega_{\langle AB\rangle})$ *the following estimate holds:*

$$
\begin{aligned}
&|\langle \mathscr{A}_h(\Theta_h)v_h, z_h\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\Theta_h)v_h, z_h\rangle_{V(\Omega_{\langle AB\rangle})}| \\
&\leqslant \text{constant}_{\langle 1\rangle}\, h\|v_h\|_{V(\Omega_{\langle AB\rangle})}\|z_h\|_{V(\Omega_{\langle AB\rangle})},
\end{aligned} \tag{6.11}
$$

*where the* constant$_{\langle 1\rangle}$ *is independent of* $h$.

*Proof.* We may write

$$
\begin{aligned}
&|\langle \mathscr{A}_h(\Theta_h)v_h, z_h\rangle_{V_h(\Omega)} - \langle \mathscr{A}(\Theta_h)v_h, z_h\rangle_{V(\Omega)}| \\
&= \Big|2\pi \sum_{j=1}^{N(h)} \Big(\int_{\mathscr{O}_j} \langle \mathrm{d}^2 v_h/\mathrm{d}r^2, (1/r)\mathrm{d}v_h/\mathrm{d}r\rangle \\
&\times([\mathbb{K}(\Theta_h(a_j))] - [\mathbb{K}(\Theta_h(r))])\langle \mathrm{d}^2 z_h/dr^2, (1/r)\mathrm{d}z_h/dr\rangle^T r\mathrm{d}r\Big| \\
&\leqslant 2\pi \sum_{j=1}^{N(h)} \Big(\int_{\mathscr{O}_j} |r(\mathrm{d}^2 v_h/\mathrm{d}r^2)(\mathrm{d}^2 z_h/\mathrm{d}r^2)[D_{11}(\Theta_h(a_j)) - D_{11}(\Theta_h(r))]|\mathrm{d}r \\
&+ \int_{\mathscr{O}_j} |[(\mathrm{d}^2 v_h/\mathrm{d}r^2)(\mathrm{d}z_h/\mathrm{d}r) + (\mathrm{d}v_h/\mathrm{d}r)(\mathrm{d}^2 z_h/\mathrm{d}r^2)] \\
&\times[D_{12}(\Theta_h(a_j)) - D_{12}(\Theta_h(r))]|r\mathrm{d}r \\
&+ \int_{\mathscr{O}_j} |(1/r)(\mathrm{d}v_h/\mathrm{d}r)(\mathrm{d}z_h/dr)[D_{22}(\Theta_h(a_j)) - D_{22}(\Theta_h(r))]|\mathrm{d}r\Big).
\end{aligned} \tag{6.12}
$$

Further, taking into account the estimate

$$
\begin{aligned}
&|\Theta_h^3(a_j) - \Theta_h^3(r)| \leqslant (3/2)\|\Theta_h^2 \mathrm{d}\Theta_h/\mathrm{d}r\|_{C(\bar{\mathscr{O}}_j)} h \\
&\leqslant (3/2)\Theta_{\text{MAX}}^2 \text{constant}_{\langle\Theta\rangle} h = \text{constant}_{\langle Q\rangle} h, \quad r \in \mathscr{O}_j
\end{aligned} \tag{6.13}
$$

for (6.12), we conclude that

$$
\begin{aligned}
&|\langle \mathscr{A}_h(\Theta_h)v_h, z_h\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\Theta_h)v_h, z_h\rangle_{V(\Omega_{\langle AB\rangle})}| \\
&\leqslant \max[E_{11}, E_{22}, E_{12}] \sum_{j=1}^{N(h)} \text{constant}_{\langle Q\rangle}\, h\Big(\int_{\mathscr{O}_j} |r(\mathrm{d}^2 v_h/\mathrm{d}r^2)(\mathrm{d}^2 z_h/dr^2)|\mathrm{d}r \\
&\quad + \int_{\mathscr{O}_j} |(\mathrm{d}^2 v_h/\mathrm{d}r^2)(\mathrm{d}z_h/\mathrm{d}r) + (\mathrm{d}v_h/dr)(\mathrm{d}^2 z_h/\mathrm{d}r^2)|\mathrm{d}r \\
&\quad + \int_{\mathscr{O}_j} |(1/r)(\mathrm{d}v_h/\mathrm{d}r)(\mathrm{d}z_h/dr)|\mathrm{d}r\Big) \leqslant \text{constant}_{\langle 1\rangle}\, h\|v_h\|_{V(\Omega_{\langle AB\rangle})}\|z_h\|_{V(\Omega_{\langle AB\rangle})}.
\end{aligned}
$$

Here the subspace $V_h(\Omega_{\langle AB\rangle})$ is finite-dimensional, $V_h(\Omega_{\langle AB\rangle})$ is a closed subspace of $V(\Omega_{\langle AB\rangle})$. Then, the bilinear form $\langle \mathscr{A}_h(\mathcal{O}_h).,.\rangle_{V_h(\Omega_{\langle AB\rangle})}$ is bounded and $V_h(\Omega_{\langle AB\rangle})$-elliptic on $V_h(\Omega_{\langle AB\rangle}) \times V_h(\Omega_{\langle AB\rangle})$ and $\langle \boldsymbol{\mathcal{L}},.\rangle_{V_h(\Omega)}$ is continuous linear form on $V_h(\Omega_{\langle AB\rangle})$

The estimates

$$\begin{cases} |\langle \mathscr{A}_h(\mathcal{O}_h)v_h, z_h\rangle_{V_h(\Omega_{\langle AB\rangle})}| \leqslant \text{constant}_{\langle \Pi_1\rangle} ||v_h||_{V(\Omega_{\langle AB\rangle})} ||z_h||_{V(\Omega_{\langle AB\rangle})}, \\ \langle \mathscr{A}_h(\mathcal{O}_h)v_h, v_h\rangle_{V_h(\Omega_{\langle AB\rangle})} \geqslant \text{constant}_{\langle \Pi_2\rangle} ||v_h||^2_{V(\Omega_{\langle AB\rangle})}, \end{cases} \tag{6.14}$$

follow immediately from (6.10) and the bounds for $\mathcal{O}_h$. Hence the following variational inequality (6.8) has a unique solution $u_h(\mathcal{O}_h)$ for any $h$ and any $\mathcal{O}_h \in \mathscr{U}_{ad\langle h\rangle}(\Omega)$. ■

Furthermore, the problem that remains is to get an estimate

$$||u(\mathcal{O}) - u_{h_n}(\mathcal{O}_{h_n})||_{V(\Omega)},$$

where $u(\mathcal{O})$ is the solution of (6.4) and $u_h(\mathcal{O}_h)$ is the solution of (6.8). Next, we may write (substituting the zero function for $v_h$ in (6.8))

$$\begin{aligned} &\text{constant}\, ||u_{h_n}(\mathcal{O}_{h_n})||^2_{V(\Omega_{\langle AB\rangle})} \leqslant \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V_h(\Omega_{\langle AB\rangle})} \\ &\leqslant \langle \boldsymbol{\mathcal{L}}, u_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega_{\langle AB\rangle})} \leqslant ||\boldsymbol{\mathcal{L}}||_{V^*(\Omega_{\langle AB\rangle})} ||u_{h_n}(\mathcal{O}_{h_n})||_{V(\Omega_{\langle AB\rangle})}. \end{aligned}$$

Hence, we conclude that

$$||u_{h_n}(\mathcal{O}_{h_n})||_{V(\Omega_{\langle AB\rangle})} \leqslant \text{constant}. \tag{6.15}$$

Then, a subsequence of $\{u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\}_{n\in N}$ exists such that

$$u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) \to u_\diamondsuit \text{ weakly in } V(\Omega_{\langle AB\rangle}). \tag{6.16}$$

But the embedding of $H^2(\Omega_{\langle AB\rangle}) \subset C^1(\bar{\Omega}_{\langle AB\rangle})$ is completely continuous, this means that

$$u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) \to u_\diamondsuit \text{ strongly in } C^1(\bar{\Omega}_{\langle AB\rangle}). \tag{6.17}$$

Here, we may pass to the limit with $h_{n_k} \to 0^+$ in the inequality

$$u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) \geqslant 0 \text{ for } r = R_B,$$

to obtain that

$$u_\diamondsuit \geqslant 0 \text{ for } r = R_B.$$

Thus, we deduce that $u_\diamondsuit \in \mathscr{K}(\Omega_{\langle AB\rangle})$.

Let us verify that $u_\diamondsuit = u(\mathcal{O})$ is the solution of (6.4). Given a function $v \in \mathscr{K}(\Omega_{\langle AB\rangle})$ (due to the Lemma 6.1 for any $\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB\rangle})$ the set $\mathscr{K}(\Omega_{\langle AB\rangle}) \cap C^\infty(\bar\Omega_{\langle AB\rangle})$ is dense in $\mathscr{K}(\Omega_{\langle AB\rangle})$) there exists a sequence $v_{\mathcal{O}_k} \in \mathscr{K}(\Omega_{\langle AB\rangle}) \cap C^\infty(\bar\Omega_{\langle AB\rangle})$ such that $v_{\mathcal{O}_k} \to v$ strongly in $V(\Omega_{\langle AB\rangle})$. If we denote by $\vartheta_{h_k} = H_{h_k} v_{\mathcal{O}_k}$ the Hermite cubic interpolate of $v_{\mathcal{O}_k}$ over the partition $\mathscr{T}_{h_{n_k}}$, the one has $\vartheta_{h_k} \in V_{h_{n_k}}(\Omega)$ and even $\vartheta_{h_k} \in \mathscr{K}_{h_{n_k}}(\Omega_{\langle AB\rangle})$.

Furthermore, we have

$$||H_{h_k} v_{\mathcal{O}} - v_{\mathcal{O}}||_{V(\Omega_{\langle AB\rangle})} \leqslant \text{constant}_{\langle 1\rangle} h_{n_k}^2 ||v_{\mathcal{O}}||_{H^4(\Omega_{\langle AB\rangle})} \tag{6.18}$$

and therefore

$$||\vartheta_{h_k} - v||_{H^2(\Omega_{\langle AB\rangle})} \to 0, \text{ for } h_k \to 0^+. \tag{6.19}$$

In view of the inequality (6.8), we may write (inserting $v_{h_{n_k}} = \vartheta_{h_k}$)

$$\begin{aligned}&\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), \vartheta_{h_k} - u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V_h(\Omega_{\langle AB\rangle})}\\ &\geqslant \langle \boldsymbol{\mathscr{L}}, \vartheta_{h_k} - u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V_h(\Omega_{\langle AB\rangle})}.\end{aligned} \tag{6.20}$$

Here we can show that for $h_{n_k} \to 0^+$

$$\liminf_{k\to\infty} \langle \mathscr{A}(\mathcal{O}_{h_{n_k}}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})} \geqslant \langle \mathscr{A}(\mathcal{O}) u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega_{\langle AB\rangle})}. \tag{6.21}$$

In fact, since $\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB\rangle}) \to \langle \mathscr{A}(\mathcal{O}) u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega_{\langle AB\rangle})}$ is a lower semicontinuous functional on $V(\Omega_{\langle AB\rangle})$ and from (6.16) we conclude that

$$\lim_{k\to\infty} \langle \mathscr{A}(\mathcal{O}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})} \geqslant \langle \mathscr{A}(\mathcal{O}) u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega_{\langle AB\rangle})}.$$

Further, since

$$\begin{aligned}&|\langle \mathscr{A}(\mathcal{O}_{h_{n_k}}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V_h(\Omega_{\langle AB\rangle})}\\ &\quad -\langle \mathscr{A}(\mathcal{O}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})}|\\ &\leqslant \text{constant } (||\mathcal{O}^3_{h_{n_k}} - \mathcal{O}||_{C(\bar\Omega_{\langle AB\rangle})} + ||\mathcal{O}^2_{h_{n_k}} - \mathcal{O}^2||_{C(\bar\Omega_{\langle AB\rangle})}\\ &\qquad + ||\mathcal{O}_{h_{n_k}} - \mathcal{O}||_{C(\bar\Omega_{\langle AB\rangle})}) ||u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})||^2_{V(\Omega_{\langle AB\rangle})} \to 0,\end{aligned} \tag{6.22}$$

we may write

$$\begin{aligned}&\liminf_{k\to\infty} (\langle \mathscr{A}(\mathcal{O}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})}\\ &+[\langle \mathscr{A}(\mathcal{O}_{h_{n_k}}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})}\\ &-\langle \mathscr{A}(\mathcal{O}) u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})}]) \geqslant \langle \mathscr{A}(\mathcal{O}) u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega_{\langle AB\rangle})},\end{aligned}$$

which gives the estimate (6.21).

Then due to the estimates (6.12) and (6.22), we conclude that

$$\liminf_{k\to\infty}(\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})}$$
$$+[\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V_h(\Omega_{\langle AB\rangle})}$$
$$-\langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(O_{h_{n_k}})\rangle_{V(\Omega_{\langle AB\rangle})}])$$
$$\geqslant \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, u_{\diamondsuit}\rangle_{V(\Omega_{\langle AB\rangle})}. \tag{6.23}$$

For any $v \in V(\Omega_{\langle AB\rangle})$

$$\lim_{k\to 0}\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), v\rangle_{V(\Omega_{\langle AB\rangle})} = \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, v\rangle_{V(\Omega_{\langle AB\rangle})}. \tag{6.24}$$

In fact, we may write

$$|\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), v\rangle_{V(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, v\rangle_{V(\Omega_{\langle AB\rangle})}|$$
$$\leqslant |\langle \mathscr{A}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), v\rangle_{V(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\mathcal{O})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), v\rangle_{V(\Omega_{\langle AB\rangle})}|$$
$$+|\langle \mathscr{A}(\mathcal{O})(u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) - u_{\diamondsuit}), v\rangle_{V(\Omega_{\langle AB\rangle})} \to 0,$$

since $u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})$ are bounded and (6.17) holds.

On the other hand, in view of (6.24) and Lemma 6.2, we derive that

$$\lim_{k\to\infty}\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), v\rangle_{V_h(\Omega_{\langle AB\rangle})} = \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, v\rangle_{V(\Omega_{\langle AB\rangle})}. \tag{6.25}$$

Further, due to the estimates (6.14) and (6.16), (6.19), we conclude that

$$|\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), \vartheta_{h_{n_k}} - v\rangle_{V_h(\Omega_{\langle AB\rangle})}|$$
$$\leqslant \text{constant}_{\langle\Pi_1\rangle}||u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})||_{V(\Omega_{\langle AB\rangle})}||\vartheta_{h_{n_k}} - v||_{H^2(\Omega_{\langle AB\rangle})} \to 0. \tag{6.26}$$

Hence (combining (6.26) with (6.25)), we arrive at

$$|\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), \vartheta_{h_{n_k}}\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, v\rangle_{V(\Omega_{\langle AB\rangle})}|$$
$$\leqslant |\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), \vartheta_{h_{n_k}} - v\rangle_{V_h(\Omega_{\langle AB\rangle})}|$$
$$+|\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), v\rangle_{V_h(\Omega_{\langle AB\rangle})}$$
$$-\langle \mathscr{A}(\mathcal{O})u_{\diamondsuit}, v\rangle_{V(\Omega_{\langle AB\rangle})}| \to 0. \tag{6.27}$$

Finally, we can write

$$|\langle \boldsymbol{\mathscr{L}}, u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \boldsymbol{\mathscr{L}}, u_{\diamondsuit}\rangle_{V(\Omega_{\langle AB\rangle})}| \to 0 \tag{6.28}$$

making use of (6.16)

Now, the inequality (6.20) can be rewritten as follows

$$\langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), u_{h_{n_k}}(\mathcal{O}_{h_{n_k}})\rangle_{V_h(\Omega_{\langle AB\rangle})}$$
$$\leqslant \langle \mathscr{A}_{h_{n_k}}(\mathcal{O}_{h_{n_k}})u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}), \vartheta_{h_n}\rangle_{V_h(\Omega_{\langle AB\rangle})}$$
$$+\langle \boldsymbol{\mathscr{L}}, u_{h_{n_k}}(\mathcal{O}_{h_{n_k}}) - \vartheta_{h_{n_k}}\rangle_{V_h(\Omega_{\langle AB\rangle})}. \tag{6.29}$$

Let us pass to the $\liminf_{n\to\infty}$ on both sides of (6.29). Here, due to (6.23), the left-hand side is bounded below by $\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit\rangle_{V(\Omega_{\langle AB\rangle})}$.

Moreover, the right hand side possesses the following limit

$$\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, v\rangle_{V(\Omega_{\langle AB\rangle})} + \langle \boldsymbol{\mathcal{L}}, u_\diamondsuit - v\rangle_{V(\Omega_{\langle AB\rangle})},$$

as follows from (6.27), (6.28) and (6.16), (6.19). Thus, we arrive at the inequality

$$\langle \mathscr{A}(\mathcal{O})u_\diamondsuit, u_\diamondsuit - v\rangle_{V(\Omega_{\langle AB\rangle})} \leqslant \langle \boldsymbol{\mathcal{L}}, u_\diamondsuit - v\rangle_{V(\Omega_{\langle AB\rangle})}.$$

Since $u_\diamondsuit \in \mathscr{K}(\Omega)$ was arbitrary and the inequality (6.4) has a unique solution, $u_\diamondsuit = u(\mathcal{O})$ and the whole sequence $\{u_{h_n}(\mathcal{O}_{h_n})\}_{n\in N}$ tends to $u_\diamondsuit$ weakly in $V(\Omega_{\langle AB\rangle})$.

Finally, it remains to prove the strong convergence. Note that due to (6.20) and (6.4), we may write

$$\begin{aligned}
&|\langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{V(\Omega_{\langle AB\rangle})}|\\
&\leqslant |\langle \boldsymbol{\mathcal{L}}, u_{h_n}(\mathcal{O}_{h_n}) - \vartheta_{h_n}\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \boldsymbol{\mathcal{L}}, u(\mathcal{O}) - v\rangle_{V(\Omega_{\langle AB\rangle})}|\\
&+|\langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), \vartheta_{h_n}\rangle_{V_h(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), v\rangle_{V(\Omega_{\langle AB\rangle})}|.
\end{aligned}$$

But, the first term on the right hand side has the zero limit (by (6.28) and (6.19)), the second term has the zero limit (due to (6.27)).

Hence, we conclude that

$$\lim_{n\to\infty} \langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V_h(\Omega_{\langle AB\rangle})} = \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{V(\Omega_{\langle AB\rangle})}. \tag{6.30}$$

Then, taking into account (6.30), (6.11) and (6.22), we have

$$\begin{aligned}
&|\langle \mathscr{A}(\mathcal{O})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega_{\langle AB\rangle})} - \langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{V(\Omega_{\langle AB\rangle})}|\\
&\leqslant |\langle \mathscr{A}(\mathcal{O})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega_{\langle AB\rangle})}\\
&\quad -\langle \mathscr{A}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega_{\langle AB\rangle})}|\\
&\quad +|\langle \mathscr{A}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V(\Omega_{\langle AB\rangle})}\\
&\quad -\langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V_h(\Omega_{\langle AB\rangle})}|\\
&\quad +|\langle \mathscr{A}_{h_n}(\mathcal{O}_{h_n})u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})\rangle_{V_h(\Omega_{\langle AB\rangle})}\\
&\quad -\langle \mathscr{A}(\mathcal{O})u(\mathcal{O}), u(\mathcal{O})\rangle_{V(\Omega_{\langle AB\rangle})}| \to 0,
\end{aligned} \tag{6.31}$$

for $h_n \to 0$.

Next, we define the scalar product $(.,.)_{\mathscr{A}} = \langle \mathscr{A}(\mathcal{O}).,.\rangle_{V(\Omega_{\langle AB\rangle})}$ on $V(\Omega_{\langle AB\rangle})$. Then (6.31) implies that the associated norms $||u_{h_n}(\mathcal{O}_{h_n})||_{\mathscr{A}}$ tend to $||u(\mathcal{O})||_{\mathscr{A}}$. Since the norms $||.||_{\mathscr{A}}$ and $||.||_{V(\Omega_{\langle AB\rangle})}$ are equivalent, we are led to the strong convergence by the following estimate:

$$\begin{aligned}&\alpha_{\mathscr{A}}||u_{h_n}(\mathcal{O}_{h_n})-u(\mathcal{O})||^2_{V(\Omega_{\langle AB\rangle})}\leqslant||u_{h_n}(\mathcal{O}_{h_n})-u(\mathcal{O})||^2_{\mathscr{A}}\\&=(u_{h_n}(\mathcal{O}_{h_n})-u(\mathcal{O}),u_{h_n}(\mathcal{O}_{h_n})-u(\mathcal{O}))_{\mathscr{A}}\\&=||u_{h_n}(\mathcal{O}_{h_n})||^2_{\mathscr{A}}+||u(\mathcal{O})||^2_{\mathscr{A}}-2(u(\mathcal{O}),u_{h_n}(\mathcal{O}_{h_n}))_{\mathscr{A}}\to 0,\ \text{for } h_n\to 0_+.\end{aligned}\tag{6.32}$$

Here, we have used the weak convergence $u_{h_n}(\mathcal{O}_{h_n})\to u(\mathcal{O})$, the convergence of the norm $||u_{h_n}(\mathcal{O}_{h_n})||^2_{\mathscr{A}}$ and the continuity of the linear functional $(u(\mathcal{O}),.)_{\mathscr{A}}$.

Let the appropriate optimal control problems (6.5) be defined in the following way: Find $\mathcal{O}_{*\langle h\rangle}\in\mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB\rangle})$ such that

$$\begin{cases}\mathscr{L}_{\text{DESIRED DEFLECTION}}(u_h(\mathcal{O}_{*\langle h\rangle}))=\min\limits_{\mathcal{O}_h\in\mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB\rangle})}\mathscr{L}_{\text{DESIRED DEFLECTION}}(u_h(\mathcal{O}_{\langle h\rangle}))\\ \text{or}\\ J_{\text{INTENSITY OF SHEAR STRESS}\langle h\rangle}(u_h(\mathcal{O}_{*\langle h\rangle}))=\mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}\langle h\rangle}(\mathcal{O}_{*\langle h\rangle},u_h(\mathcal{O}_{*\langle h\rangle}))\\ =\min\limits_{\mathcal{O}_h\in\mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB\rangle})}J_{\text{INTENSITY OF SHEAR STRESS},\langle h\rangle}(\mathcal{O}_{\langle h\rangle}),\end{cases}\tag{6.33}$$

where $u_h(\mathcal{O}_{\langle h\rangle})$ solves (6.8) and the functional $\mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O},u(\mathcal{O}))$ is approximated by the functional

$$\mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}\langle h\rangle}(\mathcal{O}_h,u_h(\mathcal{O}_h))=\mathcal{Q}\sum_{j=1}^{N(h)}\mathcal{O}_h^2(a_j)\int\limits_{e_j}r\mathscr{S}(u_h(\mathcal{O}_h),u_h(\mathcal{O}_h))\mathrm{d}r.$$

LEMMA 6.3 *The optimal control problems (6.33) have at least one solution for any sufficiently small and positive $h$.*

*Proof.* Here we employ Theorem 3.2 for the operator $\mathscr{A}_h(\mathcal{O}_h)$ with $\langle h\rangle$ fixed. Let us choose $\mathscr{U}(\Omega_{\langle AB\rangle})=C(\bar{\Omega}_{\langle AB\rangle})$, $V(\Omega_{\langle AB\rangle})=V_h(\Omega_{\langle AB\rangle})$. The set $\mathscr{U}_{ad,\langle h\rangle}(\Omega_{\langle AB\rangle})$ is closed. Then $\mathscr{U}_{ad,\langle h\rangle}(\Omega_{\langle AB\rangle})\subset\mathscr{U}(\Omega_{\langle AB\rangle})$ is compact set and the form $\mathscr{A}_h(\mathcal{O}_h)$ fulfil ((H1),3$^o$), (see the proof of the relation (6.14). Let us verify ((H1),4$^o$). Let us assume $[\mathcal{O}_h,\mathcal{O}_{h\langle n\rangle}]\in\mathscr{U}_{ad}(\Omega_{\langle AB\rangle})$, $\mathcal{O}_{h\langle n\rangle}\to\mathcal{O}_h$ in $\mathscr{U}(\Omega_{\langle AB\rangle})$ and $u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle})\to u_h(\mathcal{O}_h)$ in $V_h(\Omega_{\langle AB\rangle})$ for $n\to\infty$. Then, we may write (analogy to (3.20))

$$\begin{aligned}&|\langle\mathscr{A}_h(\mathcal{O}_{h\langle n\rangle})u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle}),\theta_h\rangle_{V_h(\Omega_{\langle AB\rangle})}-\langle\mathscr{A}_h(\mathcal{O}_h)u_h(\mathcal{O}_h),\theta_h\rangle_{V_h(\Omega_{\langle AB\rangle})}|\\&\leqslant|\langle\mathscr{A}_h(\mathcal{O}_{h\langle n\rangle})u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle}),\theta_h\rangle_{V_h(\Omega_{\langle AB\rangle})}-\langle\mathscr{A}_h(O_h)u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle}),\theta_h\rangle_{V_h(\Omega_{\langle AB\rangle})}|\\&+|\langle\mathscr{A}_h(\mathcal{O}_h)u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle}),\theta_h\rangle_{V_h(\Omega_{\langle AB\rangle})}-\langle\mathscr{A}_h(\mathcal{O}_h)u_h(\mathcal{O}_h),\theta_h\rangle_{V_h(\Omega_{\langle AB\rangle})}|\to 0,\end{aligned}$$

for $n\to\infty$, for any $\theta_h\in V_h(\Omega_{\langle AB\rangle})$,

$$\langle\boldsymbol{\mathscr{L}},u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle})\rangle_{V_h(\Omega_{\langle AB\rangle})}\to\langle\boldsymbol{\mathscr{L}},u_h(\mathcal{O}_h)\rangle_{V(\Omega_{\langle AB\rangle})}.$$

Moreover one has

$$\mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}_{h\langle n\rangle}, u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle})) \to \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}_h, u_h(\mathcal{O}_h))$$
$$\mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}_{h\langle n\rangle}, u_{h\langle n\rangle}(\mathcal{O}_{h\langle n\rangle})) \to \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}_h, u_h(\mathcal{O}_h)).$$

Then (6.8) coincides with (6.4) and since all the assumptions of Theorem 3.1 are fulfilled, the existence of a solution of (6.33) follows. ∎

LEMMA 6.4 *Assume that a sequence* $\{\mathcal{O}_{h_n}\}_{n\in N}$, $\mathcal{O}_{h_n} \in \mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB\rangle})$ *converges to a function* $\mathcal{O} \in \mathscr{U}_{ad}(\Omega_{\langle AB\rangle})$ *for* $h_n \to 0_+$. *Then one has*

$$\begin{cases} \lim\limits_{n\to\infty} \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}_{h_n}, u_{h_n}(\mathcal{O}_{h_n})) = \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}, u(\mathcal{O})), \\ \lim\limits_{n\to\infty} \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS},\langle h_n\rangle}(\mathcal{O}_{h_n}, u_{h_n}(\mathcal{O}_{h_n})) = \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}, u(\mathcal{O})). \end{cases} \tag{6.34}$$

*Proof.* Due to (6.32), we obtain the assertion ($6.34, 2^o$). Indeed, we can write

$$\begin{aligned} &|\mathscr{L}_{\text{INTENSITY OF SHEAR STRESS},\langle h_n\rangle}(\mathcal{O}_{h_n}, u_{h_n}(\mathcal{O}_{h_n})) - \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}, u(\mathcal{O}))| \\ &\leqslant |\mathsf{Q} \sum_{j=1}^{N(h)} \mathcal{O}^2_{h_n}(a_j) \int_{\mathscr{O}_j} r[\mathscr{S}(u_{h_n}(\mathcal{O}_{h_n}), u_{h_n}(\mathcal{O}_{h_n})) - \mathscr{S}(u(\mathcal{O}), u(\mathcal{O}))]\mathrm{d}r| \\ &\qquad + |\mathsf{Q} \sum_{j=1}^{N(h)} \int_{\mathscr{O}_j} (\mathcal{O}^2_{h_n}(a_j) - \mathcal{O}^2(r)) r \mathscr{S}(u(\mathcal{O}), u(\mathcal{O}))]\mathrm{d}r| \\ &\leqslant \text{constant}_{\langle A\rangle}(||u_{h_n}(\mathcal{O}_{h_n})||_{V(\Omega_{\langle AB\rangle})} + ||u(\mathcal{O})||_{V(\Omega_{\langle AB\rangle})})||u_{h_n}(\mathcal{O}_{h_n}) - u(\mathcal{O})||_{V(\Omega_{\langle AB\rangle})} \\ &\qquad + \text{constant}_{\langle H\rangle}(||\mathcal{O}_{h_n} - \mathcal{O}||_{C(\bar{\Omega}_{\langle AB\rangle})}||u(\mathcal{O})||^2_{V(\Omega_{\langle AB\rangle})} \to 0, \end{aligned}$$

for $n \to \infty$, since the sequence $\{||u_{h_n}(\mathcal{O}_{h_n})||_{V_h(\Omega_{\langle AB\rangle})}\}_{n\in N}$ is bounded and $\mathcal{O}_{h_n} \to \mathcal{O}$ strongly in $C(\bar{\Omega}_{\langle AB\rangle})$.

Next we have

$$\begin{aligned} &|\mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}_{h_n}, u_{h_n}(\mathcal{O}_{h_n})) - \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}, u(\mathcal{O}))| \\ &\leqslant ||u_{h_n}(\mathcal{O}_{h_n}) + u(\mathcal{O}) - 2z_{ad}||_{L_2(\Omega_{\langle AB\rangle})}||u_{h_n}(\mathcal{O}_{h_n}) - u(\mathcal{O})||_{V(\Omega_{\langle AB\rangle})} \to 0. \end{aligned}$$

∎

LEMMA 6.5 *Let* $\{\mathcal{O}_{*\langle h_n\rangle}\}_{n\in N}$, *for* $h_n \to 0_+$ *be a sequence of solutions of the approximate problems (6.9). Then there exists a subsequence* $\{\mathcal{O}_{*\langle h_{n_k}\rangle}\}_{k\in N}$ *such that for* $h_{n_k} \to 0_+$, $\mathcal{O}_{*\langle h_{n_k}\rangle} \to \mathcal{O}_*$ *in* $C(\bar{\Omega}_{\langle AB\rangle})$, $u_{h_{n_k}}(\mathcal{O}_{*\langle h_{n_k}\rangle}) \to u(\mathcal{O}_*)$ *strongly in* $V(\Omega_{\langle AB\rangle})$, *where* $\mathcal{O}_* \in \mathscr{U}_{ad}(\Omega_{\langle AB\rangle})$ *is the solution of the optimization problem (6.6) and* $u(\mathcal{O}_*) \in \mathscr{K}(\Omega_{\langle AB\rangle})$ *is the corresponding solution of (6.4).*

*Proof.* Let us consider a function $\mathscr{H} \in \mathscr{U}_{ad}(\Omega_{\langle AB \rangle})$. There exists a sequence $\{\mathscr{H}_{h_n}\}_{n\in N}$, $h_n \to 0_+$ such that $\mathscr{H}_{h_n} \in \mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB \rangle})$ and $\mathscr{H}_{h_n} \to \mathscr{H}$ strongly in $C(\bar{\Omega}_{\langle AB \rangle})$. Here, we denote by $u_h(\mathscr{H}_{h_n})$ the solution of (6.8) where $\mathcal{O}_{h_n}$ is replaced by $\mathscr{H}_{h_n}$. Further, since $\mathscr{U}_{ad\langle h\rangle}(\Omega_{\langle AB \rangle}) \subset \mathscr{U}_{ad}(\Omega_{AB})$ and $\mathscr{U}_{ad}(\Omega_{\langle AB \rangle})$ is compact in $C(\bar{\Omega}_{\langle AB \rangle})$, hence we conclude that a subsequence of $\{\mathcal{O}_{h_n}\}_{n\in N}$ exists such that $\mathcal{O}_{*\langle h_{n_k}\rangle} \to \mathcal{O}_{\langle * \rangle}$ uniformly in $\bar{\Omega}_{\langle AB \rangle}$ for $h_{n_k} \to 0_+$, so that $\mathcal{O}_{\langle * \rangle} \in \mathscr{U}_{ad}(\Omega_{\langle AB \rangle})$. Next, in view of the definition of the problem (6.33), we conclude that

$$\begin{cases} \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS},\langle h_n\rangle}(\mathcal{O}_{*\langle h_n\rangle}, u_{h_n}(\mathcal{O}_{*\langle h_n\rangle})) \\ \quad \leqslant \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS},\langle h_n\rangle}(\mathscr{H}_{h_n}, u_{h_n}(\mathscr{H}_{h_n})), \\ \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}_{*\langle h_n\rangle}, u_{h_n}(\mathcal{O}_{*\langle h_n\rangle})) \leqslant \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathscr{H}_{h_n}, u_{h_n}(\mathscr{H}_{h_n})). \end{cases} \tag{6.35}$$

Let us pass to the limit with $h_{n_k} \to 0_+$ in (6.35) and apply Lemma 6.4 and (6.32) to the sequences $\{\mathcal{O}_{*\langle h_{n_k}\rangle}\}_{k\in N}$ and $\{\mathscr{H}_{h_{n_k}}\}_{k\in N}$.

Thus we come to the inequality

$$\begin{cases} \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathcal{O}_{\langle * \rangle}, u(\mathcal{O}_{\langle * \rangle})) \leqslant \mathscr{L}_{\text{INTENSITY OF SHEAR STRESS}}(\mathscr{H}, u(\mathscr{H})), \\ \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathcal{O}_{\langle * \rangle}, u(\mathcal{O}_{\langle * \rangle})) \leqslant \mathscr{L}_{\text{DESIRED DEFLECTION}}(\mathscr{H}, u(\mathscr{H})). \end{cases} \tag{6.36}$$

On the other hand from (6.36), we conclude that

$$\langle \mathcal{O}_{\langle * \rangle,\text{INTENSITY OF SHEAR STRESS}}, \mathcal{O}_{\langle * \rangle,\text{DESIRED DEFLECTION}} \rangle$$

are a solution of the problem (6.36). Moreover, due to (6.32), we may write

$$||u_{h_n}(\mathcal{O}_{*\langle h_n\rangle}) - u(\mathcal{O}_{\langle * \rangle})||_{V(\Omega_{\langle AB \rangle})} \to 0, \text{ for } h_n \to 0_+.$$

and each uniformly convergent subsequence of $\{\mathcal{O}_{h_n}\}_{n\in N}$ has the same property. ■

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