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Dual dynamic approach to shape optimization

by

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Abstract: We consider the control problem with multidimensional integral functional where state and control satisfy a system of the first order hyperbolic PDE. Next, a type of deformation with control of the domain is described and then we define suitable shape functional. Having defined trajectory and control of deformation the dual dynamic programming tools are applied to derive optimality condition for the shape functional with respect to such a deformation.

Keywords: control, deformation, dual dynamic programming, shape functional.

1. Introduction

The aim of this paper is to present some new approach to shape optimization. Shape functionals are difficult to study by dynamic programming methods. The main difficulty appears since there are no classical dynamic programming tools which allow to consider multidimensional domains. The second difficulty is how to choose a suitable deformation of the domain in order to get new functional (shape functional) depending on some new quantities. Such a functional should allow for applying known mathematical tools so as to have possibilities to determine some optimality conditions with respect to chosen deformation. The most popular method is to distinguish one parameter in the deformation and then try to calculate different derivatives (see, e.g., Nazarov, Sokolowski, 2003). In last few years we find in literature a notion of topological derivative applied to shape functionals (see, e.g., Sokolowski, Zochowski, 1999; Nazarov, Sokolowski, 2003). It uses the variation of the geometrical domain resulting in the change of the topological characteristic by removing a small ball from that domain - the parameter is the radius of that ball.

Our approach is close to the classical control problem. We consider as a control problem the multidimensional integral functional with the state and control subject to satisfy a system of first order hyperbolic PDE (see, e.g., *Encyclopedia of Mathematics* of Kluwer). For that problem we apply our earlier result concerning sufficient optimality conditions in terms of dual dynamic programming PDE (Section 2). Next we describe a type of deformation of the domain following Zolesio (see, e.g., Sokolowski, Zolesio, 1992) but adding to that deformation a control which allow to control (to some extent) that deformation and then we define suitable shape functional. Having defined trajectory and control of deformation we are able to apply the dual dynamic programming tools (see Nowakowski, 1992) to derive optimality condition for our shape functional with respect to that deformation.

2. Control problem

Consider the following optimal control problem (P):

$$\text{minimize } J(x,u) = \int_{\Omega} L(t, x(t), u(t))dt \tag{1}$$

subject to

$$x_{t_i}(t) = f_i(t, x(t), u(t)) \quad \text{a. e. on} \quad \Omega, \; i = 1, ..., n \tag{2}$$

$$u(t) \in U \quad \text{a. e. on} \quad \Omega \tag{3}$$

$$x(t) = \varphi(t) \quad \text{on} \quad \partial\Omega \tag{4}$$

where Ω is a given bounded subset of R^n with Lipschitz boundary and U is a given nonempty set in R^m; $L : \Omega \times R \times R^m \to R$, $f_i : \Omega \times R \times R^m \to R$ for $i = 1, ..., n$ and $\varphi : R^n \to R$ are given functions; $x : \Omega \to R$, $x \in W^{1,2}(\Omega)$ and $u : \Omega \to R^m$ is a Lebesgue measurable function. We assume that for all s in R, the functions $(t, u) \to L(t, s, u)$, $(t, u) \to f_i(t, s, u)$ for $i = 1, ..., n$ are $(L \times B)$-measurable, where $L \times B$ is the σ-algebra of subsets of $\Omega \times R^m$ generated by products of Lebesgue measurable subsets of Ω and Borel subsets of R^m, and that for each $(t, u) \in \Omega \times R^m$, the functions $s \to L(t, s, u)$, $s \to f_i(t, s, u)$ for $i = 1, ..., n$ are continuous. We would like to stress that we assume the existence of solutions to (2)-(4).

A pair $x(t)$, $u(t)$ is called admissible, if it satisfies (2)-(4) and $L(t, x(t), u(t))$ is summable; then the corresponding trajectory $x(t)$ is called admissible.

3. A sketch of the dual approach

Let us sketch the main idea of the dual approach. Let $V(t, p)$ be a C^1 function defined on a subset P of R^{n+2} of the variables $(t, p) = (t, y^0, y)$, $y^0 \leq 0$, and satisfying the following conditions:

$$V(t, p) = y^0 V_{y^0}(t, p) + yV_y(t, p) = V_p(t, p)p \tag{5}$$

with $V_y(t,p) = -x(t,p)$ for $(t,p) \in P$, where $x(t,p)$ is such a function defined on P that for each admissible trajectory $x(t)$ there exists a function $p(t) = (y^0, y(t))$, $p \in W^{1,2}(\Omega)$, $(t,p(t)) = \big(t, y^0, y(t)\big) \in P$, such that $x(t) = x(t,p(t))$;

$$\int_{\partial\Omega} V(s,p(s))\nu(s)ds = -\int_{\partial\Omega} y(s)x(s,p(s))\nu(s)ds - S_D \tag{6}$$

and

$$y^0 \int_{\partial\Omega} V_{y^0}(s,p(s))\nu(s)ds = -S_D,$$

where $\nu(\cdot)$ is the exterior unit normal vector to $\partial\Omega$ and

$$S_D := \inf\left\{-y^0 \int_{\Omega} L(t,x(t),u(t))dt\right\} \tag{7}$$

over admissible pairs $x(t)$, $u(t)$, $t \in \Omega$, such that there are a function $p(t) = (y^0, y(t))$, $p \in W^{1,2}(\Omega)$, $(t,p(t)) \in P$, and a function $\psi : R^n \to R$ satisfying the conditions: $x(t) = x(t,p(t))$ for $t \in \Omega$, $x(t,\psi(t)) = \varphi(t)$ and $p(t) = \psi(t)$ on $\partial\Omega$. By

$$\int_{\partial\Omega} h(s)\nu(s)ds,$$

where $h : \Omega \to R$ is a real valued function and $\nu(\cdot)$ is the exterior unit normal vector to $\partial\Omega$, we mean in the paper the integral taken from the scalar product of two n-dimensional vectors $(\underbrace{h(\cdot),...,h(\cdot)}_{n})$ and $\nu(\cdot) = (\nu_1(\cdot),...,\nu_n(\cdot))$. The function $V(t,p)$ satisfies the partial differential equation

$$\sum_{i=1}^{n} V_{t_i}(t,p) + H(t, -V_y(t,p), p) = 0, \tag{8}$$

where $H(t,v,p) = y\sum_{i=1}^{n} f_i(t,v,u(t,p)) - y^0 L(t,v,u(t,p))$ and $u(t,p)$ is an optimal dual feedback control, and the dual partial differential equation of multidimensional dynamic programming (DPDEMDP)

$$\begin{aligned} \max\{&-\textstyle\sum_{i=1}^{n}[V_{t_i}(t,p) + y f_i(t,-V_y(t,p),u)] \\ &+y^0 L(t,-V_y(t,p),u) : u \in U\big\} = 0. \end{aligned} \tag{9}$$

REMARK 3.1 *We would like to stress that the duality, which is sketched in this section is not a duality in the sense of convex optimization. It is a new nonconvex duality, first time described in Nowakowski (1992) for which we do not have the relation* $\sup(D) \le \inf(P)$ *(D – meaning the dual problem, P – the primal one). But, instead of it, we have other relations, namely (5) and (6) – they are* *a generalization of transversality conditions from classical mechanics. Different duality, different relation. If we find a solution to (8) then checking the relation (5) for concrete problems is not very difficult. How to do that for most ceases, is described in the section Example and Conclusions of this paper. In our opinion there are no relations between duality described here and the duality of Kloetzler as the latter is in fact a kind of generalization of the convex duality to nonconvex problems (more details about that approach see Pickenhain, 2001).*

4. Sufficiency theorem

The theorems below, proved in Galewska, Nowakowski (2003) provide the sufficient conditions for optimality, which a solution $V(t,p)$ of the dual partial differential equation of multidimensional dynamic programming and an optimal dual feedback control, respectively, should satisfy.

THEOREM 4.1 *Let $V(t,p)$ be a C^1 solution of DPDEMDP (9) on P such that (5) holds. Let $\overline{x}(t)$, $\overline{u}(t)$, $t \in \Omega$, be an admissible pair and let $\overline{p}(t) = (\overline{y}^0, \overline{y}(t))$, $\overline{p} \in W^{1,2}(\Omega)$, $(t, \overline{p}(t)) \in P$, be such a function that $\overline{x}(t) = -V_y(t, \overline{p}(t))$ for $t \in \Omega$. Suppose that for almost all $t \in \Omega$,*

$$-\sum_{i=1}^{n} [V_{t_i}(t,\overline{p}(t)) + \overline{y}(t) f_i(t, -V_y(t,\overline{p}(t)), \overline{u}(t))] + \overline{y}^0 L(t, -V_y(t,\overline{p}(t)), \overline{u}(t)) = 0. \tag{10}$$

Then $\overline{x}(t)$, $\overline{u}(t)$, $t \in \Omega$, is an optimal pair relative to all admissible pairs $x(t)$, $u(t)$, $t \in \Omega$, for which there exists such a function $p(t) = (\overline{y}^0, y(t))$, $p \in W^{1,2}(\Omega)$, $(t, p(t)) \in P$, that $x(t) = -V_y(t, p(t))$ for $t \in \Omega$.

We now define the concept of an optimal dual feedback control.

DEFINITION 4.1 *A function $u = u(t,p)$ from a subset P of R^{n+2} of the points $(t,p) = (t, y^0, y)$, $y^0 \leq 0$, into U is called a dual feedback control, if there exists the solution $x(t,p)$, $(t,p) \in P$, of the system of partial differential equations*

$$x_{t_i} = f_i(t, x, u(t,p)),\ i = 1, ..., n \tag{11}$$

such that for each admissible trajectory $x(t)$, $t \in \Omega$, there is a function $p(t) = (y^0, y(t))$, $p \in W^{1,2}(\Omega)$, $(t, p(t)) \in P$, such that $x(t) = x(t, p(t))$ for $t \in \Omega$. (It is clear that the solution $x(t,p)$ is not unique).

DEFINITION 4.2 *A dual feedback control $\overline{u}(t,p)$ is called an optimal dual feedback control, if there exist a function $\overline{x}(t,p)$, $(t,p) \in P$, corresponding to $\overline{u}(t,p)$ as in Definition 4.1, and a function $\overline{p}(t) = (\overline{y}^0, \overline{y}(t))$, $\overline{p} \in W^{1,2}(\Omega)$, $(t, \overline{p}(t)) \in P$, such that, for*

$$S_D = -\overline{y}^0 \int_\Omega L(t, \overline{x}(t, \overline{p}(t)), \overline{u}(t, \overline{p}(t)))dt \tag{12}$$

defining $V_{y^0}(t, \overline{p}(t))$ *by*

$$\overline{y}^0 \int_{\partial\Omega} V_{y^0}(s, \overline{p}(s))\nu(s)ds = -S_D$$

and for $V_y(t,p) = -\overline{x}(t,p)$*, there is* $V(t,p)$ *satisfying (5).*

The following fundamental theorem gives sufficient optimality conditions for the existence of an optimal dual feedback control.

THEOREM 4.2 *Let* $\overline{u}(t,p)$ *be a dual feedback control in* P*. Suppose that there exists a* C^1 *solution* $V(t,p)$ *of DPDEMDP (9) on* P *and satisfying (5). Let* $\overline{p}(t) = (\overline{y}^0, \overline{y}(t))$*,* $\overline{p} \in W^{1,2}(\Omega)$*,* $(t, \overline{p}(t)) \in P$*, be such a function that* $\overline{x}(t) = \overline{x}(t, \overline{p}(t))$*,* $\overline{u}(t) = \overline{u}(t, \overline{p}(t))$*,* $t \in \Omega$*, is an admissible pair. Assume further that:*

$$V_y(t,p) = -\overline{x}(t,p) \quad \textit{for} \quad (t,p) \in P,\ (t,p) \in P, \tag{13}$$

$$\overline{y}^0 \int_{\partial\Omega} V_{y^0}(s, \overline{p}(s))\nu(s)ds = \overline{y}^0 \int_{\Omega} L(t, \overline{x}(t, \overline{p}(t)), \overline{u}(t, \overline{p}(t)))dt. \tag{14}$$

Then $\overline{u}(t,p)$ *is an optimal dual feedback control.*

5. Shape optimization problem

Let Ω_1 be a given C^2 class simply connected subset of $\mathbb{R}^n$. From now on we assume that $\mathbf{U}$ is a given nonempty, compact set in $(R^m)^+$ i.e. all $u \in \mathbf{U}$ have coordinates $u_i \geq 0$. In order to construct the deformation of Ω_1 the following boundary value problem is introduced:

For a given control $v(t) \in \mathbf{U}$, $t \in \Omega_1$, being a Hoelder continuous function find $z \in H^1(\Omega_1)$ satisfying (in weak sense)

$$\begin{cases} \Delta z = v & \text{in} \quad \Omega_1, \\ z = 1 & \text{on} \quad \Gamma_1 = \partial\Omega_1. \end{cases} \tag{15}$$

Put

$$z^{-1}(\rho) = \{t \in \Omega_1 : z(t) = \rho\}, \quad 0 \leq \rho \leq 1$$

and

$$\Gamma_1 = z^{-1}(1)$$

and

$$\Omega_\rho = \{t \in \Omega_1 : 0 < z(t) < \rho\}.$$

Since $\mathbf{U}$ is bounded, Ω_1 is domain of class C^2, it follows that a solution to (15) is a function $z(t,v)$ of class $C^2(\Omega_1)$. Thus boundary of Ω_ρ is locally Lipschitz.

Following Zolesio (Sokolowski, Zolesio, 1992) we introduce the field

$$Z(t,v) = \|\nabla z(t,v)\|^{-2} \nabla z(t,v).$$

Then the deformation is defined by $T_\rho(w,v) = t(\rho,w,v)$, where $t(\cdot,\cdot,\cdot)$ is a solution to

$$\frac{d}{d\rho}t(\rho,w,v) = Z(t(\rho,w,v),v),$$

with $w \in \Gamma_1$ and the initial condition

$$t(1,w,v) = w.$$

In the control theory we write it as:

$$\frac{d}{d\rho}t(\rho,w) = Z(t(\rho,w),v), \qquad (16)$$
$$t(1,w) = w.$$

Any trajectory $t(\rho)$ corresponding to control v satisfying (16) is called admissible and the pair $(t(\rho),v)$ is an admissible pair. The set of graphs of all admissible trajectories is denoted by T. We can deform Ω_1 by changing ρ. Thus we get the deformed domains:

$$\Omega_\rho = \{t \in \Omega_1 : 0 < z(t,v) < \rho\}.$$

We should stress that our deformation Ω_ρ depends on the control $v(t)$ too, i.e. we should, rather, write $\Omega_\rho(v)$. We note that in (15) the control $v(t)$ is defined in Ω_1. However if we consider (15) in $\Omega_\rho(v)$ then the solution $z(t,v)$ exists in $\Omega_\rho(v)$ and it agrees with that of (15) but with new boundary conditions i.e. $z = \rho$ on $\partial\Omega_\rho$. Of course, v is now also considered only in $\Omega_\rho(v)$ and to underline that, we shall write in the next part of the paper that v is defined in $[\rho,1]$, i.e. $v(\tau)$, $\tau \in [\rho,1]$. For each domain $\Omega_\rho(v)$ we can consider the optimal control problem (P). For each problem with the domain $\Omega_\rho(v)$, according to the sufficiency Theorem 4.1, there exists an optimal value depending on ρ and trajectory $t(\rho)$ (see also Definition 4.2)

$$J(\rho,t(\rho)) = -S_D(\rho,v) = y^0 \int_{\partial\Omega_\rho(v)} V_{y^0}(s,\overline{p}(s))n(s)ds,$$

where $n(\cdot)$ is the exterior unit normal vector to $\partial\Omega_\rho(v)$. In this way we get the new functional $J(\rho,t(\rho))$ depending only on ρ and $t(\rho)$ i.e. we can treat $J(\rho,t(\rho))$ as a terminal functional at $(\rho,t(\rho))$ with state $t(s)$ and control $v(s)$ defined in $[\rho,1]$. However, the starting point is now $(1,t(1))$ and the terminal point is $(\rho,t(\rho))$. Therefore, as the boundary condition for partial dynamic programming we will assume the value

$$J(\rho,t(\rho)). \qquad (17)$$

For the functional $J(\rho,t(\rho))$ we can formulate dual dynamic construction as in Nowakowski (1992), where now our functional does not depend explicitly on a state $t(s)$, $s \in [\rho, 1]$. Thus, we should treat that problem as to minimize $J(\rho, t(\rho))$ with respect to control $v : [\rho, 1] \to \mathbf{U}$ and state $t(s)$, $s \in [\rho, 1]$. So, by applying construction from Nowakowski (1992) let $Y(s,p)$ be a function defined on a set $P \subset [\rho, 1] \times R^{n+1}$, $(s,p) = (s, y^0, y)$, $y^0 \leq 0$ and satisfying

$$Y(s,p) = y^0 Y_{y^0}(s,p) + y Y_y(s,p) = Y_p(s,p)p \qquad \text{on P.} \tag{18}$$

We require that for each admissible trajectory $t(s)$, $s \in [\rho, 1]$ satisfying (16) there exist a $p(s) = (y^0, y(s))$ – absolutely continuous such that $t(s) = -Y_y(s, p(s))$, $s \in [\rho, 1]$, $p(1) = p_0$; p_0 is fixed for all admissible trajectory $t(s)$, $s \in [\rho, 1]$.

Let

$$y^0 Y_{y^0}(\rho, p) = -J_D(\rho, p)$$

with dual value function defined by

$$J_D(\rho, p) = \inf\{-y^0 S_D(\rho, v)\} = \inf\{-y^0 J(\rho, t(\rho))\}$$

where infimum is taken over all admissible pairs $(t(s), v(s))$, $s \in [\rho, 1]$ whose trajectories are starting at $(1, -Y_y(1,p))$. Then, by (17),

$$y^0 Y_{y^0}(\rho, p) = y^0 J(\rho, -Y_y(\rho, p)), \;\; (\rho, p) \in P \tag{19}$$

and $Y(s,p)$ satisfies

$$Y_s(s,p) + \overline{H}(s, -Y_y(s,p), p) = 0, \; (s,p) \in P,$$

where $\overline{H}(s,x,p) = yZ(x, u(s,p))$ and $u(s,p)$ is an optimal dual feedback control and the partial differential equation of dynamic programming is:

$$\min\{Y_s(s,p) + yZ(-Y_y(s,p), v) : \; v \in \mathbf{U}\} = 0, \; (s,p) \in P. \tag{20}$$

6. Sufficiency theorem

In this section, we formulate sufficiency conditions which allow us to determine, at least theoretically, the optimal state $\overline{t}(s)$ under control $\overline{v}(s)$, $s \in [\rho, 1]$.

THEOREM 6.1 *Let $Y(s,p)$ be a C^1 solution of the partial differential equation (20) on P and such that (18) and (19) hold. Let $(\overline{t}(s), \overline{v}(s))$ be an admissible pair, $s \in [\rho, 1]$ and let $\overline{p}(s)$, $s \in [\rho, 1]$, $\overline{p}(1) = p_0$ – absolutely continuous be such that $\overline{t}(s) = -Y_y(s, \overline{p}(s))$, $s \in [\rho, 1]$ and satisfies*

$$Y_s(s, \overline{p}(s)) + \overline{y}(s) Z(-Y_y(s, \overline{p}(s)), \overline{v}(s)) = 0, \qquad \text{in } [\rho, 1]. \tag{21}$$

Then, $\overline{t}(s), \overline{v}(s)$, $s \in [\rho, 1]$ is an optimal pair relative to all admissible pairs $t(s)$, $v(s)$, $s \in [\rho, 1]$ for which there exists such absolutely continuous function $p(s) = (y^0, y(s))$, $p(1) = p_0$, that $t(s) = -Y_y(s, p(s))$, $s \in [\rho, 1]$.

Proof. Take any admissible pair $t(s)$, $v(s)$, $s \in [\rho, 1]$, whose graph of trajectory is contained in T and for which there exists an absolutely continuous function $p(s) = \big(\overline{y}^0, y(s)\big)$, $p(1) = p_0$ lying in P such that $t(s) = -Y_y(s, p(s))$ for $s \in [\rho, 1]$. Then, from (18), we have , for almost all $s \in [\rho, 1]$

$$Y_s(s, p(s)) = \overline{y}^0\,(d/ds)\,Y_{y^0}(s, p(s)) + y(s)\,(d/ds)\,Y_y(s, p(s)). \tag{22}$$

Let $W(s, p(s))$ be a function defined on P by the formula

$$W\,(s, p\,(s)) := -\overline{y}^0 Y_{y^0}(s, p(s)). \tag{23}$$

Since

$$\overline{y}^0\,(d/ds)\,Y_{y^0}(s, p(s)) = -\,(d/ds)\,W\,(s, p\,(s))$$

and

$$(d/ds)\,Y_y(s, p(s)) = -Z(-Y_y(s, p(s)), v(s))$$

a. e. on $[\rho, 1]$, it follows, by (22) and (41), that for almost all $s \in [\rho, 1]$,

$$(d/ds)\,W\,(s, p\,(s)) = -Y_s(s, p(s)) - y(s)Z(-Y_y(s, p(s)), v(s)) \quad . \tag{24}$$

Thus, by (24) and (20), we get

$$(d/ds)\,W\,(s, p\,(s)) \leq 0 \;\text{ a. e. on }\; [\rho, 1]\,. \tag{25}$$

Similarly, by (24) and (21), we obtain

$$(d/ds)\,W(s, \overline{p}(s)) = 0 \;\text{ a. e. on }\; [\rho, 1]\,. \tag{26}$$

By (25) and (26) we get that function $W(s, p\,(s))$ is a nonincreasing function of s and $W(s, \overline{p}(s))$ is constant on $[\rho, 1]$ and equals $-\overline{y}^0 Y_{y^0}(\rho, \overline{p}(\rho)) = -\overline{y}^0 Y_{y^0}(1, p_0))$. Thus, by (23) and $-\overline{y}^0 Y_{y^0}(1, p(1)) = -\overline{y}^0 Y_{y^0}(1, p_0))$, we get

$$-\overline{y}^0 Y_{y^0}(\rho, \overline{p}(\rho)) \leq -\overline{y}^0 Y_{y^0}(\rho, p(\rho)), \tag{27}$$

$$-\overline{y}^0 J(\rho, \overline{t}(\rho)) \leq -\overline{y}^0 J(\rho, t(\rho)), \tag{28}$$

which proves the assertion of the theorem. ■

6.1. Example

Let us consider an example meant to explain what is the real value of the above theory. We consider the following nonlinear optimization problem (P): put $L(t, x, u) := x\sqrt[4]{x}u$ and $f_i(t, x, u) := 13\sqrt[4]{x}u^2/(72)$ for $i = 1, 2$, $(t, x, u) \in \Omega \times R \times R$, where Ω is a ball in R^2 with radius equals $1/2$ and center $(1/2, 1/2)$. Let further $P := \left\{(y^0, y) \in R^2 : y^0 \leq 0,\;\; y > 0\right\}$. Thus by the verification of

Theorem 4.1 we obtain analogously as in Galewska, Nowakowski (2003) that for $\overline{c} = (1/2, 9/13)$, $V(t, \overline{c}p) := -(9y/13)^{13/9} + 9(y^0/2)^2 \sum_{i=1}^{2} t_i \Big/ (26)$ one obtains

$$(1/2)\,\overline{y}^0 \int_{\partial\Omega} V_{y^0}(s, \overline{cp}(s))\nu(s)ds \tag{29}$$
$$= (1/2)\,\overline{y}^0 \int_{\Omega} L(t, \overline{x}(t, \overline{p}(t)), \overline{u}(t, \overline{p}(t)))dt,$$

where

$$\overline{x}(t, \overline{p}(t)) = \left(\sum\nolimits_{i=1}^{2} t_i\right)^{4/13} \tag{30a}$$

$$\overline{u}(t, \overline{p}(t)) = (9/13)\overline{y}^0 \left(\sum\nolimits_{i=1}^{2} t_i\right)^{-5/13} \tag{30b}$$

and v is a normal to $\partial\Omega$. Therefore, by Theorem 4.2

$$\overline{u}(t, p) = (9/13)\overline{y}^0\,(y)^{-5/9} \tag{31}$$

is an optimal dual feedback control and from (12) we conclude that

$$S_D = -\frac{1}{4}(\overline{y}^0)^2\pi \tag{32}$$

is a minimal value of the problem under consideration in the subspace of $W^{1,2}(\Omega)$

$$\left\{x \in W^{1,2}(\Omega) : x(t) = \varphi(t) \quad \text{on} \quad \partial\Omega\right\} \tag{33}$$

where

$$\varphi(t) = \left(\sum_{i=1}^{2} t_i\right)^{4/13} \quad \text{for} \quad t \in \partial\Omega.$$

According to Section 5 next step is to solve, for a given control $v(t) \in \mathbf{U} = [0, 10]$, $t \in \Omega$, linear elliptic equation (14). It is well known that there exists explicit formula for the solution:

$$z(t, v) = \int_{\partial\Omega} K(t, h) \cdot 1ds_h + \int_{\Omega} G(t, h)v(h)dh, \tag{34}$$

where, in our case ($t = (t_1, t_2)$, $h = (h_1, h_2)$)

$$K(t, h) = \frac{1 - |t - (1/2, 1/2)|^2}{4\pi\,|t - h|^2},$$

and

$$G(t,h) = \frac{1}{2\pi}\log|t-h| - \frac{1}{2\pi}\log\Big(4\,|t-(1/2,1/2)|^2\,|h-(1/2,1/2)|^2 + 1/4 - 2 < t-(1/2,1/2), h-(1/2,1/2) > \Big)^{1/2}.$$

To calculate the field $Z(t,v)$ we need the derivative z_{t_1} and z_{t_2} :

$$z_{t_1}(t,v) = -\int_{\partial\Omega}\left(\frac{t_1}{2\pi\,|t-h|^2} + \frac{(t_1-h_1)\left(1-\left(t_1^2+t_2^2\right)\right)}{2\pi\,|t-h|^4}\right)ds_h - \frac{1}{2\pi}\int_\Omega \frac{t_1 v(h)}{|t-h|^2}dh - \frac{1}{2\pi}\int_\Omega \frac{(2 < t-(1/2,1/2), h-(1/2,1/2) > -1/2)\,v(h)}{|< t-(1/2,1/2), h-(1/2,1/2) > -1/2|^2}dh$$

$$z_{t_2}(t,v) = -\int_{\partial\Omega}\left(\frac{t_2}{2\pi\,|t-h|^2} + \frac{(t_2-h_2)\left(1-\left(t_1^2+t_2^2\right)\right)}{2\pi\,|t-h|^4}\right)ds_h - \int_\Omega \frac{t_2 v(h)}{|t-h|^2}dh - \frac{1}{2\pi}\int_\Omega \frac{(2 < t-(1/2,1/2), h-(1/2,1/2) > -1/2)\,v(h)}{|< t-(1/2,1/2), h-(1/2,1/2) > -1/2|^2}dh.$$

Then

$$Z(t,v) = (Z_1(t,v), Z_2(t,v)) = \frac{(z_{t_1}(t,v), z_{t_2}(t,v))}{|(z_{t_1}(t,v), z_{t_2}(t,v))|^2}$$

and the partial differential equation of dynamic programming is:

$$\min\{Y_s(s,p) + y_1 Z_1(-Y_y(s,p),v) + y_2 Z_2(-Y_y(s,p),v) :\ v\in\mathbf{U}\} = 0, \quad (s,p)\in P_1 \tag{35}$$

where $P_1 = \left\{(y^0,y_1,y_2) : y^0 \le 0,\ y_1,y_2\in(-1/2,0)\right\}$. We easily check that the minimum in (35) is attained at $v\equiv 0$. Therefore, equation (35) takes the form

$$\begin{aligned} Y_s(s,p) = {} & yZ(-Y_y(s,p),0) = y_1\int_{\partial\Omega}\frac{-Y_{y_1}(s,p)}{2\pi\,|-Y_y(s,p)-h|^2}ds_h \\ & + y_1\int_{\partial\Omega}\frac{(-Y_{y_1}(s,p)-h_1)\left(1-\left(Y_{y_1}(s,p)^2+Y_{y_2}(s,p)^2\right)\right)}{2\pi\,|-Y_y(s,p)-h|^4}ds_h \\ & + y_2\int_{\partial\Omega}\frac{-Y_{y_2}(s,p)}{2\pi\,|-Y_y(s,p)-h|^2}ds_h \\ & + y_2\int_{\partial\Omega}\frac{(-Y_{y_2}(s,p)-h_2)\left(1-\left(Y_{y_1}(s,p)^2+Y_{y_2}(s,p)^2\right)\right)}{2\pi\,|-Y_y(s,p)-h|^4}ds_h. \end{aligned} \tag{36}$$

Since the function $Y(s,p)$ has to satisfy (18) too, in order to find such a function we can help ourselves by defining $Y_y(s,p)$ and then try to find $Y(s,p)$ satisfying

(18) and (35). To this effect let us take $Y_{y_1}(s,p) = y_1 - 1/2$, $Y_{y_2}(s,p) = y_2 - 1/2$, $s \in [\rho, 1]$, $y_1, y_2 \in (-1/2, 0)$. Then, we see that for all $(s,p) \in P_1$ $Y_s(s,p) \leq 0$. Taking into account (18) and (19) we get

$$Y_s(s,p) = \frac{\partial}{\partial s} y^0 J(s, -Y_y(s,p)) \text{ at } s = \rho.$$

Let us observe that ρ was arbitrarily chosen and fixed and the above equality is true for all such ρ. Since $y^0 \leq 0$ thus we infer that

$$\frac{\partial}{\partial s} J(s, -Y_y(s,p)) \geq 0 \text{ at } s = \rho \text{ for each } 0 < \rho \leq 1.$$

Hence it follows that the maximum value of $J(\rho, -Y_y(\rho,p))$ is taken for $\rho = 1$. This result is intuitively obvious as $J(\rho, -Y_y(\rho,p)) > 0$ for all $0 < \rho \leq 1$ and the volume of Ω_ρ increases with ρ in that case i.e. for $v = 0$.

6.2. Conclusions

Let us write down, in consecutive steps, what we did to solve the above example. First it is necessary to solve the original control problem with given domain Ω. It is (32) in the example. The next step is to solve, for a given control $v(t) \in \mathbf{U}$, $t \in \Omega$, linear elliptic equation (14). It is easy, if the domain Ω is a ball, for then we have an explicit formula (34). Now we build the field $Z(t,v)$ and having that we form the partial differential equation of dynamic programming (35). We do not need, very often, to solve it explicitly. In many cases it is enough to have information on $Y_s(s,p)$ only. It is the case because of our construction: $Y_s(s,p) = \frac{d}{ds} y^0 J(s, -Y_y(s,p))$ at $s = \rho$. $J(\rho, -Y_y(\rho,p))$ is the value of our functional considered on the domain Ω_ρ. Thus, knowing the sign of $\frac{d}{ds} J(s, -Y_y(s,p))$ in some neighborhood of ρ we have information on the increase or decrease of the value of the functional $J(\rho, -Y_y(\rho,p))$ when the domain Ω_ρ is changing together with ρ. It is worth to stress that, generally, we do not need to know that the function $\rho \to J(\rho, t(\rho))$ is differentiable. One more aspect should be underlined: Theorem 6.1 states sufficient optimality conditions in terms of the optimal pair $\overline{t}(s), \overline{v}(s)$, $s \in [\rho, 1]$, however, if we look carefully at the construction of Section 5 then we see that a trajectory $t(\rho)$ must satisfy initial conditions on the boundary of Ω i.e. $t(1,w) = w$ for all $w \in \partial\Omega$. In most cases the pair $\overline{t}(s), \overline{v}(s)$, $s \in [\rho, 1]$ will be optimal for all $w \in \partial\Omega$. Hence in those cases it is enough to investigate only $Y_s(s,p) = yZ(-Y_y(s,p), \overline{v})$, $(s,p) \in P_1$.

Therefore the theory described in Sections 5 and 6 gives us a new tool for studying a certain type of shape optimization problems. We believe that the method described here can be also developed for a different type of deformations of the domain Ω.

7. ϵ-Value function

In practice it is more important to have a function, which may be calculated effectively. This is why we will now deal with objects that we call $\epsilon-$value functions. These functions are very useful when we want to construct numerical approximation of the value function. They allow to check whether our approximate function is already good enough or not.

Definition 7.1 *By a dual $\epsilon-$value function we mean any function $J_{\epsilon D}(s,p)$ defined in $P \subset [\rho,1] \times R^{n+1} \in P$, $(s,p)=(s,y^0,y)$, $y^0 \le 0$, which satisfies the following inequalities:*

$$J_D(\rho,p) \le J_{\epsilon D}(\rho,p) \le J_D(\rho,p) - \epsilon y^0,\ (\rho,p) \in P,$$

where $J_D(\cdot,\cdot)$, is the dual value function and $\epsilon > 0$ is any given number.

Definition 7.2 *An admissible trajectory $t_\epsilon(s)$ under control $u_\epsilon(s)$, $s \in [\rho,1]$ is called an $\epsilon-$optimal trajectory if there exists an absolutely continuous function $p_\epsilon(s)$, $s \in [\rho,1]$, $p(1)=p_0$ lying in P, such that*

$$-y^0_\epsilon J(\rho,t_\epsilon(\rho)) \le -y^0_\epsilon J(\rho,t(\rho)) - y^0\epsilon$$

relative to all admissible pairs $t(s)$, $v(s)$, $s \in [\rho,1]$ for which there exists such a function $p(s)=(y^0_\epsilon,y(s))$, p-absolutely continuous, that $t(s)=-Y_y(s,p(s))$, $s \in [\rho,1]$, $p(1)=p_0$, where $Y(s,p)$ is a solution to (37).

Theorem 7.1 *Let $Y(s,p)$, $(s,p) \in P$ be a C^1 solution of the following partial differential inequality*

$$0 \le \max\{Y_s(s,p) + yZ(-Y_y(s,p),v) :\ v \in \mathbf{U}\} \le -\frac{1}{2}y^0\epsilon,\ (s,p) \in P, \quad (37)$$

and such that

$$Y(s,p) = Y_p(s,p)p - \frac{1}{2}y^0\epsilon(1-s),\ (s,p) \in P \quad (38)$$

and (19) holds. Let $(t_\epsilon(s),v_\epsilon(s))$ be an admissible pair, $s \in [\rho,1]$ and let $p_\epsilon(s)$, $s \in [\rho,1]-$ absolutely continuous, be such that $t_\epsilon(s)=-Y_y(s,p_\epsilon(s))$, $s \in [\rho,1]$, $p(1)=p_0$ and satisfy

$$0 \le Y_\rho(\rho,p_\epsilon(\rho)) + y_\epsilon(\rho)Z(-Y_y(\rho,p_\epsilon(\rho)),v_\epsilon(\rho)) \le -\frac{1}{2}y^0_\epsilon\epsilon, \qquad in\ [0,1]. \quad (39)$$

Then $t_\epsilon(s),v_\epsilon(s)$, $s \in [\rho,1]$ is an ϵ-optimal pair for the dual ϵ-value function $J_{\epsilon D}(\rho,p) = -y^0_\epsilon Y_{y^0}(\rho,p)$ relative to all admissible pairs $t(s)$, $v(s)$, $s \in [\rho,1]$ for which there exists such a function $p(s)=(y^0_\epsilon,y(s))$, p-absolutely continuous that $t(s)=-Y_y(s,p(s))$, $s \in [\rho,1]$, $p(1)=p_0$.

Proof. Take any admissible pair $t(s)$, $v(s)$, $s \in [\rho, 1]$, whose graph of trajectory is contained in T and for which there exists an absolutely continuous function $p(s) = \left(y^0_\epsilon, y(s)\right)$ lying in P such that $t(s) = -Y_y(s, p(s))$ for $s \in [\rho, 1]$. Then, from (38), we have, for almost all $s \in [\rho, 1]$

$$Y_s(s, p(s)) = y^0_\epsilon \,(d/ds)\, Y_{y^0}(s, p(s)) + y(s)\,(d/ds)\, Y_y(s, p(s)) + \frac{1}{2} y^0_\epsilon \epsilon. \quad (40)$$

Let $W(s, p(s))$ be a function defined on P by the formula

$$W\,(s, p\,(s)) := -y^0_\epsilon Y_{y^0}(s, p(s)). \quad (41)$$

Since

$$y^0_\epsilon \,(d/ds)\, Y_{y^0}(s, p(s)) = -\,(d/ds)\, W\,(s, p\,(s))$$

and

$$(d/ds)\, Y_y(s, p(s)) = -Z(-Y_y(s, p(s)), v(s))$$

a. e. on $[\rho, 1]$, it follows, by (40) and (41), that for almost all $s \in [\rho, 1]$,

$$(d/ds)\, W\,(s, p\,(s)) = -Y_s(s, p(s)) - y(s) Z(-Y_y(s, p(s)), v(s)) + \frac{1}{2} y^0_\epsilon \epsilon. \quad (42)$$

Thus, by (42) and (37), we get

$$y^0_\epsilon \epsilon \le (d/ds)\, W\,(s, p\,(s)) \;\text{ a. e. on } [\rho, 1]. \quad (43)$$

Similarly, by (42) and (39), we obtain

$$y^0_\epsilon \epsilon \le (d/ds)\, W(s, p_\epsilon(s)) \le 0 \;\text{ a. e. on } [\rho, 1]. \quad (44)$$

By (43) and (44) we get that function $W(s, p\,(s))$ and $W(s, p_\epsilon(s))$ are nonincreasing functions of s on $[\rho, 1]$. Thus, by (41) and (19), we get

$$y^0_\epsilon \epsilon - y^0_\epsilon Y_{y^0}(\rho, p_\epsilon(\rho)) \le -y^0_\epsilon Y_{y^0}(\rho, p(\rho)),$$
$$-y^0_\epsilon J(\rho, t_\epsilon(\rho)) \le -y^0_\epsilon J(\rho, t(\rho)) - y^0_\epsilon \epsilon$$

which proves the assertion of the theorem. ■

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