Control and Cybernetics

vol. **35** (2006) No. 2

Book review:

VARIATIONAL ANALYSIS AND GENERALIZED DIFFERENTIATION
Vol. I: Basic Theory, Vol. II: Applications

by

Boris S. Mordukhovich

It would be hard to find an author more competent than Boris Mordukhovich to write a synthesis of the modern variational analysis in infinite dimensional spaces. In the 1970s and 1980s, he was one of the pioneers in the development of nonsmooth analysis, which, together with set-valued analysis, constitutes the basis of the modern variational analysis. Since that, his personal contribution to the development of mathematical tools of variational analysis has been crucial.

Therefore, the new book "*Variational Analysis and Generalized Differentiation*", written by Boris Mordukhovich and published by Springer, has been met with great interest by a broad community of mathematicians, not only working directly in variational problems and their applications, but also interested in foundations of nonsmooth and set-valued mathematical analysis, as well as in generalized differentiation.

The work consists of two volumes. The first volume contains the basic theory, whereas the second one presents some important applications. As the author states it in the preface, "*the primary goals of the book are to present basic concepts and principles of variational analysis unified in finite-dimensional and infinite-dimensional settings, to develop a comprehensive generalized differential theory...*". Accordingly, the first volume of the book contains the following four chapters:

1. Generalized Differentiation in Banach Spaces,
2. Extremal Principle in Variational Analysis,
3. Full Calculus in Asplund Spaces,
4. Characterization of Well-Posedness and Sensitivity Analysis.

In *Chapter 1* the basic notions of generalized normals to nonconvex sets, coderivatives of set-valued mappings and subdifferentials of nonsmooth functions in abstract Banach spaces are introduced and their properties are derived. A characteristic feature of these notions is that they are difined in dual spaces via sequential weak * limits. This approach turns out to be very convenient. The compatible concept of sequential normal compactness (SNC) is introduced.

In *Chapter 2* the extremal principle is proved, which is the main mathematical tool in the further variational analysis. The principle can be viewed as a variational counterpart of the convex separation principle, for nonconvex settings. It turns out that Asplund spaces constitute the environment, where the extremal principle fits the best. The extremal principle is compared with some known variational principles, including the famous Ekeland's principle.

Chapter 3 presents full set of calculus rules for normal cones, coderivatives, subdifferentials and multivalued mappings, which constitute a complete theory of generalized differentiation in Asplund spaces.

In *Chapter 4* the basic properties of set-valued mappings, such as covering properties, metric regularity and Lipschitz behaviour are studied, using the tools developed in previous chapters. The results obtained are applied to well-posedness and Lipschitzian stability analysis for constrained and variational processes.

The second part of the book, devoted to applications of the basic theory, consists of the following chapters:

5. Constrained Optimization and Equilibria,
6. Optimal Control and Evolution Systems in Banach Spaces,
7. Optimal Control of Distributed Systems,
8. Applications to Economics.

In *Chapter 5* the approach based on extremal principle and generalized differentiation is applied in the study of nonsmooth constrained optimization problems, including problems with equilibrium constraints and multiobjective criteria. Necessary optimality and suboptimality conditions are derived.

In *Chapter 6* optimal control control problems for differential inclusions are investigated. The discrete approximations for continuous-time problems are introduced. Well-posedness and convergence properties of these approximations are analysed and necessary optimality conditions are formulated. In the same chapter, Pontryagin's type maximum principle for smooth dynamics is derived and properties of approximate maximum principle are discussed.

Chapter 7, devoted to optimal control, partially concerns hereditary systems and partially problems described by partial differential equations of parabolic and hyperbolic types. Employing approximation and perturbation methods, necessary optimality conditions are derived for various problems of this class, including boundary control and problems subject to control and state constraints.

The last chapter presents the application of infinite-dimensional variational analysis to economic modelling. In particular, the extremal principle is used to study Pareto allocations and price equilibria of nonconvex welfare economics models.

The above short outline of the contents shows that the first part of the book provides a complete and thorough presentation of mathematical tools of nonsmooth and nonconvex variational analysis in infinite-dimension, based on a comprehensive theory of generalized differentiation. The second part of the book

shows how the basic theory can be applied to various important constrained optimization problem, including optimal control.

The book is self-contained, in the sense that the complete proofs of all major results are included. It is very carefully prepared from the editorial view point. Each chapter is concluded with comprehensive and instructive comments, where the historical background is outlined, with detailed references, the choice of basic idea is motivated and some open problems are listed. A list of statements and a glossary of notions, as well as a subject index for each of the volumes is provided. An extesive bibliography contains almost 1400 items, including a rich choice of Russian books and papers, which often are not well known to western readers.

To conclude, I would like to stress that the new book by Boris Mordukhovich is a comprehensive monography on generalized differentiation and variational problems. It will no doubt be an important source of mathematical tools and concepts for all researchers and doctoral students working in variational problems, optimization and related topics. I am convinced that the book should be made available in any mathematical library.

Kazimierz Malanowski

Boris S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I. Basic Theory, II. Applications*, Series of Comprehensive Studies in Mathematics, Volumes 330 and 331, Springer-Verlag, Berlin–Heidelberg 2006, 580 and 610 pages, ISBN 3-540-25437-4 and ISBN 3-540-25438-2, EUR 85.55 per volume.