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Fuzzy satisfactory evaluation method for covering the ability comparison in the context of DEA efficiency

by

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Abstract: Evaluation of efficiency of each of the DMUs (Decision Making Units) in a company is a very important task. Thus, the studies of evaluation of efficiency are being actively carried out, based on production function. Until quite recently, the loglinear production function (the Cobb-Douglas function) has been used for evaluation purposes. The loglinear model evaluates the DMUs by measuring the average efficiency. Of late, the DEA (Data Envelopment Analysis) focussed the interest as the available method, in the form of either the CCR (Charnes-Cooper-Rhodes) or the BCC (Banker-Charnes-Cooper) model. However, the DEA approach does not provide for the lower limit of the production set, but only for the upper one. Hence, considering the fact that in the real-life problems the production set ranges between the lower and the upper limit, it is proposed that the possibility production function be constructed by introducing fuzziness into the loglinear production function. When we try to evaluate efficiency with the help of this possibility function, we can obtain from it two efficiency ratings, corresponding to the upper and lower limits. The DEA and the fuzzy loglinear models perform evaluation in the sense of inclusion of all the DMU data and provide a dual possibility image of efficiency in the sense that the DEA assesses the lower limit of inputs for the given output, while the fuzzy loglinear model assesses the maximum output for the given inputs. Hence, by making full use of this duality, we try to fuse the DEA and the fuzzy loglinear model in the evaluation of DMU efficiency by introducing a fuzzy goal. We propose to construct the fuzzy goal by evaluating the ratings for individual outputs with the help of fuzzy loglinear analysis, and introduce this fuzzy goal into the DEA. This approach can yield both efficiency and ability as obtained from the comparison of the CCR-based efficiencies.

Keywords: DEA, Charnes-Cooper-Rhodes model, fuzzy loglinear model, fuzzy goal.

1. Introduction

Evaluation of efficiency of each of the DMUs (Decision Making Units) in a company is a very important task. Thus, the studies of evaluation of efficiency are being actively carried out, based on the production functions. Until quite recently, the loglinear production function (the Cobb-Douglas function) has been used for evaluation purposes, Sato (1975). The loglinear model evaluates the DMUs by measuring the average efficiency. Of late, the DEA (Data Envelopment Analysis) focussed the interest as the available method, in the form of either the CCR (Charnes-Cooper-Rhodes) or the BCC (Banker-Charnes-Cooper) model, see Charnes, Cooper and Rhodes (1978). However, the DEA approach does not provide for the lower limit of the production set, but only for the upper one. Hence, upon considering the fact that in the real-life problems the production set ranges between the lower limit and the upper limit, Watada and Morimoto (1992) proposed a possibility production function, in which fuzziness is introduced into the loglinear production function. When we try to evaluate efficiency with the help of this possibility function, we can obtain from it two efficiency ratings, corresponding to the upper and lower limits (Uemura, Kobayashi and Hiro, 1996). The DEA and the fuzzy loglinear models provide evaluations in the sense of inclusion of all the DMU data. The primary difference between the two is that in the DEA we obtain the lower limit inputs for the given output, while from the fuzzy loglinear model we obtain the maximum possibility output for the given inputs, see Uemura (1998a). In order to make a possibly full use of this duality of the two approaches, we try to fuse DEA and fuzzy loglinear model in the evaluation of efficiency of the DMUs by introducing into the approach the concept of a fuzzy goal, Uemura (1998b,c, 2003). We propose to construct the fuzzy goal on the basis of evaluation ratings for individual outputs obtained from the fuzzy loglinear analysis, and to introduce this fuzzy goal into the DEA. The thus formulated method, referred to as satisfactory method, provides both efficiency and ability by comparing the DEA-generated efficiency measures. In another paper, Uemura (2001), a fuzzy mixed CCR model is proposed.

2. Evaluation of efficiency by the loglinear model

The loglinear regression analysis provides the evaluation on the basis of the distances of points located away from the loglinear regression line. The Cobb-Douglas production function (Sato, 1975) has the following general form:

$$Q = X_0 \cdot A_1^{x_1} \cdot A_2^{x_2} \cdot \ldots \cdot A_n^{x_n} \tag{1}$$

where Q is the single output from a DMU, and $A_i, (i = 1, \ldots, n)$ are multiple inputs into the DMU.

By applying logarithms to both sides of (1) we obtain the model in the loglinear form:

$$q = x_0 + a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \tag{2}$$

in which $x_0 = log\, X_0$, and $a_i = log\, A_i (i = 1, \ldots, n)$.

Now, let us denote the data for individual DMUs indexed $j\,(j = 1, \ldots, m)$ as $(Q_j,\, A_{1j},\, A_{2j},\, \ldots,\, A_{nj})$, and $q_j = log\, Q_j, a_{ij} = log\, A_{ij} (i = 1, \ldots, n; j = 1, \ldots, m)$.

The parameters x_i, appearing in (1) and (2), are identified with the least squares (LSQ) method. The here presented loglinear model serves to evaluate the DMUs by measuring the LSQ average with respect to the loglinear production function, whose parameters are identified on the basis of data.

3. Evaluation of efficiency with the fuzzy loglinear model

We will introduce now fuzziness into the loglinear model. The parameters x_i of the loglinear model from (1) are extended as the $L - L$ fuzzy parameters $\tilde{x}_i = (x_i, d_i)_L$. As already indicated, given that the object of our study is to construct an evaluation of the efficiency of the DMUs, we assume that the possibility production function has an upper limiting curve and a lower limiting curve. These two curves define the area, which contains all the outputs of the DMUs obtained for the same source point on the output axis, corresponding to one point of the possibility function. The latter amounts to taking X_0 as a non-fuzzy parameter. Using the notation from Section 2 in order to avoid complications we obtain therefore the possibility production function in the form

$$Q = X_0 \cdot A_1^{\tilde{x}_1} \cdot A_2^{\tilde{x}_2} \cdot \ldots \cdot A_n^{\tilde{x}_n}. \tag{3}$$

Again, by taking logarithms of both sides of (3), we transform the model into the loglinear one:

$$q = x_0 + a_1\tilde{x}_1 + a_2\tilde{x}_2 + \ldots + a_n\tilde{x}_n. \tag{4}$$

Similarly as before, we introduce the notation for individual DMUs, indexed $j\,(j = 1, \ldots, m)$, that is, $(Q_j,\, A_{1j},\, \ldots\, ,\, A_{nj})$, and $q_j = log\, Q_j,\ a_{ij} = log\, A_{ij}$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$. By the extension principle, the resulting model takes the form

$$x_0 + a_1\tilde{x}_1 + a_2\tilde{x}_2 + \ldots + a_n\tilde{x}_n = (x_0 + \sum_{i=1}^{n} a_{ij}\, x_i \ \ \sum_{i=1}^{n} a_{ij}\, d_i)_L. \tag{5}$$

Due to the procedure of identification of fuzzy parameters in fuzzy regression analysis of Tanaka (1991), the fuzzy parameters from (4) are obtained as the solution to the following mathematical programming problem:

$$\min \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}\, d_i$$

$$\text{subject to} \quad x_0 + \sum_{i=1}^{n} a_{ij} x_i + \mid L^{-1}(h) \mid \cdot \sum_{i=1}^{n} a_{ij}\, d_i \geq q_j \tag{6}$$
$$x_0 + \sum_{i=1}^{n} a_{ij} x_i - \mid L^{-1}(h) \mid \cdot \sum_{i=1}^{n} a_{ij}\, d_i \geq q_j$$
$$x_i \geq 0,\ i = 1, \ldots, n$$

in which h is the level of possibility, specified by the decision maker, and $L(.)$ is the reference function for the $L-L$ fuzzy numbers. Though x_i is usually not constrained in fuzzy regression analysis, it is assumed to be bigger than 0 in problem (6), because we aim at the construction of the possibility production set.

It is generally assumed in obtaining the possibility interval from the fuzzy regression analysis that $h = 0$. Yet, the object of our study is to determine the efficiency ratings, which are defined as the ratios of output to the respective upper and lower limits. As we shall see further on in the paper, the ratios of output to the upper and lower limit are independent of the value of h. In the considerations of this paper the level of h is fixed at 0.95.

DEFINITION 3.1 *(the efficiency ratings of the DMUs in the possibility production function) The efficiency rating of the DMU indexed j will be defined as $Q_j/\widehat{QU}_j$, and the respective inefficiency rating will be defined as $\widehat{QL}_j/Q_j$, where $[\widehat{QL}_j, \widehat{QU}_j]$ is the interval of output values, estimated with the help of (6).*

4. DEA based on the possibility production set

The initial formulation of the DEA referred to fractional mathematical programming as the model for the direct estimation of efficiency of the DMUs, see Charnes, Cooper and Rhodes (1978). Later on, the DEA approach has been broadened by the formulation involving the construction of the possibility production set, see Tone (1993). In the present section we explain the approach to DEA modelling based upon the possibility production set.

Assume that we have the input set $X_j = (x_{j1}, \ldots, x_{jm})$ and the output set $Y_j = (y_{j1}, \ldots, y_{js})$ for the DMU indexed $j\,(j = 1, \ldots, n)$. We denote the input vector as $\mathbf{X} = [X_1, \ldots, X_n]$ and the output vector as $\mathbf{Y} = [Y_1, \ldots, Y_n]$. The input and output vectors are supposed to take positive values. Using the data set corresponding to the DMUs, $(\mathbf{X}, \mathbf{Y})$, we define the possibility production set (x, y) as the set satisfying the following constraints:

$$x \geq \mathbf{X}\lambda$$
$$y \leq \mathbf{Y}\lambda$$
$$\lambda \geq 0$$
$$L \leq e^T \lambda \leq U, \tag{7}$$

in which $x \in R^m,\ y \in R^s,\ \lambda \in R^n$, and $e^T = (1, \ldots, 1)$.

By virtue of (7), the possibility production set of the DEA is defined. In this context, various DEA-associated models are connected with different assumptions concerning the values of L and U. And thus, when $L = 0, U = \infty$, we have the CCR (Charnes-Cooper-Rhodes) model, while for $L = U = 1$, the respective model is the BCC (Banker-Charnes-Cooper) model, see Tanaka (1991). In this paper we focus on the CCR model, as we compare the DEA and the fuzzy loglinear model in the sense of the respective possibility production sets. The CCR model is formulated as follows:

The CCR model

$$\begin{aligned} & \min \theta - \epsilon(e^T s^+ + e^T s^-) \\ \text{subject to} \quad & \theta x_0 - s^+ = \mathbf{X}\lambda \\ & y + s^- = \mathbf{Y}\lambda \\ & \lambda,\ s^+,\ s^- \geq 0 \end{aligned} \tag{8}$$

where $\theta = \mathbf{Y}_0\lambda$, and $s^+,\ s^-$ are the slack vectors. If $\theta^* = 1$, which is the solution of (8), the corresponding DMU is called D-efficient.

5. Fusion of evaluation with the fuzzy loglinear model and the CCR model

After having evaluated every single output with the help of the fuzzy loglinear function, let us denote the maximum efficiency rating as θ_1 and the minimum efficiency rating as θ_2 across all the efficiency ratings of all outputs. Assume that the satisfactional efficiency exists between θ_1 and θ_2, and the decision maker sets the fuzzy goal with respect to (8), $\mu_G : \theta \to [0, 1]$, in the following manner:

$$\mu_G(\theta) = \begin{cases} 0, \theta & \leq \quad \theta_2 \\ (\theta - \theta_2)/(\theta_1 - \theta_2), \theta_2 & \leq \quad \theta_1 \\ 0, \theta_1 & \leq \quad \theta. \end{cases} \tag{9}$$

On the basis of the maximising decision, we can formulate the Fuzzy Satisfactional DEA Model as follows (see Sakawa, 1993):

$$\begin{aligned} & \min \alpha \\ \text{subject to} \quad & \alpha \leq \mu_G(\theta) \\ & \theta x_0 \geq \mathbf{X}\lambda \\ & y \leq \mathbf{Y}\lambda \\ & \lambda, s^- \geq 0 \end{aligned} \tag{10}$$

with the problem (10) solved by LP. Now, let us denote the fuzzy satisfactional solution $\hat{\alpha}$ as $\hat{\theta}$. The nature of the fuzzy satisfactional solution is given in the following theorem:

Theorem 5.1 *(the nature of the fuzzy satisfactional solution)*
$\theta_1 \leq \theta$ (DEA efficiency), $\theta = \theta_1$. (See Uemura, 1998b).

The solution, $\hat{\theta}$, can compensate for an important weak point in the DEA, which often evaluates one output and ignores the other outputs. With the new solution we can obtain the satisfactional evaluation ratings, accounting for the possibility outputs for the given inputs.

On the basis of the fusional evaluation rating $\hat{\theta}$ we can obtain the improvement plan for the non-efficient DMUs in the following form

Improvement for inputs in the non-efficient DMUs:

$$x := \hat{\theta}\, x$$

Theorem 5.2 *(comparison of ability through DEA efficiency)*

1. *if $\hat{\theta} \geq \theta$ (DEA efficiency), then we can hardly expect that it is possible to obtain higher outputs with the given inputs; in this case we can say that the DMU considered is making sufficient efforts, that is - the ability is used appropriately.*
2. *if $\hat{\theta} \geq \theta$ (DEA efficiency), we know that the current outputs are lower than the possibility resulting from the current inputs; in this case DEA efficiency is higher, indicating that the given DMU is inactive in terms of using the ability offered; this is the case, in which the previously specified improvement of inputs ought to be performed.*

6. An example of application

First, the DEA efficiency values for eleven national-level banks are shown in Table 1. The table contains the data on three inputs: total asset value (in billion Yen), number of employees, and number of branch offices; and on two outputs: gross profit (in billion Yen), and net return (in billion Yen). The table shows also the DEA efficiencies with $\mu^* = (\mu_1^*, \mu_2^*) = (a, 0)$, and $\theta^* = (\mu_1^*, \mu_2^*) = (0, b)$. This means that the CCR approach analyses total efficiency only with respect to one output and ignores the other output. This exactly is the most serious problem with the CCR model.

Then, we apply the fuzzy loglinear model to individual outputs. We denote the gross profit by y_1 and the net return by y_2. After having identified the respective parameters with the data available we obtain two following fuzzy loglinear functions:

$$y_1 = 1 \cdot x_1^{(0.735,0.017)_L} \cdot x_2^{(0.00,0.00)_L} \cdot x_3^{(0.162,0.00)_L} \tag{11}$$
$$y_2 = 1 \cdot x_1^{(0.675,0.035)_L} \cdot x_2^{(0.00,0.00)_L} \cdot x_3^{(0.002,0.00)_L} \tag{12}$$

in which x_1 denotes total assets, x_2 - number of employees, and x_3 - number of branch offices.

Let us now denote the efficiency rating defined with (11) by θ_1, and the efficiency rating defined with (12) by θ_2, as shown in Table 2.

Finally, on the basis of θ_1 and θ_2 we construct the fuzzy goal and introduce it into the DEA. Therefrom we obtain the fusional evaluation ratings of Table 2.

Table 1. The Japanese bank data

DMUs	Total assets (million Yen)	Number of branches	Number of employees	Gross profit	Net profit
Bank 1	52,230,247	418	19,061	2,416,526	225,141
Bank 2	10,597,709	211	6,128	457,723	32,021
Bank 3	23,265,654	94	8,284	1,484,621	172,423
Bank 4	52,465,934	565	21,600	2,617,983	160,168
Bank 5	48,775,121	368	15,701	2,872,097	246,699
Bank 6	50,730,147	365	16,252	2,870,857	288,250
Bank 7	52,256,008	387	17,247	2,812,759	243,580
Bank 8	18,299,016	243	9,604	1,097,083	93,180
Bank 9	51,849,609	396	14,909	2,748,593	304,894
Bank 10	30,860,566	302	11,971	1,463,481	145,766
Bank 11	28,004,284	437	14,436	1,287,137	146,690

Table 2. DEA efficiency θ^*, optimum weights for two outputs, efficiencies from the fuzzy loglinear model and the fuzzy satisfactional efficiencies

DMU	DEA efficiency θ^*	Optimum weights (μ_1^*, μ_2^*)	Fuzzy loglinear model		Fuzzy satisfactional efficiency $\hat{\theta}$
			Output 1	Output 2	
Bank 1	0.7251	(0.300, 0.000)	0.7748	0.7345	0.7748
Bank 2	0.6768	(0.148, 0.000)	0.5851	0.3247	0.5851
Bank 3	1.0000	(0.322, 0.303)	1.0000	1.0000	1.0000
Bank 4	0.7820	(0.299, 0.000)	0.8195	0.5205	0.8195
Bank 5	1.0000	(0.343, 0.623)	1.0000	0.8453	1.0000
Bank 6	0.9678	(0.328, 0.943)	0.9648	0.9604	0.9648
Bank 7	0.8952	(0.319, 0.000)	0.9155	0.7939	0.9128
Bank 8	0.9396	(0.856, 0.000)	0.8643	0.6404	0.8643
Bank 9	1.0000	(0.348, 0.144)	0.9218	1.0000	1.0000
Bank 10	0.7432	(0.508, 0.000)	0.7508	0.6913	0.7508
Bank 11	0.7203	(0.560, 0.000)	0.6893	0.7446	0.7184

Let us consider the results for Bank 7. We can see that the fuzzy satisfactional solution not only exists, and is located between θ_1 and θ_2, but it is also bigger than the DEA efficiency θ^* (Table 2). On the other hand, in case of Bank 11, the fuzzy satisfactional solution, contained between θ_1 and θ_2, is smaller than the respective DEA efficiency. This example shows the consequences of the fact that $\hat{\theta}$ is obtained on the basis of two outputs, and makes up for the previously mentioned deficiency of the DEA. There are, therefore, two situations, as indicated at the end of Section 5 in Theorem 2.

7. Conclusions

Considering that the more realistic production function features an upper and lower limit, a fuzzy loglinear model is proposed. The fuzzy parameters of this model are obtained straightforwardly through the LSQ procedure designed for fuzzy regression analysis. The evaluation of efficiency with the help of this fuzzy loglinear production function yields two efficiency ratings, corresponding to the upper and lower limits. In the DEA approach we obtain the lower limit of inputs for the given output, while in the fuzzy loglinear approach we obtain the possibility of the maximum output for the given inputs. By making full use of the complementarity of the two approaches we attempt to fuse the DEA and the fuzzy loglinear model for purposes of evaluation of efficiency of the DMUs by introducing the concept of a fuzzy goal. We propose to construct the fuzzy goal on the basis of evaluation ratings for individual outputs by fuzzy loglinear analysis, and then to introduce the fuzzy goal into the DEA. This proposed method yields both efficiency and ability by comparison of the respective DEA efficiencies.

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