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Modeling and manipulating fuzzy regions: strategies to define the topological relation between two fuzzy regions

by

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Abstract: Topological relations between geographic objects are among the most important kinds of relations to manage in Geographic Information Systems (GIS). However, it its very expensive in storage space to keep these relations explicitly stored. Therefore, the relations are usually not directly stored, but they are inferred from the geometry of the objects. Furthermore, the inference of the topological relations is very expensive in processing time, especially when managing complex geographic objects such as fuzzy regions, or regions with multiple alpha-cuts. In this paper we argue that the topological relations between two regions with multiple alpha-cuts can be defined using the topological relations between the crisp regions that compose these two regions. In addition, we present strategies to define the topological relation between two regions with multiple alpha-cuts, with the intent to minimize the number of overlays of crisp regions to be executed to define the topological relationship between the two regions.

Keywords: GIS, fuzzy regions, alpha-cuts, topological relations.

1. Introduction

There are mainly two kinds of geographic objects: those for which boundaries are well-defined, such as buildings, books, and soccer fields; and those for which there is no well-defined boundary, also known as objects with ill-defined boundaries, such as ocean shores, limits between mountains and valleys, and soil types (Erwig and Schneider, 1997). The first kind of geographic objects is usually referred to as crisp geographic objects, or simply crisp objects, and the second kind is usually referred to as fuzzy or uncertain geographic objects.

Ill-defined boundaries essentially originate from two groups of objects (Bordogna and Chiesa, 1993; Erwig and Schneider, 1997; Schneider, 1999, 2003): objects that have well-defined boundaries, but whose position and shape are unknown or cannot be measured precisely (uncertain objects) and objects with boundaries that are not well-defined or there is no need to define them precisely (fuzzy objects). In the last case, fuzziness is an intrinsic feature of the objects themselves (Guesgen, 2005; Schneider, 2000).

The modeling and manipulation of these two groups of geographic objects in spatial databases and GIS are essentially the same. Therefore, throughout this paper both groups are going to be referred to as fuzzy geographic objects, or simply fuzzy objects. Geographic objects with uncertain or fuzzy boundaries are sometimes also called objects with broad boundaries (Burrough and Frank, 1996; Clementine and DiFelice, 1994). Informally speaking, objects with broad boundaries differ from crisp ones with regard to the boundary definition (Clementine and DiFelice, 1999), that is, objects with crisp boundaries contain well-defined boundaries, and objects with broad boundaries contain ill-defined boundaries.

Crisp geographic objects have been successfully modeled and implemented in spatial databases and GIS for years. Even though the same success has not been achieved for fuzzy geographic objects, at least four models have been developed to implement and manipulate these kinds of objects (Shibaski, 1993; Schneider, 2000):

1. Exact models, which transfer type systems and concepts for spatial objects with crisp boundaries to objects with ill-defined boundaries (Clementine and DiFelice, 1994; Cohn and Gotts, 1994a; Erwig and Schneider, 1997; Schneider, 1996);
2. Models based on rough sets, which work with lower and upper approximations of spatial objects (Beauboeuf et al., 2004; Worboys, 1998a, 1998b);
3. Probabilistic models, which model positional and measurement uncertainty (Burrough, 1996; Finn, 1993; Shibaski, 1993);
4. Models based on fuzzy sets, which predominantly model fuzziness (Altman, 1994; Burrough, 1996; Dutta, 1989; Guesgen, 2005; Schneider, 1999).

Models based on fuzzy set theory (Zadeh, 1965) have received a lot of credit in the GIS community lately (Altman, 1994; Bordogna and Chiesa, 1993; Burrough and Frank, 1996; Burrough, 1996; DeCaluwe et al., 1997; Guesgen, 2005; Morris, 2003; Morris and Jankowski, 2000, 2004; Petry et al., 1999; Robinson, 2003; Zhan, 1997, 1998; Zhan and Lin, 2003). Robinson (2003) argues that what makes fuzzy logic superior to classical logic is the fact that it allows us to express irreducible observation and measurement uncertainties in their various manifestations, and make these uncertainties intrinsic in the data. He also argues that fuzzy sets present greater expressive power and they have the ability to capture human common-sense reasoning and decision making.

Clementine and DiFelice (1999) describe the use of the fuzzy set theory to model fuzziness in spatial data. Here is a brief example of how their model works:

> Let V be a bounded subset of R^2, a fuzzy set $A \subset V$ is defined as the set of ordered pairs $A = \{x, \mu_A(x)\}$, where x is an element of V and $\mu_A(x)$ denotes the degree of membership of x in A. The membership function $\mu_A(x)$ associates to each $x \in V$ a real number in the interval [0,1]. The closer the value of $\mu_A(x)$ is to 1, the more x belongs to A.
>
> An α-cut of a fuzzy set A is a crisp set A_α that contains all the elements of V having a membership degree in A greater that or equal to the specified value of α, that is, $A_\alpha = \{x \in V| \quad \mu_A(x) \geqslant \quad \alpha\}$. The set of points having $\mu_A(x) = 1$ defines the core of the object, while the set of points having $\mu_A(x) = 0$ defines its exterior. A crisp boundary is obtained by connecting all the points having a given degree. If we consider various values of membership function, we can get a finite number of nested crisp boundaries all contained between the core and the exterior.

Fuzzy set theory has been combined with the object-oriented approach to model fuzziness in spatial databases (Burrough, 1996; George et al., 1992, 1997; Morris and Jankowski, 2004; Morris and Petry, 1998; Morris et al., 1998; Petry et al., 1999). Among the advantages of using the object oriented approach to model geographic objects is that it provides a higher order of abstraction in the data model (DeCaluwe et al., 1997), and brings considerable benefits to raster-vector integration, generalization of spatial data and scale-free databases, implementation of rules-based procedures, and data integrity (Woodsford, 1995).

Manipulation of geographic objects usually consists of the definition of several spatial relations between these objects. The variety of spatial relations can be grouped into three different categories (Egenhofer and Franzosa, 1991; Güting, 1994):

- Topological relations such as *inside*, *disjoint*, and *meet*, which are invariant under topological transformations of the objects such as *translation*, *rotation* and *scaling* (Zhan, 1997);
- Metric relations such as *distance between two objects* and *size of the area common to two objects*; and
- Relations concerning the partial and total order of spatial objects as described by prepositions such as *in front of*, *behind*, *above*, and *below*.

Topological relations between geographic objects are purely qualitative, that is, they are independent of any quantitative measure, and they represent some of the most important kind of relations to manage in spatial databases and GIS (Zhan and Lin, 2003). Since it is very expensive in storage space to keep these relations explicitly stored in spatial databases, they are usually not directly stored, but they are inferred from the geometry of the objects (Clementine et al., 1994). Inference of the topological relation between two geographic objects is achieved by a process called *overlay*, which generally defines *intersection*, *union*, and/or *difference* of two geometric objects.

Furthermore, inference of topological relations is very expensive in processing time, especially when managing complex geographic objects, such as fuzzy regions or regions with holes. Therefore, strategies to define topological relations between such objects have to be developed to define these relations in a minimum or optimal processing time.

It is the objective of this paper to present a model in which a fuzzy region is decomposed into two or more crisp regions based on fuzzy set theory, as suggested in Burrough (1996), Morris (2003), Morris and Jankowski (2000, 2004), Zhan (1997, 1998), Zhan and Lin (2003). This approach essentially uses fuzzy set theory to define alpha-cut level regions, in which each alpha-cut region is treated as a crisp region, and the set of alpha-cuts of a fuzzy region represent the fuzzy boundary of the region. Fuzzy regions represented as a set of crisp regions allows us to define the topological relation between two fuzzy regions by considering the topological relation between the crisp regions that compose the fuzzy regions. In addition, the fact that we have the fuzzy regions represented as a set of crisp regions allows us to use algorithms already developed to define the overlay of crisp regions to define the overlay of fuzzy regions (Zhan, 1997, 1998).

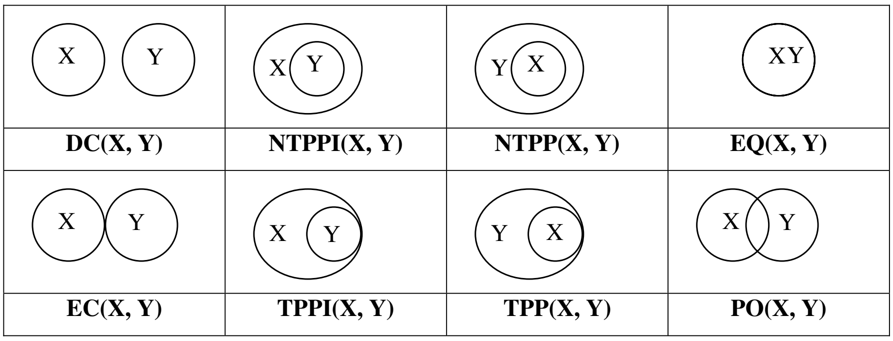
Since the number of crisp regions that compose the fuzzy regions might be high, the cost in processing time to define the topological relations between the two fuzzy regions is high as well, and it will grow as we add more alpha-cuts to geographic objects. Therefore, strategies to reduce the amount of overlays that have to be performed to define the topological relation between the regions need to be developed.

This paper is organized as follows: the present section introduces fuzzy geographic objects, and describes the importance of managing them in spatial databases and GIS. It also introduces topological relations between geographic objects, and how fuzzy regions have been modeled. Section 2 describes topological relations between crisp objects using RCC (Region-Connection Calculus) theory and point set theory. In addition, it describes how these two approaches have been extended to define the topological relations between objects with broad boundaries, and how they have been combined with fuzzy set theory to define finer distinctions between points lying within the boundaries of fuzzy regions. In Section 3, a definition of a region with multiple alpha-cuts is given based on the multi-level crisp alpha-cuts approach. In Section 4, strategies to define the topological relation between two regions with multiple alpha-cuts are presented, as well as strategies to reduce the processing time for the overlay of two such regions. In Section 5, some special cases of the model described in Section 4 are presented. Section 6 describes how the model presented in Section 4 could be implemented, and the attributes of the resulting regions could be calculated. Finally, in Section 7, conclusions and a list of future work are presented.

2. Related work

Among other approaches to model crisp objects and define their topological relations are RCC theory (Cohn et al., 1994; Randell et al., 1992) and point set theory (Egenhofer, 1991; Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991; Egenhofer and Sharma, 1991). The RCC theory is a qualitative reasoning technique for spatial data based on regions rather than on points (Beauboeuf et al., 2004). The RCC theory assumes a primitive dyadic relation $C(x, y)$, read "x connects y" (where x and y are regions), and defines 8 possible relations between these two regions in a 2-dimensional space: DC (DisConnected), EC(Externally Connected), PO (Partially Overlapping), TPP (Tangential Proper Part), NTPP (Non-Tangential Proper Part), EQ(Equal), TPPI (Tangential Proper Part Inverse), NTPPI (Non Tangential Proper Part Inverse). This is called the RCC-8 approach, and a graphical representation of the list of the 8 topological relations defined by this approach is presented in Table 1. Table 1 is based on Beauboeuf et al. (2004). RCC-5 puts together a few relations defined in the RCC-8, reducing the set of topological relations to five: DR (Distinct Regions), which is join of DC and EC, PO (Partially Overlapping), PP (Proper Part), which is a join of NTPP and TPP, PPI (Proper Part Inverse), which joins NTPPI and TPPI, and EQ (Equal). The RCC-5 assumes that the boundary lines and points at which regions meet are not themselves considered regions (Beauboeuf et al., 2004).

Table 1. The eight topological relations between two crisp regions in a two-dimensional environment as defined by the RCC-8 approach

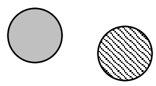
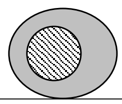
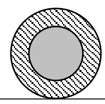
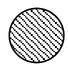
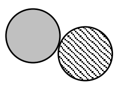
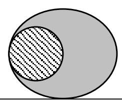
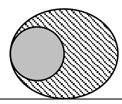
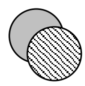


In the point set theory (Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991) objects are divided into interior ($A^{\bullet}$), boundary (A^{∂}), and exterior (A^{-}), and the relation between two objects can be defined by considering the intersection between these three parts of the two objects (Egenhofer and Herring, 1991), as can be seen in Matrix 1. The intersection of any of these three parts might be either empty or non-empty, which results in 8 realizable topological relations in a 2-dimensional space: Disjoint, Meet, Overlap, CoveredBy, Covers, Inside, Contains, and Equal. This is called the 9-intersection approach, and the 8 topological relations defined by this approach are listed in Table 2. Table 2 is based on Clementine et al. (1994). The 4-intersection approach considers the intersection of interior ($A^{\bullet}$) and boundary (A^{∂}) instead of the intersection of interior ($A^{\bullet}$), boundary (A^{∂}) and exterior (A^{-}) of crisp regions, as seen in Matrix 2. This results in exactly the same number of realizable topological relations between two regions as the 9-intersection approach.

Matrix 1. Intersection of interior, boundary, and exterior of two crisp objects A and B as defined by the 9-intersection approach

$$\begin{pmatrix} A^{\bullet} \cap B^{\bullet} & A^{\bullet} \cap B^{\partial} & A^{\bullet} \cap B^{-} \\ A^{\partial} \cap B^{\bullet} & A^{\partial} \cap B^{\partial} & A^{\partial} \cap B^{-} \\ A \cap B^{\bullet} & A \cap B^{\partial} & A \cap B^{-} \end{pmatrix}$$

Table 2. The eight topological relations between two crisp regions in a two-dimensional environment as defined by the 9-intersection approach

			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
DISJOINT	**CONTAINS**	**INSIDE**	**EQUAL**
			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
MEET	**COVERS**	**COVERED BY**	**OVERLAP**

Matrix 2. Intersection of interior and boundary of two crisp objects A and B as defined by the 4-intersections approach

$$\begin{pmatrix} A^{\bullet} \cap B^{\bullet} & A^{\bullet} \cap B^{\partial} \\ A^{\partial} \cap B^{\bullet} & A^{\partial} \cap B^{\partial} \end{pmatrix}$$

The RCC theory and the point set theory name the topological relations differently, but they mean basically the same thing. For each topological relation in RCC theory there is a similar relation in point set theory. Since they are the same, we are only going to use the nomenclature used in point set theory (Egenhofer, 1991; Egenhofer and Franzosa, 1991; Egenhofer and Herring, 1991; Egenhofer and Sharma, 1991) to describe topological relations between regions throughout this paper. However, it should be clear that instead of using a particular representation of a region, for example, the point set representations of a region, a generalized idea of a region is going to be used, independent of its representation as a raster-based or vector-based model.

A few approaches have been proposed to extend the RCC theory and the point set theory to model and define the topological relations between fuzzy regions as well. One was proposed by Cohn and Gotts (Cohn and Gotts, 1994a,b; Lehmann and Cohn, 1994), in which they developed the "egg-yolk" representation of regions with indeterminate boundaries (fuzzy regions). By considering the yolk as the core of the region and the white as the fuzzy boundary of the regions, this representation extends the five relations defined by RCC-5 approach to model the relation between any egg-egg or yolk-yolk, or any egg and yolk belonging to different eggs. This approach produces 46 possible topological relations between an egg-yolk pair.

Another approach was proposed by Clementine and Di Felice (1994), who applied the methods developed by Egenhofer and Herring (1991) to construct all relations differentiated by the 9-intersection approach to regions with a broad boundary (region with an inner and an outer boundary). This approach results in 45 topological relations between objects with broad boundaries.

The egg-yolk approach and the broad boundaries approach were written simultaneously and entirely independently. They try to solve the same problem, but they use different approaches to achieve it. The egg-yolk approach is based on previous logical formulation of spatial representation and reasoning (RCC theory), and the broad boundary approach is based on the point set theory developed in Egenhofer (1991), Egenhofer and Franzosa (1991), Egenhofer and Herring (1991), Egenhofer and Sharma (1991).

Zhan (Zhan, 1997, 1998; Zhan and Lin, 2003) argues that the two models of representing fuzzy regions described above treat the indeterminate boundary just as a thick boundary, and therefore no finer distinctions between points (pixels) lying within the thick boundary can be made in these models. In the same work, he developed a fuzzy representation of spatial regions with indeterminate boundaries, as well as methods for computing spatial relations between spatial regions with indeterminate boundaries. This representation is based on alpha-cuts, defined by fuzzy set theory.

Similar to the approach suggested by Zhan is the FOOSBALL framework developed by Morris (Morris, 2003; Morris and Jankowski, 2000, 2004). This framework suggests the use of the object oriented data model, allowing both crisp and fuzzy objects to be stored in the database. It allows GIS modelers to store the data in whatever format they want, rather than enforcing a specific model. It is the job of the system to adapt to the data, not the job of the user to force the data to adapt to the system. By allowing a varying number of alpha-cuts to represent varying degrees of membership for every feature, it allows the GIS modeler to use as much or as little fuzziness as is required. Any number of alpha-cuts may be represented, and objects with no core are supported as well.

In both approaches, alpha-cut regions are crisp regions that can be represented as either polygons (vector-based models) or image regions (raster-based models). Therefore, the overlay of two fuzzy regions can be determined using existing algorithms in either a vector-based or raster based system (Zhan, 1998).

3. Definition of the region with multiple alpha-cuts (RMAC)

A **R**egion with **M**ultiple **A**lpha-**C**uts (RMAC) is a fuzzy region represented using fuzzy set theory, that is, a RMAC A is a fuzzy region using alpha-cut level crisp regions that represent the fuzzy boundaries of the region. In general (see Morris, 2003; Morris and Jankowski, 2000, 2004; Zhan, 1997, 1998; Zhan and Lin, 2003), A can be decomposed into three major parts (considering a two-dimensional environment): core area, denoted by $A^{\bullet}$, fuzzy boundary, denoted by A^{∂}, and exterior area, denoted by A^{-}. An alpha-cut is denoted by $A_i^{\alpha}(0 \leqslant i \leqslant n)$, where n is the number of alpha-cuts of the RMAC. Each RMAC might have a different number of alpha-cuts ($n \geqslant 0$), and the definition of the alpha-cuts for each RMAC can be different. Both core and alpha-cuts might be empty regions, but not all the alpha-cuts and the core can be empty at the same time. Otherwise, it would characterize an empty RMAC.

The core area, also known as interior of a RMAC, if it exists, is the area of the RMAC whose membership in the region is 1. The core area is usually located in the center of the RMAC, and it can be surrounded by zero or more alpha-cuts. Alpha-cuts are used to construct areas of a region whose membership in the region is less than 1 and greater than 0. If an object has no alpha-cuts it is simply treated as a crisp object. A crisp region representation is simply a special case of a RMAC in which case the number of alpha-cuts is zero ($n = 0$). This is because not all geographic regions are fuzzy regions, as argued in Morris (2003). If an object has no core, it means that no part of the region associated with the object has membership value 1.

The exterior of the RMAC is the area of $\Re^2$ whose membership in the region A is 0. The exterior of a RMAC is empty only if $A = \Re^2$. For the concept of a RMAC being presented here, it is assumed that both core and alpha-cuts are crisp regions without holes and can be represented either as a polygon or an image region. Fig. 1 shows the general concept of a RMAC. It contains one core, n alpha-cuts, and one exterior. Notice that this is based on a 2-dimensional environment.

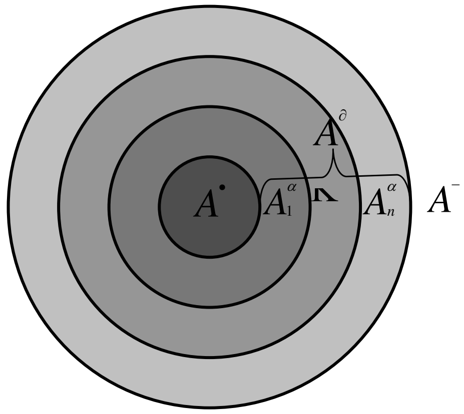


Figure 1. Region with Multiple Alpha-Cuts (RMAC)

Formal definitions

Definition 3.1 *The core of a RMAC A is the region of the RMAC for which the membership value μ is equal to 1:*

$$A^{\bullet} = \{x|\mu_A(x) = 1\}. \tag{1}$$

Definition 3.2 *An alpha-cut of a RMAC A is a region of the RMAC for which the membership value μ is greater than α, where $0 < \alpha < 1$:*

$$A_i^{\alpha} = \{x|\mu_A(x) > \alpha_i\}. \tag{2}$$

It is assumed here that $1 > \alpha_1 > \alpha_2 > \ldots > \alpha_{n-1} > \alpha_n > 0$, that is, the corresponding sets are distinct such that A_i^{α} is a proper subset of A_{i+1}^{α}.

Definition 3.3 *The boundary of a RMAC A is the region of the RMAC for which the membership value μ is in $]0,1[$. It corresponds to the union of all alpha-cuts of A:*

$$A^{\partial} = \{x|0 < \mu_A(x) < 1\} \tag{3}$$

$$A^{\partial} = \bigcup_{i=1}^{n} A_i^{\alpha}\,. \tag{4}$$

Definition 3.4 *The RMAC A can be defined as the union of its core (if it exists), and all its alpha-cuts (if there is one or more).*

$$A = A^{\bullet} \cup \bigcup_{i=1}^{n} A_i^{\alpha} \tag{5}$$

$$A = A^{\bullet} \cup A^{\partial}\,. \tag{6}$$

DEFINITION 3.5 *The exterior of a RMAC A is the area in $\Re^2$ whose membership μ in A is equal to zero. It can also be defined as $\Re^2$ less the region covered by A:*

$$A^- = \{x|\mu(x) = 0\} \tag{7}$$
$$A^- = R^2 - A\,. \tag{8}$$

Notice that the exterior of A (A^-) is empty only if $A = \Re^2$.

Furthermore, due to assumptions here adopted we know that the following properties have to hold for a RMAC (Zhan, 1997):

Property 1: Both core and alpha-cuts are crisp regions without holes;

Property 2: Core, boundary (union of the alpha-cuts), and exterior are nested.

That is:

$$\mathbf{Core} = 1 > \alpha_1 > \alpha_2 > \ldots > \alpha_{\mathrm{n}-1} > \alpha_{\mathrm{n}} > 0 = \mathbf{Exterior} \tag{9}$$
$$A^{\bullet} \supseteq A_1^{\alpha} \supseteq A_2^{\alpha} \supseteq \ldots \supseteq A_{n-1}^{\alpha} \supseteq A_n^{\alpha} \supseteq A^- \tag{10}$$

4. Defining the topological relation between two RMACs

Once the concept of a RMAC has been presented and considering the topological relations in the point set theory, the topological relation between two RMACs can be defined based on the topological relations between the crisp regions that compose the two RMACs. That is, the topological relations among cores and alpha-cuts of the two RMACs can be used to define the topological relation of the two RMACs. This can be implemented using existing algorithms used to define the topological relations between crisp objects. However, an implementation of this should not just calculate the topological relations between all the crisp regions that compose the fuzzy regions, because this would be very expensive in processing time. Instead, it should make use of some strategies to reduce the processing time of this calculation. These strategies are based on constraints imposed by the definition of a RMAC and by other redundancies imposed by the model.

Based on the constraints imposed by the definition of a RMAC and its properties, this section presents strategies to define the topological relation between two RMACs, each with an arbitrary number of alpha-cuts. Definition of the topological relation between two RMACs is similar to the definition of the topological relation between crisp regions with holes (Egenhofer et al., 1994). Even though a RMAC might not have a core or some of its alpha-cuts might be empty, in this section we are going to assume that none are empty. The case in which the core or some of the alpha-cuts are empty is discussed in the next section, where some special cases of RMACs are described.

Let A and B be two RMACs with m and n alpha-cuts, respectively. Based on the representation of RMACs given in Section 3, the crisp regions that compose these two RMACs consist of the set $S = \{A^\bullet, A_1^\alpha, \ldots, A_m^\alpha, B^\bullet, B_1^\alpha, \ldots, B_n^\alpha\}$ with

1. The core of region A, denoted by $A^\bullet$;
2. The core of region B, denoted by $B^\bullet$;
3. Each alpha-cut of A, denoted by $A_1^\alpha, \ldots, A_m^\alpha$; and
4. Each alpha-cut of B, denoted by $B_1^\alpha, \ldots, B_n^\alpha$.

Since this set contains $(m + n + 2)$ crisp regions, the total number r of topological relations between the crisp regions that compose the two RMACs is determined by:

$$r = (m + n + 2)^2 \, .$$

4.1. Eliminating redundant relations

For this representation, several elements of the set $(m + n + 2)^2$ of topological relations are redundant (Egenhofer and Sharma, 1991). The relations that can be immediately eliminated are those that are enforced by the node consistency and the arc consistency rules (Egenhofer et al., 1994), as described in Mackworth (1997):

Redundancy 1: *The relation between a crisp region A and itself must be equal (node consistency).*

This rule reduces the number of topological relations to be defined between the crisp regions that compose the RMACs by $(m + n + 2)$.

Redundancy 2: *The relation between two crisp regions A and B must be the converse relation between B and A (arc consistency).*

This rule reduces the number of topological relations to be defined between the crisp regions that compose the RMACs by half.

By eliminating the redundant relations defined by the node consistency and the arc consistency rules the new total number of topological relations between the crisp objects r' is given by:

$$r' = \frac{(m + n + 2)^2 - (m + n + 2)}{2} \, .$$

Furthermore, there are certain constraints about the topological relations that must hold between each core and alpha-cuts, and between alpha-cuts of the same RMAC because of the definition of a RMAC and RMAC redundancy assumptions stated earlier:

Redundancy 3: *The topological relation between the core of a RMAC ($A^\bullet$) and the alpha-cuts of the same RMAC (A_i^α) must always be INSIDE, except for A_1^α, in which case the relation is always COVERED BY.*

This means that $A^\bullet$ is INSIDE A_i^α, iff $i > 1$, and $A^\bullet$ is COVERED BY A_i^α iff $i = 1$. This reduces the number of topological relations by $(m + n)$.

Redundancy 4: The topological relation between any pair of alpha-cuts that belong to the same RMAC must be either INSIDE, COVERED BY, CONTAINS or COVERS.

Even though this rule lists four out of eight possible topological relations, the topological relation that is going to hold can be determined in a very straightforward way. The definition of the topological relations can be made by comparing the indexes of the two alpha-cuts:

A_i^α is INSIDE A_j^α, if $i < (j - 1)$;
A_i^α is COVERED BY A_j^α, if $i = (j - 1)$;
A_i^α is CONTAINS A_j^α, if $i > (j + 1)$;
A_i^α is COVERS A_j^α, if $i = (j + 1)$.

As initially stated, the topological relation of each alpha-cut of $A(1, \dots, i, \dots, m)$ would be defined to be related to all other alpha-cuts of $A(1, \dots, i, \dots, m)$, which results in the definition of m^2 topological relations. The same happens with the topological relations among B's alpha-cuts. There are n^2 of them. Redundancy 4 reduces the number of topological relations by $(m^2 + n^2)$ because we can infer those relations from the definition of a RMAC and respective assumptions.

After eliminating the topological relations imposed by the definition and constraints of the RMACs the new number of topological relations is given by:

$$r'' = \frac{(m + n + 2)^2 - (m + n + 2) - (m + n) - (m^2 + n^2)}{2} \,.$$

Simplifying this formula we get:

$$r'' = mn + m + n + 1 \,.$$

The topological relations between two RMACs, A and B, with a distinct number of alpha-cuts, can therefore be defined by the topological relations between:

1. A's core and B's core (1 relation);
2. A's core and each of B's alpha-cuts (n relations);
3. B's core and each of A's alpha-cuts (m relations);
4. Each of A's alpha-cuts and each of B's alpha-cuts (mn relations).

This applies to *any* configuration independent of the particular values the topological relations may have. These are called *explicit* relations as opposed to implicit relations, which are those relations that can be inferred from the definition and properties of a RMAC. Examples of implicit topological relations are:

- The topological relation between the core of a RMAC and the alpha-cuts of the same RMAC is always INSIDE, except for A_1^α, in which case the relation is always COVERED BY;
- The topological relation between any pair of alpha-cuts that belong to the same RMAC is either INSIDE, COVERED BY, CONTAINS or COVERS, which can be defined in a straightforward way.

4.2. Strategies to minimize the number of relations to be defined

There are two main assumptions on which the strategies described in this section are based:

- INSIDE and CONTAINS are inverse relations, that is, whenever a crisp region A is INSIDE a crisp region B, that means also that B CONTAINS A.
- COVERED BY and COVERS are inverse relations, that is, whenever a crisp region A is COVERED BY a crisp region B that means also that B COVERS A.

Since each RMAC is composed of one or more crisp regions and the constraints of a RMAC can be used to define the relations between these regions, we can ignore the exteriors of the regions and work only with the interiors and the boundaries. In the point set theory, this is defined by the 4-intersection approach.

Depending on the particular values, shape and size a RMAC may have, further strategies might be used to reduce the number of topological relations between two RMACs. These strategies are listed below, and they assume that at some point the topological relation between one of the crisp regions of RMAC A and another crisp region of RMAC B is known or has been calculated.

Strategy 1: *If DISJOINT is the relation between A_i^α and B_j^α, then DISJOINT is the relation between each $A_{i'}^\alpha$ $(i' \leqslant i)$ and $A^\bullet$ and each $B_{j'}^\alpha$ $(j' \leqslant j)$ and $B^\bullet$.*

An example of how this strategy applies can be seen in Fig. 2. Suppose we are defining the topological relation between A_3^α and B_1^α. Once we have defined that DISJOINT is the topological relation between these two crisp regions, we can infer that A_3^α, A_2^α, A_1^α, and $A^\bullet$ are DISJOINT from B_1^α and $B^\bullet$. This is because the alpha-cuts and core are nested, and the indexes of the last ones are smaller that the indexes of the two crisp regions for which we originally defined the topological relation.

Strategy 2: *If MEETS is the relation between A_i^α and B_j^α, then DISJOINT is the relation between each $A_{i'}^\alpha$ $(i' < i)$ and $A^\bullet$ and each $B_{j'}^\alpha$ $(j' < j)$ and $B^\bullet$.*

The application of this strategy can be seen in Fig. 2 by considering the definition of the topological relation between A_2^α and B_2^α. Once we have defined MEET as being the topological relation between these two crisp regions, we can infer that A_1^α and $A^\bullet$ are DISJOINT from B_1^α and $B^\bullet$.

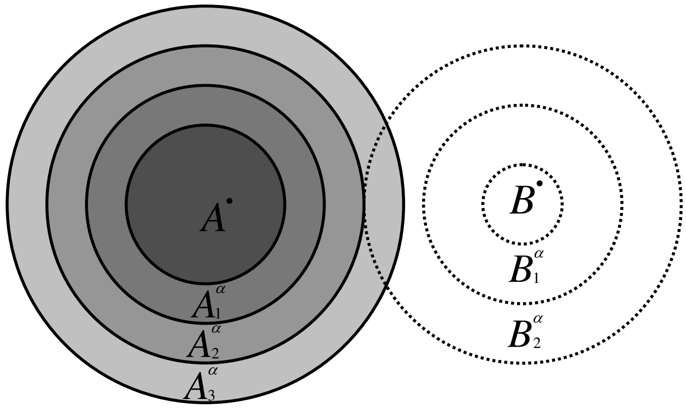


Figure 2. DISJOINT and MEETS reduction strategies (strategies 1 and 2, respectively

Strategy 3: *If INSIDE is the relation between A_i^α and B_j^α, then INSIDE is the relation of each $A_{i'}^\alpha$ $(i' \leqslant i)$ and $A^\bullet$ and each $B_{j'}^\alpha$ $(j' \geqslant j)$.*

Note that this also means that CONTAINS is the relation between each $B_{j'}^\alpha$ $(j' \geqslant j)$ and each $A_{i'}^\alpha$ $(i' \leqslant i)$ and $A^\bullet$.

Strategy 4: *If CONTAINS is the relation between A_i^α and B_j^α, then CONTAINS is the relation between each $A_{i'}^\alpha$ $(i' \geqslant i)$ and each $B_{j'}^\alpha$ $(j' \leqslant j)$ and $B^\bullet$.*

Note that this also means that INSIDE is the relation between each $B_{j'}^\alpha$ $(j' \leqslant j)$ and $B^\bullet$ and each $A_{i'}^\alpha$ $(i' \geqslant i)$.

Strategy 5: *If EQUAL is the relation between A_i^α and B_j^α, then INSIDE is the relation between each $A_{i'}^\alpha$ $(i' < i)$ and $A^\bullet$ and each $B_{j'}^\alpha$ $(j' \geqslant j)$ and between each $B_{j'}^\alpha$ $(j' < j)$ and $B^\bullet$ and each $A_{i'}^\alpha$ $(i' \geqslant i)$.*

An example of the application of this strategy can be seen in Fig. 3 by analyzing the crisp regions A_2^α and B_2^α. Once we have defined that these two crisp regions are EQUAL, then we can infer that A_1^α and $A^\bullet$ are INSIDE B_2^α and B_1^α and $B^\bullet$ are INSIDE A_2^α.

Strategy 6: *If COVERS is the relation between A_i^α and B_j^α, then CONTAINS is the relation between each $A_{i'}^\alpha$ $(i' \geqslant i)$ each $B_{j'}^\alpha$ $(j' < j)$ and $B^\bullet$.*

The application of this strategy can be seen in Fig. 4 by analyzing A_3^α and B_2^α. Once we have defined that A_3^α COVERS B_2^α, then we can infer that A_3^α CONTAINS both B_1^α and $B^\bullet$.

Strategy 7: *If COVERED BY is the relation between A_i^α and B_j^α, then INSIDE is the relation between each $A_{i'}^\alpha$ $(i' < i)$ and $A^\bullet$ and each $B_{j'}^\alpha$ $(j' \geqslant j)$.*

The application of this strategy can be seen in the same example used from strategy 6, in Fig. 4, by analyzing B_2^α and A_3^α. Once we have defined that B_2^α is COVERED BY A_3^α, we can infer that B_1^α and $B^\bullet$ are INSIDE A_3^α.

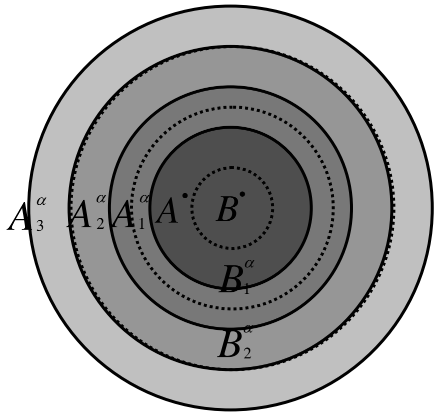


Figure 3. EQUAL reduction strategy (strategy 5)

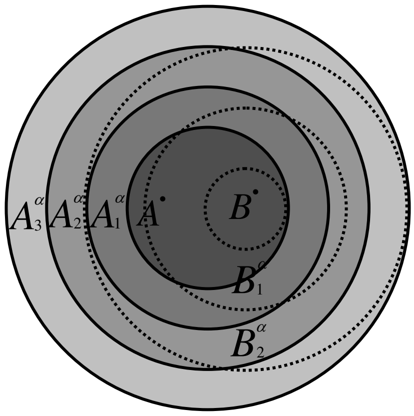


Figure 4. COVERS and COVERED BY reduction strategy

Strategy 8: *If OVERLAP is the relation between A_i^α and B_j^α, then more tests have to be done to infer further relations.*

Strategy 8 does not reduce the number of topological relations between crisp objects that have to be defined to define the topological relation between two RMACs. Since we do not know the shape and size of the core and alpha-cuts of the two RMACs, it is difficult to define a strategy to infer topological relations without more calculations.

The benefits in terms of processing time when applying these strategies will vary depending on the particular configuration of the RMACs. Either none or nearly all explicit relations may be found using these strategies. One example in which no further topological relations can be inferred without more calculations is the case of the OVERLAP relation, as shown in Fig. 5, by considering A_3^α and B_2^α. Even though we know that A_3^α and B_2^α OVERLAP, no further topological relations can be inferred.

On the other hand, there are calculations in which almost all other relations can be inferred once we know one or two of them. One such case is shown in Fig. 6. Once we know that B_1^α is INSIDE A_2^α and DISJOINT from A_1^α, all other topological relations can be inferred from there, and no other overlay operation needs to be calculated.

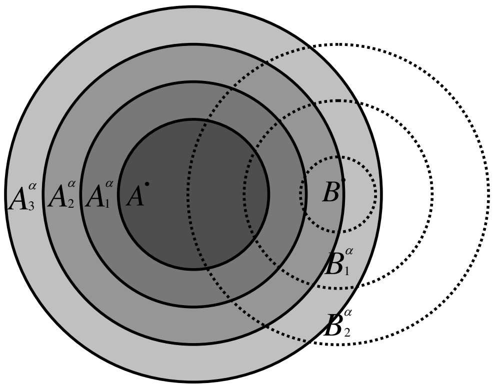


Figure 5. No further reduction in the number of topological relations is possible

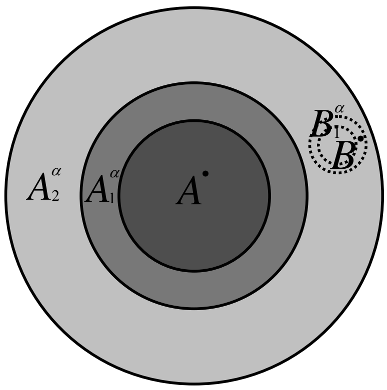


Figure 6. No further topological relations need to be defined. All of them can be inferred

5. Special cases

5.1. RMACs with no core or missing alpha-cuts

Since the concept of an RMAC allows regions with no core or missing alpha-cuts, a small change would be needed on the strategies described in Section 4 to accommodate this feature. Before defining any topological relation between any two crisp regions, they should be tested to see if they are empty or not. If one or both are empty, then their topological relation definition can just be ignored because their intersection is going to be empty. This would reduce the initial number of crisp topological relations definitions to

$$r''' = m'n' .$$

Here m' is the number of non empty crisp regions in RMAC A including the core, if not empty; and n' is the number of non-empty crisp regions in RMAC B, including the core, if not empty. The strategies described in Section 4 would be exactly the same, except that the core could be ignored since it would be used simply as an alpha-cut, and would have an index as well, which would typically be 0.

5.2. RMAC with more than one core or holes

Based on the actual concept of a RMAC, a fuzzy region with a disconnected core has to be divided into two RMACs. Also, only crisp regions with no holes are considered. A tentative idea to extend this model to also consider disconnected core and core and alpha-cuts with holes would be to combine it with the approach described in Egenhofer et al. (1994), which describes an approach to define the topological relation between crisp regions with holes. More work has to be done to combine these two approaches.

6. Implementation and definition of the resulting regions

6.1. Implementation

To implement the strategies described in Section 4 for determining the topological relation between two RMACs, existing algorithms to determine the overlay of crisp regions can be used, either in a vector-based or raster based system. This is because both the core and alpha-cuts of a RMAC are crisp regions. There are several such algorithms. One is proposed in Clementine et al. (1994). In this work, an algorithm to determine the smallest subset of the nine intersections that have to be evaluated to define the topological relation between two crisp regions is presented, as well as a greedy algorithm that constructs a decision tree that partitions the search space at each node, thereby progressively excluding all other relations. This algorithm improves the processing time to define the topological relation between two crisp regions. In addition, this algorithm is based on a well defined formalism, it is customizable, and it can take into account statistical information about data.

6.2. Determination of the attributes of the resulting regions

The strategies just described are purely qualitative, which means that they do not consider the size and the characteristics of the crisp regions that compose a RMAC. In the case of defining the characteristics of each of the resulting regions, several techniques already defined for crisp regions could be used (DeCaluwe et al., 1997; Schneider, 1999; Zhan, 1997, 1998; Zhan and Lin, 2003). Zhan and Lin (2003) define the attributes of the resulting regions using four sets of rules: enumeration rules, dominance rules, contributory rules, and interaction rules.

7. Conclusions and future work

Regions with fuzzy boundaries are complex structures to model and manipulate. We do not claim that this is a complete solution to the problem of modeling these kind of regions and defining the topological relations between two such regions. We also do not claim that the model and strategies described here optimize either the space needed to store regions with fuzzy boundaries, or the processing time to define their topological relations, but we believe that we have made considerable progress toward an optimal solution.

Throughout this paper a model based on fuzzy sets was presented to implement and manipulate fuzzy regions, which were referred to as Regions with Multiple Alpha-Cuts (RMACs). Such a model is needed because fuzzy regions are, as described above, complex structures to manage in spatial databases and GIS. One of the most important kinds of relations between geographic objects, topological relations, is expensive to be kept explicitly stored in spatial databases. They are usually inferred from the geographic object's geometry, which is expensive in processing time.

After the definition of a RMAC was presented to model fuzzy regions using fuzzy sets, strategies were described to define the topological relation between two RMACs. In this model the topological relation between two RMACs can be defined by defining the topological relations between the crisp regions that compose them. The strategies described intend to minimize the number of topological relations between crisp regions that have to be defined to define the topological relation between two RMACs. This is achieved by the inference of some, or almost all of the topological relations between the crisp objects that compose the fuzzy objects, based on the constraints imposed by the definition and properties of a RMAC.

One advantage of this approach is that it is based on solid formalism already defined and used by spatial databases and GIS to model regions with well-defined boundaries. An implementation of this approach would make use of the models for regions with crisp objects and combine them in such a way to implement the constraints and properties imposed by the definition of a RMAC.

Another advantage is that since RMACs are modeled as sets of crisp regions, algorithms already defined to manipulate this kind of objects can be used to manipulate them and define the topological relations between them. Again, these algorithms need to be used in such a way to implement the constraints and properties imposed by the definition and properties of RMACs.

In summary, to implement the approach described in this paper, there is no need to develop a completely new spatial database and or GIS. Several, commercially available GIS would allow the implementation of this approach.

There are several ways in which the approach presented in this paper might be extended. One consists of implementing this approach in a commercially available spatial database and/or a GIS and using it in a real-world environment. This would be needed to evaluate the value of this approach for the GIS community.

A second extension consists of eliminating some of the topological relations among crisp objects. Since not all applications need to distinguish between topological relations such as *contains* and *covers*, *inside* and *covered by*, and *disjoint* and *meet*, we could reduce the number of crisp topological relations to be defined to define the topological relations between two fuzzy regions. To do that, our model should not consider the tangential boundary lines and tangential points at which regions meet as *regions*.

Another extension consists of developing an object-oriented model to define the classes needed to implement the constraints imposed by the definition and properties of RMACs. The methods defined in this model would implement the strategies to minimize the number of overlays of crisp regions to define the topological relation between two RMACs, as described in Section 4 of this paper.

Finally, this approach can be extended by combining it with the approach presented in Egenhofer et al. (1994) in which regions with holes are allowed, and therefore even more complex objects can be stored and manipulated by spatial databases and GIS. Indeed, regions with disconnected cores can be modeled.

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