Control and Cybernetics

vol. **34** (2005) No. 4

Some results on the ageing class

by

Leszek Knopik

Faculty of Mechanical Engineering, University of Technology and Agriculture
Bydgoszcz, ul.Kaliskiego 7, Poland

Abstract: In this paper, we investigate an ageing class of life distributions. We show that the family of life distributions is closed under formation of parallel and series systems. This class is situated between IFR and NBUE.

Keywords: reliability, ageing class, DMRL, IFR, IFRA, NBUE, parallel and series systems.

1. Introduction

Let T be a nonnegative, absolutely continuous random variable (life time) with a distribution function $F(x) = P\{T \leq x\}$, a reliability (survival) function $R(x) = 1 - F(x)$, a density function $f(x)$, a failure rate function $\lambda(x) = f(x)/R(x)$ and a finite mean value ET.

Reliability theory (see Barlow and Proschan, 1981; Bryson and Siddiqui, 1969; Korczak, 2001; Marschall and Proschan, 1972) deals with the following ageing classes of distributions:

IFR - nondecreasing failure rate $\lambda(x)$ for $x \in S = \{t : R(t) > 0\}$,

NBU - new better than used, if $R(x+y) \leq R(x)R(y)$ for $x, y \geq 0$,

DMRL - decreasing mean residual life, if

$$\frac{\int_x^\infty R(t)dt}{R(x)}$$

is nonincreasing for $x \in S$,

IFRA - increasing failure rate average, if $\ln R(x)/x$ is increasing for $x \in S - \{0\}$,

NBUE - new better than used in expectation, if $\int_0^x R(t)dt \geq F(x)ET$ for $x \geq 0$.

In particular, it is well known that

$$\text{IFR} \subset \text{NBU} \subset \text{NBUE}$$

and

$$\text{IFR} \subset \text{DMRL} \subset \text{NBUE}\,.$$

NBUE is the weakest of all these classes. In this paper, we introduce a new ageing class MTFR such that

$$\text{IFR} \subset \text{MTFR} \subset \text{NBUE}\,.$$

We prove that the MTFR class is preserved under formation of any parallel system and series system (with identically distributed components). In the respective theorems we assume that the life times of the components are independent and absolutely continuous random variables.

The class MTFR is the outcome of the formation of criteria for existing maximum profit per a unit of time in semi-Markov systems (Knopik, 2003).

2. Definitions and basic properties

If we consider an age replacement policy as the one in which a unit is replaced x time units after installation or at failure, whichever occurs first, then the expected value for the first time to an in-service is (see Barlow, Proschan, 1980):

$$M(x) = \frac{\int_0^x R(t)dt}{F(x)} \quad \text{for} \quad x \in \{t : F(t) > 0\}\,.$$

The case when $M(x)$ is monotonic was considered by Barlow, Campo (1975), Marschall, Proschan (1972) and Klefsjö (1982).

Definition 2.1 *The random variable T belongs to the MTFR class (mean time to failure with replacement) if a function $M(x)$ is non-increasing for $x \in \{t : F(t) > 0\}$.*

If T is absolutely continuous variable then $T \in MTFR$ if only if

$$r(x) = f(x)\int_0^x R(t)dt - F(x)R(x) \geq 0 \quad \textit{for} \quad x \geq 0 \tag{1}$$

or

$$\lambda(x) \geq \frac{1}{M(x)} \quad \textit{for} \quad x \in \{t : F(t) > 0\}\,.$$

The properties of the function $r(x)$ were described by Klimaszewski and Knopik (2003) and Knopik (2003). These properties were used for characterization of exponential distributions.

It has been shown in Barlow (1965), Klefsjö (1982) that

$$\text{IFR} \subset \text{MTFR} \subset \text{NBUE}.$$

It is also known that if T is an absolutely continuous variable, then (Barlow, 1975)

$$\text{IFRA} \subset \text{MTFR}.$$

3. Preservation of life distribution classes under reliability operations

THEOREM 3.1 *If $X_1, X_2, \ldots, X_n$ are independent nonnegative absolutely continuous random variables and $X_i \in$ MTFR for $i = 1, 2, \ldots, n$, then $Y_n = \max(X_1, X_2, \ldots, X_n) \in$ MTFR.*

Proof. (for $n = 2$) Let $X_1, X_2 \in$ MTFR be independent random variables with distribution functions $F_1(x)$ and $F_2(x)$.

Let us define $Y_2 = \max(X_1, X_2)$,

$$r_Y(x) = f_Y(x) \int_0^x R_Y(t)dt - F_Y(x)R_Y(x), \tag{2}$$

where

$$F_Y(x) = F_1(x)F_2(x), \quad R_Y(x) = 1 - F_Y(x), \tag{3}$$

$$f_Y(x) = f_1(x)F_2(x) + F_1(x)f_2(x), \tag{4}$$

$$\int_0^x R_Y(t)dt = \int_0^x R_1(t)dt + \int_0^x R_2(t)dt - \int_0^x R_1(t)R_2(t)dt. \tag{5}$$

From (2), (3), (4) and (5) we obtain

$$\begin{aligned} r_Y(x) &= [f_1(x) \int_0^x R_1(t)dt - F_1(t)R_2(t)]F_2(x) + [f_2(x) \int_0^x R_2(t)dt \\ &- F_2(t)R_2(t)]F_1(x) + f_1(x)F_2(x)[\int_0^x R_1(t)dt - \int_0^x R_1(t)R_2(t)dt] \\ &+ F_1(x)f_2(x)[\int_0^x R_2(t)dt - \int_0^x R_1(t)R_2(t)dt] + F_1(x)F_2(x)R_1(x)R_2(x). \end{aligned} \tag{6}$$

It is easy to verify that

$$\int_0^x R_i(t)dt \geq \int_0^x R_1(t)R_2(t)dt \quad \text{for} \quad i = 1, 2\,. \tag{7}$$

From the definition of MTFR class and (7), we obtain

$$r_Y(x) \geq 0 \quad \text{for} \quad x \geq 0 \quad \text{and} \quad Y_2 \in \text{MTFR}\,.$$

Using induction, this result can be generalized to n independent variables from the class MTFR. ■

LEMMA 3.1 *If $X_1, X_2, \ldots, X_n \in MTFR$ are independent absolutely continuous random variables, identically distributed with a common distribution function $F(x)$, then*

$$\int_0^x R^n(t)dt \geq \frac{\int_0^x R(t)dt}{n}\,[1 + R(x) + R^2(x) + ... + R^{n-1}(x)]. \tag{8}$$

Proof. It is easy to check that for $n = 1$ the inequality (8) is true. If $X_i \in$ MTFR, then

$$f(x)R^{n-1}(x)\int_0^x R(t)dt \geq F(x)R^n(x)) \tag{9}$$

for $n \geq 1$, $x \geq 0$.

Integrating the left side of the equation (9) by parts we obtain

$$\int_0^x f(t)R^{n-1}(t)dt \int_0^x R(t)dt = -\frac{1}{n}R^n(x)\int_0^x R(t)dt + \frac{1}{n}\int_0^x R^{n+1}(t)dt\,. \tag{10}$$

Using (9) and (10) we get

$$\frac{1}{n}\int_0^x R^{n+1}(t)dt - \frac{1}{n}R^n(x)\int_0^x R(t)dt \geq \int_0^x F(t)R^n(t)dt$$

and

$$\int_0^x R^{n+1}(t)dt \geq \frac{n}{n+1}\int_0^x R^n(t)dt + \frac{1}{n+1}R^n(x)\int_0^x R(t)dt.$$

From (8) we have

$$\int_0^x R^{n+1}(t)dt \geq \frac{1}{n+1}\int_0^x R(t)dt[1+R(x)+R^2(x)+...+R^{n-1}(x)]$$
$$+\frac{1}{n+1}R^n(x)\int_0^x R(t)dt\,.$$

This results in

$$\int_0^x R^{n+1}(t)t \geq \frac{1}{n+1}\int_0^x R(t)dt\,[1+R(x)+R^2(x)+...+R^n(x)]\,. \qquad (11)$$

From (11) and by induction, we obtain the thesis of the lemma for $n+1$. ∎

THEOREM 3.2 *If $X_1, X_2, \ldots, X_n \in MTFR$ are independent nonnegative absolutely continuous and identically distributed random variables, then $Z_n = \min(X_1, X_2, \ldots, X_n \in MTFR$.*

Proof. Let $F(x)$ be a common distribution function of X_i, $F_n(x)$ a distribution function of Z_n and $f_n(x)$ a density function of Z_n.

Let

$$r_n(x) = f_n(x)\int_0^x R_n(t)dt - F_n(x)R_n(x), \quad \text{where} \quad R_n(x) = 1 - F_n(x).$$

It is well known that

$$F_n(x) = 1 - R^n(x) \quad \text{and} \quad f_n(x) = nf(x)R^{n-1}(x),$$

where $R_n(x) = 1 - F_n(x)$ and $f(x)$ is a density function of X_i.
Hence

$$r_n(x) = nf(x)R^{n-1}(x)\int_0^x R_n(t)dt - R^n(x)\,[1 - R^n(x)]\,.$$

From Lemma 3.1, we have

$$r_n(x) \geq R^{n-1}(x)\{f(x)\int_0^x R_n(t)dt\,[1+R(x)+R^2(x)+\ldots+R^{n-1}(x)]$$
$$-R(x)\,[1-R^n(x)]\}$$
$$= \frac{R^{n-1}(x)[1-R^n(x)]}{F(x)}\,\{f(x)\int_0^x R(t)dt - R(x)F(x)\} \geq 0 \quad \text{for} \quad x \in \{t : F(t) > 0\}.$$

Since $X_i \in$ MTFR, the above results shows that $Z_n \in$ MTFR. ∎

Acknowledgements

The author is very grateful to the referee for the valuable comments and constructive suggestions, which have greatly improved the presentation of this paper.

References

Barlow, R.E. and Campo, R. (1975) Total time on test processes and applications to failure data analysis. In: R.E. Barlow, J. Fussel and N.P. Singpurwalla, eds., *Reliability and Fault Tree Analysis*. SIAM, Philadelphia, 451-481.

Barlow, R.E. and Proschan, F. (1965) *Mathematical Theory of Reliability*. Wiley, New York.

Barlow, R.E. and Proschan, F. (1981) *Statistical Theory of Reliability and Life Testing. Probabilistic Models*. Silver Spring, New York.

Bryson, M.C. and Siddiqui, M.N. (1969) Some criteria for ageing. *J. Amer. Statist. Assoc.* **64**, 1472-1483.

Klefsjö, B. (1982) On ageing properties and total time on test transforms. *Scandinavian Journal of Statistics. Theory and Applications* **9** (1), 37-41.

Klimaszewski, S. and Knopik, L. (2003) Stability of a certain characterization of the exponential distribution (in Polish). *Exploitation Problems of Machines* **38**, 2(134), 151-161.

Knopik, L.(2003) Maximization methods of profit in diagnostic system of vehicle exploitation with usage of semi-Markov processes (in Polish). *Exploitation Problems of Machines* **39**, 1(133), 153-168.

Korczak, E. (2001) Upper bund on the integrated tail of reliability function, with applications. *Control and Cybernetics* **30** (3), 356-366.

Marschall, A.W. and Proschan, F. (1972) Classes of distributions applicable in replacement with renewal theory implications. In: L. LeCam et al., eds., *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistisc and Probability*. Volume I, University of California Press, Berkeley, 395-415.