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Superlinear elliptic systems with distributed and boundary controls

by

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Abstract: The paper investigates the nonlinear partial differential equations of the superlinear elliptic type with the Dirichlet boundary data. Some sufficient conditions, under which the solutions of considered equations depend continuously on distributed and boundary controls, are proved. The proofs of the main results are based on variational methods.

Keywords: boundary value problems, continuous dependence, stability, variational methods.

1. Introduction

In the paper we investigate the elliptic systems of nonlinear partial differential equations with variable distributed parameters (controls) and variable boundary conditions (controls). The systems considered are of the form

$$-\Delta z(x) = G_z(x, z(x), u(x)) \tag{1}$$

with the boundary condition

$$z(x) = v(x) \text{ on } \partial\Omega \tag{2}$$

where $x \in \Omega \subset \mathbb{R}^n$, $n \geq 2$, Ω is a bounded domain with Lipschitzian boundary $\partial\Omega$, $z(\cdot) \in H^1(\Omega, \mathbb{R}^N)$. We shall assume that the distributed control $u(\cdot)$ varies in the space $L^\infty(\Omega, \mathbb{R}^m)$ and the boundary control $v(\cdot)$ belongs to the space of traces $H^{1/2}(\partial\Omega, \mathbb{R}^N)$, N, p, $m \geq 1$ (for details see Section 2).

In the paper the terms: distributed parameters and distributed controls as well as variable boundary data (conditions), boundary controls are used interchangeably.

The main result of this paper is stated in Theorem 4.1 (Section 4). Under some suitable assumption we show that for an arbitrary pair of controls (u, v) there exists a solution $z_{u,v}$ to the system (1)-(2), which is stable with respect to distributed and boundary controls. By stability we understand here continuous dependence of solutions on variable controls. More precisely, we prove that $z_{u,v} \to z_{u_0,v_0}$ in $H^1\left(\Omega, \mathbb{R}^N\right)$ provided that u tends to u_0 in $L^\infty\left(\Omega, \mathbb{R}^m\right)$ and v tends to v_0 in $H^{1/2}\left(\partial\Omega, \mathbb{R}^N\right)$. Furthermore, by applying the above stability results we show the existence of optimal solution to the control problem described by (1)-(2) with an integral performance index (see Theorems 5.1 and 5.2). Similar results to Theorems 5.1 and 5.2 were proved for optimal control systems of ordinary differential equations by Macki and Strauss (1982, Chapter IV).

It is easy to notice that system (1) represents the Euler-Lagrange equation for the following integral functional (the functional of action)

$$F(z) = \int_\Omega \left[\frac{1}{2}\left|\nabla z(x)\right|^2 - G(x, z(x), u(x))\right] dx \tag{3}$$

where $z(\cdot) \in H^1\left(\Omega, \mathbb{R}^N\right)$, $z(x) = v(x)$ for $x \in \partial\Omega$ a.e., $v(\cdot) \in H^{1/2}\left(\partial\Omega, \mathbb{R}^N\right)$, $u(\cdot) \in L^\infty\left(\Omega, \mathbb{R}^m\right)$.

On the function G we shall impose, besides some technical assumptions, the following condition

$$a < pG(x, z, u) \leq \langle G_z(x, z, u), z\rangle \tag{4}$$

for some $a > 0$, $p > 2$ and $|z|$ sufficiently large, which guarantees that problem (1)-(2) is referred to as a superlinear boundary value problem, where the functional of action is unbounded from above and below.

Generally, in the theory of boundary value problems and its applications we consider, first of all, the problem of the existence of a solution and then the questions of stability, uniqueness, smoothness, etc.

R. Courant and D. Hilbert write in their monograph : "A mathematical problem which is to correspond to physical reality should satisfy the following basic requirements: (1) The solution must exist. (2) The solution should be uniquely determined. (3) The solution should depend continuously on the data (requirement of stability)" and, next, they write: "The third requirement, particularly incisive, is necessary if the mathematical formulation is to describe observable natural phenomena. Data in nature cannot possibly be conceived as rigidly fixed: the mere process of measuring them involves small errors..." (see Courant, Hilbert, 1962, Vol II Ch.III § 6.2).

A wide presentation of the methods and results related to the existence theory of variational and boundary value problems can be found, in particular, in monographs: *Partial Differential Equations* by Evans (1998), *Problèmes de Dirichlet Variationnels Non-Linéaires* by Mawhin (1987), *Critical Point Theory and Hamiltonian Systems* by Mawhin and Willem (1989), *Minimax Methods in Critical Point Theory with Applications to Differential Equations* by Rabinowitz (1986), *Variational Methods* by Struwe (1990), *Minimax Theorems* by Willem (1996). To obtain the existence result we apply the Mountain Pass Theorem presented in the above monographs.

As far as we know the question of continuous dependence of solutions on distributed and boundary controls for nonlinear partial differential equation of elliptic type has not been investigated up to now. However, in the 1970s some papers were published in which authors dealt with Dirichlet problem for scalar ordinary differential equations with two-point boundary value conditions. For example in Ingram (1972), Klaasen (1970), Lepin, Ponomariev (1973), Sedziwy (1971) some stability results are proved by means of direct methods. The question of stability of vector systems of ordinary differential equations was investigated in Walczak (1995), where the proofs of the main results were obtained by variational methods.

The first result concerning the question of continuous dependence of solutions of the linear partial differential equation of elliptic type with the variable Dirichlet boundary data and parameters was published in Oleĭnik (1952). In this work the linear partial differential systems are defined in the classical spaces. Similar results for scalar, but still linear partial differential equation with the Dirichlet boundary conditions defined in some Sobolev spaces are proved in Kok, Penning (1980/81). Stability results for nonlinear partial differential equation are presented in Walczak (1998), where the author, applying variational methods, considers coercive functional of action.

2. Formulation of the problem and basic assumptions

By $H^1\left(\Omega, \mathbb{R}^N\right)$ we shall denote the Sobolev space of functions $z = z\left(x\right)$ defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, such that $z(\cdot) \in L^2(\Omega, \mathbb{R}^N)$, whose (distributional) derivatives ∇z are elements of the space $L^2(\Omega, \mathbb{R}^{Nn})$ with the norm

$$\|z\|^2_{H^1(\Omega)} = \int_\Omega \left(\left|\nabla z\left(x\right)\right|^2 + \left|z\left(x\right)\right|^2 \right) dx.$$

By $H^{1/2}\left(\Omega,\mathbb{R}^N\right)$ we denote the space of all functions $v(\cdot)\in L^2(\Omega,\mathbb{R}^N)$ for which

$$I_0\left(v\right)=\int\limits_{\Omega}\int\limits_{\Omega}\frac{\left|v\left(x\right)-v\left(y\right)\right|^2}{\left|x-y\right|^{n+1}}dxdy<\infty,$$

equipped with the norm

$$\|v\|^2_{H^{1/2}(\Omega)}=\|v\|^2_{L^2(\Omega)}+I_0\left(v\right)$$

(see Theorem 7.48 in Adams, 1975 or Definition 6.8.2 in Kufner, John, Fucik, 1977).

Covering $\partial\Omega$ by coordinate patches, we define the space $H^{1/2}{=}H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$ as before via such charts (see 7.51 in Adams, 1975 or § 6 in Kufner, John, Fucik, 1977) with an analogous norm.

$H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$ is said to be the space of traces (boundary values or boundary controls) of functions from the space $H^1\left(\Omega,\mathbb{R}^N\right)$. Throughout the paper, we shall assume that Ω satisfies any condition which guarantees a compact embedding of $H^1\left(\Omega,\mathbb{R}^N\right)$ into $L^s\left(\Omega,\mathbb{R}^N\right)$ with $s\in[1,2n/\left(n-2\right))$ if $n\geq 3$ and $s\geq 1$ if $n=2$; for example, $\partial\Omega$ may be Lipschitzian, i.e. $\Omega\in C^{0,1}$ (see Kufner, John, Fucik, 1977).

Let us recall some facts from the trace theory. According to Theorem 6.8.13 of Kufner, John, Fucik (1977), we have the existence of a unique continuous linear mapping R acting from $H^1\left(\Omega,\mathbb{R}^N\right)$ into $H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$ such that $Rz=z\mid_{\partial\Omega}$ for all $z\in C^\infty\left(\overline{\Omega}\right)$. The value Rz is often referred to as the trace of the function z on the boundary $\partial\Omega$ and we usually write $z\mid_{\partial\Omega}$ instead of Rz. Therefore, in system (1)-(2), the equality $z=v$ on $\partial\Omega$ has to be understood in the trace sense.

In our considerations, an essential role is played by Theorem 6.9.2 from Kufner, John, Fucik (1977). We can prove that any function from the space $H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$ is a trace of a function from $H^1\left(\Omega,\mathbb{R}^N\right)$, namely, there exists a continuous linear operator T (lifting operator) acting from $H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$ into $H^1\left(\Omega,\mathbb{R}^N\right)$ such that, for $p=Tv$, we have $p=v$ on $\partial\Omega$ (in the trace sense) and there exists a positive constant c such that the following inequality holds:

$$\|Tv\|_{H^1}\leq c\,\|v\|_{H^{1/2}}$$

where c depends on the mapping T and the description of $\partial\Omega$. Besides, it is easy to check that there exists the mapping T such that

$$H^1\left(\Omega,\mathbb{R}^N\right)=H^1_0\left(\Omega,\mathbb{R}^N\right)\oplus\operatorname{Im}T,$$

i.e. for any $y\in H^1_0\left(\Omega,\mathbb{R}^N\right)$ and $v\in H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$

$$\left(y,Tv\right)_{H^1}=0,$$

where $(\cdot,\cdot)_{H^1}$ denotes the scalar product in $H^1\left(\Omega,\mathbb{R}^N\right)$.

Let v_0 be a fixed point from the space $H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$. By $\mathcal{V}$ denote the set of all boundary conditions of the form

$$\mathcal{V}=\left\{v\in H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right):\|v-v_0\|_{H^{1/2}}\le k_1\right\}$$

for $k_1>0$ and $\mathcal{U}$ the set of parameters

$$\mathcal{U}=\{u\in L^\infty\left(\Omega,\mathbb{R}^m\right):u\left(x\right)\in U\subset\mathbb{R}^m \text{ and } \|u\|_{L^\infty}\le k_2\}$$

for $k_2>0$ and some subset U of $\mathbb{R}^m$.

In this paper we shall consider a control system governed by an elliptic vector equation with variable parameters and boundary data of the form

$$\left\{\begin{array}{l}-\Delta z\left(x\right)=G_z\left(x,z\left(x\right),u\left(x\right)\right)\\ z\left(x\right)=v\left(x\right)\ \text{ on } \partial\Omega,\end{array}\right. \tag{5}$$

where $z\left(\cdot\right)\in H^1\left(\Omega,\mathbb{R}^N\right)$, $\Delta z=\left(\Delta z^1,...,\Delta z^N\right)$, $\Delta z^i=\frac{\partial^2 z^i}{\partial(x^1)^2}+...+\frac{\partial^2 z^i}{\partial(x^n)^2}$ for $i=1,2,...,N$, $v\left(\cdot\right)\in\mathcal{V}$, $u\left(\cdot\right)\in\mathcal{U}$ and $G:\Omega\times\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}$, $G_z=\left(G_{z^1},...,G_{z^N}\right)$.

Functional of action for system (5) has the form

$$F\left(z\right)=\int\limits_{\Omega}\left(\tfrac{1}{2}\left|\nabla z\left(x\right)\right|^2-G\left(x,z\left(x\right),u\left(x\right)\right)\right)dx, \tag{6}$$

where $z\in H^1\left(\Omega,\mathbb{R}^N\right)$, $z\left(x\right)=v\left(x\right)$ a.e. on $\partial\Omega$ and $\nabla z=\left(\nabla z^1,...,\nabla z^N\right)$, $\nabla z^i=\left(\frac{\partial z^i}{\partial x^1},...,\frac{\partial z^i}{\partial x^n}\right)$ for $i=1,2,...,N$.

On the function G we shall impose the following conditions:

(2.1) G, G_z are Carathéodory functions, i.e. they are measurable with respect to x for any $(z,u)\in\mathbb{R}^N\times\mathbb{R}^m$ and continuous with respect to $(z,u)\in\mathbb{R}^N\times\mathbb{R}^m$ for $x\in\Omega$ a.e.;

(2.2) for any bounded subset $U_0\subset U$, there exists $c>0$ such that

$$\left|G\left(x,z,u\right)\right|\le c\left(1+|z|^s\right),\ \left|G_z\left(x,z,u\right)\right|\le c\left(1+|z|^{s-1}\right),$$

for $z\in\mathbb{R}^N$, $u\in U_0$ and $x\in\Omega$ a.e., where $s\in\left(1,2^*\right)$ with $2^*=2n/\left(n-2\right)$ if $n\ge3$ and $2^*=\infty$ if $n=2$;

(2.3) there exist $p>2$, $a>0$ and $R>0$ such that

$$a<pG\left(x,z,u\right)\le\left\langle G_z\left(x,z,u\right),z\right\rangle$$

for $x\in\Omega$ a.e., $u\in U$ and $|z|\ge R$;

(2.4) there exist $\zeta > 0$ and $0 < b < \frac{1}{2}$ such that

$$\left| G(x, z, u) + \tfrac{1}{2} |z|^2 \right| \leq \tfrac{b}{2} |z - Tv_0(x)|^2$$

for $|z| \leq \zeta$, $u \in U$ and $x \in \Omega$ a.e., where T is a fixed inverse operator to the trace operator such that $H^1(\Omega, \mathbb{R}^N) = H_0^1(\Omega, \mathbb{R}^N) \oplus \operatorname{Im} T$.

3. Auxiliary lemmas

We begin with some definitions. Let $I(\cdot) : E \to \mathbb{R}$ be a functional of C^1−class defined on real Banach space E (in our case on $H_0^1(\Omega, \mathbb{R}^N)$). A point $y \in E$ is called a critical point of the functional $I(\cdot)$ if $I'(y) = 0$ and moreover the number $c = I(y)$ is referred to as a critical value.

We say that the functional $I(\cdot)$ satisfies the Palais-Smale (PS for short) condition if any sequence $\{y_k\} \subset E$ such that $I'(y_k) \to 0$ and $|I(y_k)| < C$ for some $C > 0$ is relatively compact in the strong topology of E.

In this part we shall use the following version of the Mountain Pass Theorem (see Mawhin, Willem, 1989; Struwe, 1990).

THEOREM 3.1 *If*
1^0 *there exist* $w_0, w_1 \in E$ *and a bounded neighborhood* B *of* w_0, *such that* $w_1 \in E \setminus \overline{B}$,
2^0 $\inf_{y \in \partial B} I(y) > \max\{I(w_0), I(w_1)\}$,
3^0 $c = \inf_{g \in M} \max_{t \in [0,1]} I(g(t))$, *where* $M = \{g \in C([0,1], E) : g(0) = w_0,\ g(1) = w_1\}$,
4^0 $I(\cdot)$ *satisfies the (PS) condition,*
then c *is a critical value and* $c > \max\{I(w_0), I(w_1)\}$.

In this section we shall use the following notations:

Let M_r denote a set of continuous mappings $g : [0,1] \to \mathbb{B}_r$ such that $g(0) = w_0$, $g(1) = w_1$ and w_0, $w_1 \in \mathbb{B}_r$, where $\mathbb{B}_r = \left\{ y \in H_0^1(\Omega, \mathbb{R}^N) : \|y\|_{H_0^1} < r \right\}$, $r > 0$.
Next, let $I_k : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$, $k = 0, 1, 2, \ldots$ denote an arbitrary sequence of functionals of C^1-class, and $c_k(r)$ be the value given by the formula

$$c_k(r) = \inf_{g \in M_r} \max_{t \in [0,1]} I_k(g(t)). \tag{7}$$

In this case, the set of all critical points $Y_k(r)$ corresponding to the value $c_k(r)$ has the form

$$Y_k(r) = \{ y \in \mathbb{B}_r : I_k(y) = c_k(r) \text{ and } I_k'(y) = 0 \} \tag{8}$$

for $k = 0, 1, 2, \ldots$.

For the sequence of the functionals $\{I_k(\cdot)\}$ we shall prove:

Lemma 3.1 *Assume that*
1^0 *the functionals* $I_k(\cdot)$, $k = 0, 1, 2, ...$ *are of* C^1-*class and* $I_0(\cdot)$ *satisfies the (PS) condition,*
2^0 *the sequences* $\{I_k(\cdot)\}$, $\{I'_k(\cdot)\}$ *tend uniformly on the ball* $\mathbb{B}_r$ *to* $I_0(\cdot)$, $I'_0(\cdot)$, *respectively,*
3^0 *the sets* $Y_k(r)$ *defined by (8) are not empty for* $k = 0, 1, 2, ...$
Then any sequence $\{y_k\}$ *such that* $y_k \in Y_k(r)$, $k = 1, 2, ...$ *is relatively compact in* $H_0^1(\Omega, \mathbb{R}^N)$, *i.e.* $Lim \sup Y_k(r)$ *is a nonempty set and* $Lim \sup Y_k(r) \subset Y_0(r)$, *where* $Lim \sup Y_k(r)$ *is the upper limit of the sets* $Y_k(r)$, $k = 1, 2, ...$, *i.e. the set of all cluster points with respect to the strong topology of* $H_0^1(\Omega, \mathbb{R}^N)$ *of a sequence* $\{y_k\}$ *such that* $y_k \in Y_k(r)$, $k = 1, 2, ...$

Proof. Let $\{y_k\}$ be an arbitrary sequence such that $y_k \in Y_k(r)$ for $k = 0, 1, 2, ...$ By assumption (2^0), we obtain $0 = \lim_{k\to\infty}(I'_k(y_k) - I'_0(y_k)) = -\lim_{k\to\infty} I'_0(y_k)$ because $I'_k(y_k) = 0$ for $k = 0, 1, 2....$ Furthermore $\|y_k\|_{H_0^1} < r$ hence the sequence $I_0(y_k)$ is bounded. Since $I_0(\cdot)$ satisfies the (PS) condition, we infer that $\{y_k\}$ is a relatively compact sequence with respect to the strong topology of $H_0^1(\Omega, \mathbb{R}^N)$, i.e. $Lim \sup Y_k(r)$ is not empty.

Let us notice that $\lim_{k\to\infty} c_k(r) = c_0(r)$. Indeed, by assumption (2^0), we obtain

$$\begin{aligned} c_k(r) &= \inf_{g\in M_r} \max_{t\in[0,1]} [(I_k(g(t)) - I_0(g(t))) + I_0(g(t))] \\ &\le \inf_{g\in M_r} \max_{t\in[0,1]} (\varepsilon + I_0(g(t))) = \varepsilon + c_0(r) \end{aligned}$$

for any $\varepsilon > 0$ and k sufficiently large.
Similar consideration is applied to $c_0(r) \le \varepsilon + c_k(r)$.
Consequently,

$$\lim_{k\to\infty} c_k(r) = c_0(r). \tag{9}$$

From (2^0) it follows that for any sequence $\{y_k\}$ such that $y_k \in Y_k(r)$ for $k = 1, 2, ...$, $A_k = I_0(y_k) - I_k(y_k) \to 0$ and moreover by (9), we conclude that $\lim_{k\to\infty} I_0(y_k) = \lim_{k\to\infty}(A_k + c_k(r)) = c_0(r)$.

We have proved that $Lim \sup Y_k(r)$ is a nonempty set. Let $\tilde{y}$ be an arbitrary point of this set. By definition of the upper limit of sequence of sets, $\tilde{y}$ is a cluster point of some sequence $\{y_k\}$ such that $y_k \in Y_k(r)$. Passing, if necessary, to a subsequence, we may assume that $y_k \to \tilde{y}$. Suppose that $\tilde{y} \notin Y_0(r)$, i.e. $I_0(\tilde{y}) \ne c_0(r)$ or $I'_0(\tilde{y}) \ne 0$. Let us observe that the second inequality is false. Indeed, assumption (2^0) and the first part of our proof imply

$$I'_0(\tilde{y}) = \lim_{k\to\infty} I'_0(y_k) = \lim_{k\to\infty}(I'_0(y_k) - I'_k(y_k)) = 0.$$

Putting $\alpha = I_0(\tilde{y}) - I_0(y_0)$, where $y_0 \in Y_0(r)$ and $\alpha \neq 0$, we see that

$$c_k(r) - c_0(r) = I_k(y_k) - I_0(y_0) = [I_k(y_k) - I_0(y_k)] + [I_0(y_k) - I_0(\tilde{y})] + \alpha.$$

By virtue of (9), assumption (2^0) and continuity of the functional $I_0(\cdot)$, we have the following convergences:

$$c_k(r) - c_0(r) \to 0,\ I_k(y_k) - I_0(y_k) \to 0 \text{ and } I_0(y_k) - I_0(\tilde{y}) \to 0 \text{ as } k \to \infty.$$

This contradicts the fact that $\alpha \neq 0$. Thus $\tilde{y} \in Y_0(r)$ and consequently

$$Lim\sup Y_k(r) \text{ is a nonempty set and } Lim\sup Y_k(r) \subset Y_0(r).$$

The proof is completed. ■

In our further consideration we need a specific form of the functional $I_k(\cdot)$, which is strongly connected with the form of the functional of action given by (6).

Let T be a fixed lifting operator such that $H^1(\Omega, \mathbb{R}^N) = H_0^1(\Omega, \mathbb{R}^N) \oplus \operatorname{Im} T$. For a fixed boundary value v, substituting $z = y + Tv$ into (6) we get

$$F_{v,u}(y) = \int_\Omega \left(\tfrac{1}{2} |\nabla y(x) + \nabla(Tv)(x)|^2 - G(x, y(x) + (Tv)(x), u(x)) \right) dx$$

where $y(\cdot) \in H_0^1(\Omega, \mathbb{R}^N)$, $v(\cdot) \in \mathcal{V}$ and $u(\cdot) \in \mathcal{U}$.
Let $\{v_k\} \in \mathcal{V}$, $k = 0, 1, 2, ...$ be a sequence of boundary controls and $\{u_k\} \in \mathcal{U}$, $k = 0, 1, 2...$ a sequence of distributed controls. Denote by $\{\mathcal{F}_k(\cdot)\}$ the following sequence of functionals

$$\mathcal{F}_k(y) = F_{v_k, u_k}(y) + \int_\Omega \left[G(x, Tv_k(x), u_k(x)) - \tfrac{1}{2} |\nabla(Tv_k)(x)|^2 \right] dx$$

for which we define the value

$$c_k = \inf_{g \in M} \max_{t \in [0,1]} \mathcal{F}_k(g(t)), \tag{10}$$

where M is a set of continuous mappings $g : [0,1] \to H_0^1(\Omega, \mathbb{R}^N)$ such that $g(0) = w_0$, $g(1) = w_1$ (w_0, w_1 are some elements from $H_0^1(\Omega, \mathbb{R}^N)$).
Let Y_k denote the set of critical points corresponding to the value c_k,

$$Y_k = \left\{ y \in H_0^1(\Omega, \mathbb{R}^N) : \mathcal{F}_k(y) = c_k \text{ and } \mathcal{F}_k'(y) = 0 \right\} \tag{11}$$

for $k = 0, 1, 2, ...$ In Section 4, we shall prove that for each $k \in \mathbb{N}$ the set Y_k is not empty and the sequence of sets $\{Y_k\}$ possesses nonempty upper limit such that $Lim\sup Y_k \subset Y_0$.

In the proof of the main theorem we shall use the following lemma:

Lemma 3.2 *If the function G satisfies conditions (2.1)-(2.3), then there exists a ball $\mathbb{B}_\rho$ in $H_0^1\left(\Omega, \mathbb{R}^N\right)$ such that $Y_k \subset \mathbb{B}_\rho = \left\{y \in H_0^1\left(\Omega, \mathbb{R}^N\right) : \|y\|_{H_0^1} < \rho\right\}$, $\rho > 0$ for any boundary controls $v_k \in \mathcal{V}$ and distributed controls $u_k \in \mathcal{U}$.*

Proof. Let us notice that the set $\{c_k : v_k \in \mathcal{V},\ u_k \in \mathcal{U}\}$ is bounded from above. Indeed, for any $k \in \mathbb{N}$, from assumptions (2.2), (2.3) we get

$$
\begin{aligned}
c_k &= \inf_{g\in M} \max_{t\in[0,1]} \mathcal{F}_k\left(g\left(t\right)\right) \le \max_{t\in[0,1]} \mathcal{F}_k\left((1-t)w_0 + tw_1\right) \\
&= \max_{t\in[0,1]} \Bigg(\int_\Omega \tfrac{1}{2}\left|(1-t)\nabla w_0 + t\nabla w_1\right|^2 + \left((1-t)\nabla w_0 + t\nabla w_1, \nabla\left(Tv_k\right)\right) \\
&\qquad\qquad -G\left(x,(1-t)w_0 + tw_1 + Tv_k, u_k\right) + G\left(x, Tv_k, u_k\right) \Bigg) dx \\
&\le \max_{t\in[0,1]} \Bigg(\int_\Omega \left(2(1-t)^2\left|\nabla w_0\right|^2 + 2t^2\left|\nabla w_1\right|^2\right) dx \\
&\qquad\qquad - \int_{\Omega_t^+} G\left(x,(1-t)w_0 + tw_1 + Tv_k, u_k\right) dx \\
&\qquad\qquad - \int_{\Omega_t^-} G\left(x,(1-t)w_0 + tw_1 + Tv_k, u_k\right) dx \Bigg) \\
&\quad + \int_\Omega \left(\tfrac{1}{2}\left|\nabla\left(Tv_k\right)\right|^2 + G\left(x, Tv_k, u_k\right)\right) dx \\
&\le \max_{t\in[0,1]} \left(2(1-t)^2\left\|w_0\right\|^2 + 2t^2\left\|w_1\right\|^2 - \tfrac{a}{p}\left|\Omega_t^+\right| + c\left(1+R^s\right)\left|\Omega_t^-\right|\right) \\
&\quad + \left\|Tv_k\right\|_{H^1}^2 + c\left|\Omega\right| + cc_1\left\|Tv_k\right\|_{H^1}^s \\
&\le 2\max\left\{\left\|w_0\right\|^2, \left\|w_1\right\|^2\right\} + c\left(1+R^s\right)\left|\Omega\right| + \left\|Tv_k\right\|_{H^1}^2 + c\left|\Omega\right| + cc_1\left\|Tv_k\right\|_{H^1}^s \\
&\le 2\max\left\{\left\|w_0\right\|^2, \left\|w_1\right\|^2\right\} + D \le \tilde{c},
\end{aligned}
$$

i.e.

$$
c_k \le \tilde{c}
$$

where $D, \tilde{c}$ are some constants, $\Omega_t^+ = \left\{x \in \Omega : \left|\left((1-t)w_0 + tw_1 + Tv_k\right)(x)\right| \ge R\right\}$ and $\Omega_t^- = \left\{x \in \Omega : \left|\left((1-t)w_0 + tw_1 + Tv_k\right)(x)\right| < R\right\}.$

Furthermore, for any $v_k \in \mathcal{V}$, $u_k \in \mathcal{U}$ and $y \in Y_k$ we obtain

$$
\begin{aligned}
p\tilde{c} \ge pc_k &= p\mathcal{F}_k\left(y\right) - \left\langle \mathcal{F}_k'\left(y\right), y + Tv_k \right\rangle \\
&= \tfrac{p-2}{2}\left\|y\right\|_{H_0^1}^2 + (p-2)\int_\Omega \left(\nabla y, \nabla\left(Tv_k\right)\right) dx - \int_\Omega \left|\nabla\left(Tv_k\right)\right|^2 dx
\end{aligned}
$$

$$+\int\limits_{\Omega}\Big(-pG\left(x,y+Tv_k,u_k\right)+pG\left(x,Tv_k,u_k\right)+\left(G_y\left(x,y+Tv_k,u_k\right),y+Tv_k\right)\Big)dx$$

$$\geq \tfrac{p-2}{2}\left\|y\right\|_{H_0^1}^2-(p-2)\left\|y\right\|_{H_0^1}c_1-c_1^2$$

$$+\int\limits_{\Omega^+}\left(-pG\left(x,y+Tv_k,u_k\right)+pG\left(x,Tv_k,u_k\right)+\left(G_y\left(x,y+Tv_k,u_k\right),y+Tv_k\right)\right)dx$$

$$+\int\limits_{\Omega^-}\left(-pG\left(x,y+Tv_k,u_k\right)+pG\left(x,Tv_k,u_k\right)+\left(G_y\left(x,y+Tv_k,u_k\right),y+Tv_k\right)\right)dx$$

$$\geq \tfrac{p-2}{2}\left\|y\right\|_{H_0^1}^2+D_1\left\|y\right\|_{H_0^1}+D_2$$

where D_1 and D_2 are some constants, $\Omega^+=\{x\in\Omega:\left|\left(y+Tv_k\right)(x)\right|\geq R\}$ and $\Omega^-=\{x\in\Omega:\left|\left(y+Tv_k\right)(x)\right|<R\}$.
Thus

$$p\tilde{c}\geq \tfrac{p-2}{2}\left\|y\right\|_{H_0^1}^2+D_1\left\|y\right\|_{H_0^1}+D_2. \tag{12}$$

Since $p-2>0$, there exists $\rho>0$ such that $y\in\mathbb{B}_\rho$. Consequently, $Y_k\subset\mathbb{B}_\rho$ for any $v_k\in\mathcal{V}$ and $u_k\in\mathcal{U}$. ∎

Without loss of generality, we can assume that $w_0=0$. We shall prove that there exist a bounded neighborhood B of w_0 and some point $w_1\notin\overline{B}$ such that the assumptions of the Mountain Pass Theorem are satisfied.

LEMMA 3.3 *Suppose that*
1^0 *conditions (2.1)-(2.4) are satisfied,*
2^0 *the sequence* $\{v_k\}\subset\mathcal{V}$ *tends to* v_0 *in* $H^{1/2}\left(\partial\Omega,\mathbb{R}^N\right)$ *and the sequence* $\{u_k\}\subset\mathcal{U}$ *tends to* u_0 *in* $L^\infty\left(\Omega,\mathbb{R}^m\right)$.
Then there exist a ball $B_\eta\subset H_0^1\left(\Omega,\mathbb{R}^N\right)$ *and element* $w_1\notin\overline{B_\eta}$ *such that* $\inf_{\partial B_\eta}\mathcal{F}_k>0$ *and* $\mathcal{F}_k\left(w_1\right)<0$ *for any* $v_k\in\mathcal{V}$ *and* $u_k\in\mathcal{U}$, *where* $B_\eta=\left\{y\in H_0^1\left(\Omega,\mathbb{R}^N\right):\left\|y\right\|_{H^1}<\eta\right\}$, *for* $\eta>0$.

Proof. In a similar way as in the proof of Theorem 3.3 in Mawhin (1987), we obtain that there exists a constant $a_0>0$ such that

$$G\left(x,z,u\right)\geq a_0\left|z\right|^p,$$

for $|z|\geq R$, $u\in U$ and $x\in\Omega$ a.e. and furthermore by (2.2) we have the existence of a positive constant a_1 such that $G\left(x,z,u\right)\geq a_0\left|z\right|^p-a_1$ for $z\in\mathbb{R}^N$, $u\in U$ and $x\in\Omega$ a.e., which lead to the fact that $p,s\in\left(2,2^*\right)$. By conditions (2.2), (2.4) we conclude that there exist $b\in\left(0,\frac{1}{2}\right)$ and $A>0$ such that $\left|G\left(x,z,u\right)+\frac{1}{2}\left|z\right|^2\right|\leq\frac{b}{2}\left|z-Tv_0\left(x\right)\right|^2+A\left|z-Tv_0\left(x\right)\right|^s$ for $z\in\mathbb{R}^N$, $u\in U$,

$x \in \Omega$ a.e. and $s \in (1, 2^*)$. For fixed $k \in \mathbb{N}$, by the equality $(y, Tv_k)_{H^1} = 0$, we get

$$\begin{aligned}
\mathcal{F}_k(y) &= \int_\Omega \Big(\tfrac{1}{2} |\nabla y|^2 + (\nabla y, \nabla (Tv_k)) - G(x, y + Tv_k, u_k) + G(x, Tv_k, u_k) \Big) dx \\
&= \int_\Omega \Big(\tfrac{1}{2} |\nabla y|^2 + (\nabla y, \nabla (Tv_k)) + \tfrac{1}{2} |y + Tv_k|^2 - \tfrac{1}{2} |y + Tv_k|^2 \\
&\qquad - G(x, y + Tv_k, u_k) + G(x, Tv_k, u_k) \Big) dx \\
&= \tfrac{1}{2} \|y\|_{H^1}^2 + (y, Tv_k)_{H^1} \\
&\quad - \int_\Omega \Big(G(x, y + Tv_k, u_k) + \tfrac{1}{2} |y + Tv_k|^2 \Big) dx + \int_\Omega \Big(G(x, Tv_k, u_k) + \tfrac{1}{2} |Tv_k|^2 \Big) dx \\
&\geq \left(\tfrac{1}{2} - b \right) \|y\|_{H^1}^2 - Ac_1 2^{s-1} \|y\|_{H^1}^s - \tfrac{3b}{2} \|Tv_k - Tv_0\|_{L^2}^2 - Ac_2 2^s \|Tv_k - Tv_0\|_{H^1}^s
\end{aligned}$$

where $c_1,\ c_2 > 0$. Since $\frac{1}{2} - b > 0$, $v_k \to v_0$ in the strong topology of $H^{1/2}\left(\partial\Omega, \mathbb{R}^N\right)$ and $s > 2$, there exists $\eta > 0$ such that $\inf_{\partial B_\eta} \mathcal{F}_k \geq \alpha > 0$ for k sufficiently large.

Now, we shall prove that for any $v_k \in \mathcal{V}$ and $u_k \in \mathcal{U}$ there exists $w_1 \notin \overline{B_\eta}$ such that $\mathcal{F}_k(w_1) < 0$.

For fixed $y(\cdot) \in H_0^1\left(\Omega, \mathbb{R}^N\right)$, $y \neq 0$ and $l > 0$, we have

$$\begin{aligned}
\mathcal{F}_k(ly) &= \int_\Omega \Big(\tfrac{1}{2} |l \nabla y|^2 + l (\nabla y, \nabla (Tv_k)) - G(x, ly + Tv_k, u_k) \\
&\qquad + G(x, Tv_k, u_k) \Big) dx \\
&\leq \tfrac{1}{2} l^2 \|y\|_{H^1}^2 + l c_1 \|y\|_{H^1} - \int_\Omega (a_0 |ly + Tv_k|^p - a_1) \, dx + \int_\Omega G(x, Tv_k, u_k) \, dx \\
&\leq \tfrac{1}{2} l^2 \|y\|_{H^1}^2 + l c_1 \|y\|_{H^1} - a_0 l^p \int_\Omega \Big(\big| y + \tfrac{Tv_k}{l} \big|^p \Big) dx + c_2.
\end{aligned}$$

Since $2 < p < 2^*$ and $a_0 > 0$ we infer that $\lim_{l \to \infty} \mathcal{F}_k(ly) = -\infty$. Accordingly, there exists $l_0 > 0$ such that for $w_1 = l_0 y$ we have $\|w_1\|_{H^1} \geq \eta$ and $\mathcal{F}_k(w_1) < 0$ for any $v_k \in \mathcal{V}$ and $u_k \in \mathcal{U}$. ■

Now we formulate some sufficient conditions, which guarantee a uniform convergence of a sequence of functionals and a sequence of derivative of functionals on any ball from the space $H_0^1\left(\Omega, \mathbb{R}^N\right)$.

Lemma 3.4 *If*
1^0 *conditions (2.1), (2.2) are satisfied,*
2^0 *the sequence* $\{v_k\} \subset \mathcal{V}$ *tends to* v_0 *in* $H^{1/2}\left(\partial\Omega, \mathbb{R}^N\right)$ *and the sequence* $\{u_k\} \subset \mathcal{U}$ *tends to* u_0 *in* $L^\infty(\Omega, \mathbb{R}^m)$,

3^0 *for any* $u_0 \in U$ *and* $\varepsilon > 0$, *there exists a constant* $c > 0$ *such that*

$$|G(x,z,u_1) - G(x,z,u_2)| \leq c\left(1 + |z|^2\right)|u_1 - u_2|$$
$$|G_z(x,z,u_1) - G_z(x,z,u_2)| \leq c(1 + |z|)|u_1 - u_2|$$

for $x \in \Omega$ *a.e.,* $z \in \mathbb{R}^N$ *and for any* $u_1,\ u_2 \in U$ *with* $|u_1 - u_0| < \varepsilon$ *and* $|u_2 - u_0| < \varepsilon$,
then the sequences $\{\mathcal{F}_k(\cdot)\}, \{\mathcal{F}_k'(\cdot)\}$ *tend uniformly on any ball from* $H_0^1\left(\Omega, \mathbb{R}^N\right)$ *to* $\mathcal{F}_0(\cdot)$ *and* $\mathcal{F}_0'(\cdot)$, *respectively.*

Proof. For any $\mathbb{B}_\rho \subset H_0^1\left(\Omega, \mathbb{R}^N\right)$ and $y \in \mathbb{B}_\rho$, we have

$$\begin{aligned}
|\mathcal{F}_k(y) - \mathcal{F}_0(y)| &\leq \int_\Omega |\nabla y|\,|\nabla(Tv_k) - \nabla(Tv_0)|\,dx \\
&+ \Bigg| \int_\Omega (G(x,Tv_k,u_k) - G(x,Tv_0,u_0))\,dx \\
&+ \int_\Omega (G(x,y+Tv_k,u_k) - G(x,y+Tv_0,u_0)) \Bigg| dx \\
&\leq \rho\,\|\nabla(Tv_k) - \nabla(Tv_0)\|_{L^2} + \int_\Omega |G(x,Tv_k,u_0) - G(x,Tv_0,u_0)|\,dx \\
&+ \int_\Omega |G(x,y+Tv_k,u_0) - G(x,y+Tv_0,u_0)|\,dx \\
&+ \|u_k - u_0\|_{L^\infty}\left(D_1 + D_2\,\|\nabla Tv_k\|_{L^2}^2\right) < \varepsilon
\end{aligned}$$

for k sufficiently large. In fact, from assumption $\left(2^0\right)$ it follows that we have the strong convergence of Tv_k to Tv_0 in $H^1\left(\Omega, \mathbb{R}^N\right)$. By virtue of assumptions (2.1), (2.2), $\left(3^0\right)$ and the Krasnosielskii theorem on continuity of Niemycki's operator, we conclude that the right hand side of the above inequality tends to 0 for any $y \in \mathbb{B}_\rho$. It means that the sequence $\{\mathcal{F}_k(\cdot)\}$ tends uniformly to $\mathcal{F}_0(\cdot)$ on any ball $\mathbb{B}_\rho$.

Similar arguments are applied to the case of a uniform convergence of the sequence $\{\mathcal{F}_k'(\cdot)\}$ to $\mathcal{F}_0'(\cdot)$ on any ball from $H_0^1\left(\Omega, \mathbb{R}^N\right)$. Let us take any ball $\mathbb{B}_\rho \subset H_0^1\left(\Omega, \mathbb{R}^N\right)$. For any $y \in \mathbb{B}_\rho$ and $h \in H_0^1\left(\Omega, \mathbb{R}^N\right)$ such that $h \in \mathbb{B}_1$ by simple calculations we have

$$\begin{aligned}
&|\langle \mathcal{F}_k'(y) - \mathcal{F}_0'(y), h\rangle| \leq \|\nabla(Tv_k) - \nabla(Tv_0)\|_{L^2} \\
&+ \int_\Omega |(G_y(x,y+Tv_k,u_k) - G_y(x,y+Tv_0,u_0), h)|\,dx < \varepsilon
\end{aligned}$$

for k sufficiently large, and the lemma follows. ∎

4. The main result

THEOREM 4.1 *Suppose that*
1^0 *the function* G *satisfies conditions (2.1)-(2.4) and assumption* $\left(3^0\right)$ *from Lemma 3.4,*
2^0 *the sequence* $\{v_k\} \subset \mathcal{V}$ *tends to* v_0 *in* $H^{1/2}\left(\partial\Omega, \mathbb{R}^N\right)$ *and the sequence* $\{u_k\} \subset \mathcal{U}$ *tends to* u_0 *in* $L^\infty(\Omega, \mathbb{R}^m)$.
Then
(a) for any k *the set of critical points* Y_k *of the functional* $\mathcal{F}_k(\cdot)$ *is nonempty and does not contain the trivial solution,*
(b) any sequence $\{y_k\}$ *such that* $y_k \in Y_k$, $k = 1, 2, \ldots$, *is relatively compact in* $H_0^1\left(\Omega, \mathbb{R}^N\right)$ *and* $Lim\sup Y_k \subset Y_0$.

Proof. Applying the Mountain Pass Theorem, we shall prove assertion (a) of our theorem. The functional $\mathcal{F}_k(\cdot)$, $k = 0, 1, 2, \ldots$ is of C^1-class on $H_0^1\left(\Omega, \mathbb{R}^N\right)$. From Lemma 3.3 it follows that there exist the ball B_η and the point $w_1 \in H_0^1\left(\Omega, \mathbb{R}^N\right)$ (independent of the choice of k) such that $w_1 \notin \overline{B_\eta}$ and $\inf_{\partial B_\eta} \mathcal{F}_k > 0 = \max\{\mathcal{F}_k(0), \mathcal{F}_k(w_1)\}$, $k = 0, 1, 2, \ldots$. Using assumptions (2.2) and (2.3), we shall demonstrate that the functional $\mathcal{F}_k(\cdot)$, $k = 0, 1, 2, \ldots$ satisfies the (PS) condition. For fixed k, let $\{y_i\}$ denote a sequence such that $\{\mathcal{F}_k(y_i)\}$ is bounded and $\mathcal{F}_k'(y_i) \to 0$ as $i \to \infty$. Thus, there exist constants C_1, $C_2 > 0$ such that $|\mathcal{F}_k(y_i)| \leq C_1$ and $\|\mathcal{F}_k'(y_i)\| \leq C_2$ for $i \in \mathbb{N}$. In the same manner as in the proof of Lemma 3.2, we obtain the following inequality

$$\begin{aligned} C_1 p + C_2\sqrt{1+d^2}\, \|y_i\|_{H_0^1} + C_2 \|Tv_k\|_{H^1} &\geq C_1 p + C_2 \|y_i + Tv_k\|_{H^1} \\ \geq p\mathcal{F}_k(y_i) - \langle \mathcal{F}_k'(y_i), y_i + Tv_k \rangle &\geq \tfrac{p-2}{2} \|y_i\|_{H_0^1}^2 + D_1 \|y_i\|_{H_0^1} + D_2, \end{aligned}$$

where D_1 and D_2 are some constants. Hence

$$\|y_i\|_{H_0^1}^2 \leq \tfrac{2}{p-2}\left(pC_1 + D_3 \|y_i\|_{H_0^1} + C_2 \|Tv_k\|_{H^1} - D_2\right) \text{ for } i \in \mathbb{N},$$

where $D_3 = C_2\sqrt{1+d^2} - D_1$ and D_1, D_2, C_1, C_2 are described above. It means that the sequence $\{y_i\}$ is bounded in $H_0^1\left(\Omega, \mathbb{R}^N\right)$ and therefore contains a subsequence, denoted by $\{y_i\}$, such that y_i tends to y_0 weakly in $H_0^1\left(\Omega, \mathbb{R}^N\right)$. It is a well-known fact that the space $H_0^1\left(\Omega, \mathbb{R}^N\right)$ is compactly embedding into the space $L^s\left(\Omega, \mathbb{R}^N\right)$ with $s \in [1, 2^*)$.
Consequently,

$$\langle \mathcal{F}_k'(y_i) - \mathcal{F}_k'(y_0), y_i - y_0 \rangle \underset{i\to\infty}{\to} 0$$

and because of

$$\begin{aligned}&\left\langle \mathcal{F}_k'\left(y_i\right)-\mathcal{F}_k'\left(y_0\right),y_i-y_0\right\rangle\\&=\left\|y_i-y_0\right\|_{H_0^1}^2+\int\limits_\Omega\left(G_y\left(x,y_0+Tv_k,u_k\right)-G_y\left(x,y_i+Tv_k,u_k\right),y_i-y_0\right)dx\end{aligned}$$

and the growth conditions (2.2), we get

$$\begin{aligned}&\left|\int\limits_\Omega\left(G_y\left(x,y_0+Tv_k,u_k\right)-G_y\left(x,y_i+Tv_k,u_k\right),y_i-y_0\right)dx\right|\\&\le\left\|y_i-y_0\right\|_{L^s}\left(\int\limits_\Omega\left|G_y\left(x,y_i+Tv_k,u_k\right)-G_y\left(x,y_0+Tv_k,u_k\right)\right|^{\frac{s}{s-1}}dx\right)^{\frac{s-1}{s}}\\&\le\left\|y_i-y_0\right\|_{L^s}c2^{\frac{s+1}{s}}\left(\int\limits_\Omega\left(1+\left|y_i+Tv_k\right|^s+\left|y_0+Tv_k\right|^s\right)dx\right)^{\frac{s-1}{s}}\end{aligned}$$

where the right hand side of the above inequality tends to 0 and, in consequence, $y_i\to y_0$ in $H_0^1\left(\Omega,\mathbb{R}^N\right)$ for any k. We have thus proved that the functional $\mathcal{F}_k\left(\cdot\right)$, $k=0,1,2,...$, satisfies the (PS) condition.
Applying the Mountain Pass Theorem with $w_0=0$ and $c=c_k$ (see (10)), we infer that for any v_k and u_k the set of critical points for which c_k, a critical value of the functional $\mathcal{F}_k\left(\cdot\right)$, is attained, is not empty, i.e.

$$Y_k=\left\{y\in H_0^1\left(\Omega,\mathbb{R}^N\right):\mathcal{F}_k\left(y\right)=c_k\text{ and }\mathcal{F}_k'\left(y\right)=0\right\}\neq\emptyset.$$

Moreover, $c_k=\inf_{g\in M}\max_{t\in[0,1]}\mathcal{F}_k\left(g\left(t\right)\right)>\max\left\{\mathcal{F}_k\left(0\right),\mathcal{F}_k\left(w_1\right)\right\}=0$, and therefore $y=0$ does not belong to the set Y_k for $k=0,1,2,...$. Accordingly, assertion (a) of our theorem is valid.
Next, we shall prove assertion (b). Applying Lemma 3.2, we get that there exists a ball $\mathbb{B}_\rho\subset H_0^1\left(\Omega,\mathbb{R}^N\right)$ such that $Y_k\subset\mathbb{B}_\rho$ for $k=0,1,2,...$, and consequently there exists a ball $\mathbb{B}_r\subset H_0^1\left(\Omega,\mathbb{R}^N\right)$ with $r\ge\rho$ such that $w_1\in\mathbb{B}_r$, i.e. $Y_k\left(r\right)=Y_k$, where $Y_k\left(r\right)$ is given by (8) with $I_k\left(\cdot\right)=\mathcal{F}_k\left(\cdot\right)$, $k=0,1,2,...$. From the previous part of the proof we know that $\mathcal{F}_0\left(\cdot\right)$ satisfies the (PS) condition and the sets $Y_k\left(r\right)=Y_k$ are nonempty for $k=0,1,2,...$. Thus we have the assertion of Lemma 3.4. Assertion (b) of our theorem follows directly from Lemma 3.1, which completes the proof. ■

Let us notice that $\mathbf{c}_k$, the critical value of the functional

$$F_{v_k,u_k}\left(y\right)=\int\limits_\Omega\left[\tfrac{1}{2}\left|\nabla y\left(x\right)+\nabla\left(Tv_k\right)\left(x\right)\right|^2-G\left(x,y\left(x\right)+\left(Tv_k\right)\left(x\right),u_k\left(x\right)\right)\right]dx$$

$k = 0, 1, 2, ...$ satisfies the following relation

$$\mathbf{c}_k = c_k + \int_\Omega \left[\tfrac{1}{2} \left| \nabla \left(T v_k \right)(x) \right|^2 - G\left(x, \left(T v_k \right)(x), u_k(x) \right) \right] dx$$

where c_k is defined in (10) and the corresponding set of critical points has the form

$$Y_{v_k,u_k} = \left\{ y \in H_0^1\left(\Omega, \mathbb{R}^N \right) : F_{v_k,u_k}(y) = \mathbf{c}_k \text{ and } F'_{v_k,u_k}(y) = 0 \right\}$$

for $k = 0, 1, 2, ...$

Corollary 4.1 *If we replace $\mathcal{F}_k(\cdot)$ by $F_{v_k,u_k}(\cdot)$ and Y_k by Y_{v_k,u_k}, then Theorem 4.1 is still valid.*

It is easy to see that for the sets $Z_{v_k,u_k} = Y_{v_k,u_k} + T v_k$, $k = 0, 1, 2, ...$ of all critical points of the functional $F(\cdot)$ given by (6) with $u = u_k$ and $v = v_k$ is a subset of the set of the weak solutions to problem (5).

Corollary 4.2 *If all assumptions of Theorem 4.1 are satisfied, then for any k the set Z_{v_k,u_k} is nonempty and does not contain $T v_k$ solution and any sequence $\{z_{v_k,u_k}\}$ such that $z_{v_k,u_k} \in Z_{v_k,u_k}$, $k = 1, 2, ...$ is relatively compact in $H^1\left(\Omega, \mathbb{R}^N\right)$ and $Lim \sup Z_{v_k,u_k} \subset Z_{v_0,u_0}$.*
Moreover, denoting by $\mathcal{Z}_{v_k,u_k}$ the set of the weak solutions to problem (5) corresponding to the critical value $\mathbf{c}_k$, we have the same assertion for the sets $\mathcal{Z}_{v_k,u_k}$.

Example 4.1 *It is easy to check that the assumptions of Theorem 4.1 are satisfied by the system*

$$\begin{cases} \triangle z^i(x) = -4 \left| z(x) \right|^2 z^i(x) + u(x) z^i(x) + 3u(x) \left| z(x) \right| z^i(x) \, sin^2 |x| \\ \qquad\qquad for \quad i = 1, ..., N \\ z(x) = v(x) \quad on \quad \partial\Omega, \end{cases} \tag{13}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ is a bounded domain of $\mathcal{C}^{0,1}$ – class, $u \in \mathcal{U} = \{u \in L^\infty(\Omega, \mathbb{R}) : u(x) \in [0,1] \;\; a.e.\}$ and $v \in \mathcal{V} = \left\{ v \in H^{1/2}\left(\partial\Omega, \mathbb{R}^N \right) : v^i(x) \in [0,1], \;\; i = 1, 2, ..., N, \;\; \|v\| \le 1 \right\}$. Let us notice that the functional of action has the following form

$$F(z) = \int_\Omega \left(\tfrac{1}{2} \left| \nabla z(x) \right|^2 - \left| z(x) \right|^4 + \tfrac{1}{2} u(x) \left| z(x) \right|^2 + u(x) \left| z(x) \right|^3 sin^2 |x| \right) dx$$

where $z(\cdot) \in H^1\left(\Omega, \mathbb{R}^N\right)$ and $z(x) = v(x)$ on $\partial\Omega$. Putting $\widetilde{z}_{v_k,u_k}(x) = (ksin\,|x|, ..., ksin\,|x|)$ and $\overline{z}_{v_k,u_k}(x) = (sin(k|x|), ..., sin(k|x|))$ we obtain that

$F(\widetilde{z}_{v_k,u_k}) \to -\infty$ as well as $F(\overline{z}_{v_k,u_k}) \to \infty$, i.e. the functional $F(\cdot)$ is unbounded from above and below and for this reason we cannot use methods applied, for example, in Walczak (1995, 1998).

Applying Corollary 4.2 we have that for any distributed control u and for any boundary control v there exists a solution $z_{v,u}$ to equation (13) and the solution continuously depends on controls u and v.

5. Existence of optimal processes for some control problem

Applying Theorem 4.1 we shall prove the existence of optimal processes for the optimal control problem described by the system of elliptic equations

$$-\Delta z(x) = G_z(x, z(x), u(x)) \tag{14}$$

with the fixed boundary condition $z(x) = v(x)$ on $\partial\Omega$ and with the integral cost functional

$$J(z,u) = \int_\Omega \Phi(x, z(x), u(x))\, dx \tag{15}$$

defined on $H^1(\Omega, \mathbb{R}^N) \times \mathcal{U}_\lambda$ where

$$\mathcal{U}_\lambda = \{u : \Omega \to \mathbb{R}^m : u(x) \in U \text{ and } |u(x_1) - u(x_2)| < \lambda\ |x_1 - x_2|\}$$

for $\lambda > 0$ fixed and the U compact subset of $\mathbb{R}^m$.

A pair (z,u) shall be called an admissible process for (5) if $u \in \mathcal{U}_\lambda$ and associative $z \in H^1(\Omega, \mathbb{R}^N)$ is a weak solution to (5) corresponding to the critical value $\mathbf{c}_k$ (see (13)). Let us denote by Δ the set of all admissible processes for (5). By virtue of Theorem 4.1 we have that $\Delta \neq \emptyset$.

On Φ we shall impose the following conditions:

(5.1) the function Φ is measurable with respect to x for any $(z,u) \in \mathbb{R}^N \times U$ and continuous with respect to $(z,u) \in \mathbb{R}^N \times U$ for $x \in \Omega$ a.e.;

(5.2) there exists $c > 0$ such that

$$|\Phi(x,z,u)| \le c(1 + |z|^s),$$

for $z \in \mathbb{R}^N$, $u \in U$ and $x \in \Omega$ a.e. where $s \in [1, 2^*)$.

We shall prove:

THEOREM 5.1 *If the functions G and Φ satisfy conditions (2.1) -(2.4) and (5.1)-(5.2), then the optimal control problem (5), (14) possesses at least one optimal process $(z^*, u^*) \in H^1(\Omega, \mathbb{R}^N) \times \mathcal{U}_\lambda$.*

Proof. By (5.1) and (5.2) the cost functional is well-defined and continuous with respect to the variables (z,u). Let (z_k,u_k), $k=1,2,...$ be a minimizing sequence for problem (5), (14), i.e. $u_k \in \mathcal{U}_\lambda$, $-\Delta z_k(x) = G_z(x, z_k(x), u_k(x))$ $z_k = v$ on $\partial\Omega$, (see (6)) and

$$\lim_{k\to\infty} J(z_k,u_k) = \inf_{(z,u)\in\Delta} J(z,u).$$

Entire class $\mathcal{U}_\lambda$ is equicontinuous and uniformly bounded, so certainly $\{u_k\}$ is also. By Ascoli's Theorem, there exist subsequence $\{u_k\}$ such that $u_k \to u_0$ uniformly on Ω and $u_0 \in \mathcal{U}_\lambda$. By Theorem 4.1 the sequence $\{z_k\}$ (or at least some its subsequence) tends to z_0 in $H^1(\Omega,\mathbb{R}^N)$ and (z_0,u_0) is the admissible pair for (5), thus $J(z_0,u_0) = \inf_{(z,u)\in\Delta} J(z,u)$. It means that process (z_0,u_0) is optimal for (5), (14). ■

We can obtain a similar result for another class of distributed control $\mathcal{U}_{\Omega(r)}$. More precisely, by $\Omega(r)$ we denote a fixed decomposition of Ω on r open subsets Ω_i such that $\bigcup_{i=1}^{r} \Omega_i \subset \Omega$, $\mu\left(\bigcup_{i=1}^{r} \Omega_i\right) = \mu(\Omega)$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $i,j = 1,...,r$. We shall say that a function u is constant on $\Omega(r)$ if u is constant on each subset from decomposition $\Omega(r)$, i.e. $u(x) = const_i$ for $x \in \Omega_i$, $i=1,...,r$. Finally

$$\mathcal{U}_{\Omega(r)} = \{u(\cdot) \in L^\infty(\Omega,\mathbb{R}^m) : u(x) \in U \text{ and } u \text{ is constant on } \Omega(r)\}$$

where $\Omega(r)$ is a fixed decomposition and U is a compact subset of $\mathbb{R}^m$.

Similarly to Theorem 5.1 we can prove:

THEOREM 5.2 *If the functions G and Φ satisfy conditions (2.1)-(2.4) and (5.1)-(5.2), then the optimal control problem (5), (14) possesses at least one optimal process $(z^*,u^*) \in H^1(\Omega,\mathbb{R}^N) \times \mathcal{U}_{\Omega(r)}$.*

Analogous results for optimal control systems described by ordinary differential equation were proved in *Introduction to Optimal Control Theory* by Macki and Strauss (1982) (see Chapter IV).

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