*Dedicated to*
*Professor Czesław Olech*

**Control and Cybernetics**

vol. **34** (2005) No. 3

# On the global asymptotic stability problem and the Jacobian conjecture[1]

by

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**Abstract:** In this survey, we recall the formulation of the problems and give a review of some nontrivial results in the area. Let $F = (F_1, ..., F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map and let $F'(x)$ and $\operatorname{Jac} F(x) = \det F'(x)$ denote the Jacobian matrix and the jacobian of $F$ at a point $x \in \mathbb{R}^n$, respectively. The Global Asymptotic Stability Problem (GASP) reads as follows: Assume that $F(0) = 0$ and at any point $x \in \mathbb{R}^n$ all eigenvalues of $F'(x)$ have negative real parts. Then consider the associated system of differential equations $x_j'(t) = F_j(x_1(t), ..., x_n(t))$, $j = 1, ..., n$. The question is whether the solution $x(t) = 0$ is globally asymptotically stable. If $n > 2$, then the answer is negative (even if $F$ is a a polynomial automorphism), so from now on (GASP) denotes (GASP) restricted to $\mathbb{R}^2$. In 1963, Olech showed that under the (GASP) assumption (i. e., Jac $F(x) > 0$ and $Trace\, F'(x) = \frac{\partial F_1}{\partial x_1}(x) + \frac{\partial F_2}{\partial x_2}(x) < 0$ for any $x \in \mathbb{R}^2$) the conclusion of (GASP) is equivalent to the injectivity of $F$. In 1994, Fessler, and independently Gutierrez, proved the injectivity of $F$ and, due to the above mentioned Olech's equivalence, gave the affirmative answer to the two-dimensional (GASP).

Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$, $n > 1$. The Jacobian Conjecture can be formulated as follows: If $F = (F_1, \cdots, F_n) : \mathbb{K}^n \to \mathbb{K}^n$ is a polynomial map with a constant nonzero jacobian, then $F$ is a polynomial automorphism (i. e. there exists $F^{-1}$ and $F^{-1}$ is also a polynomial map). Although the Jacobian Conjecture is still unsolved even in the case of $n = 2$, it is convenient to consider the so called Generalized Jacobian Conjecture (for short (GJC) ): *the Jacobian Conjecture holds for every* $n > 1$. We give a review of some interesting conditions equivalent to the Jacobian Conjecture, including Meisters and

[1]The paper contains the invited talk presented at the workshop (dedicated to Professor Czesław Olech) held in Banach Center, Warsaw, Aug. 30 - Sept. 3, 2004

Olech's result on the existence of a poly-flow solution of the associated Ważewski equation $x'(t) = [F'(x(t))]^{-1}(a)$. We also present a reduction of (GJC) to the case of $F$ of degree 3 and of special forms, then some partial results, and (JC)'s relations with other problems.

**Keywords:** global stability problem, Jacobian Conjecture.

# 1. Global Asymptotic Stability Problem

## 1.1. Basic facts on stability

Let $E$ be an open subset of $\mathbb{R}^n$ and $f : E \to \mathbb{R}^n$ be a $\mathcal{C}^1$ mapping. Consider a real autonomous system of differential equations

$$(*) \quad \dot{y} = f(y).$$

We know that solutions of $(*)$ are uniquely determined by initial conditions. We recall the definitions.

Definition 1.1 *Let $y_\circ(\cdot)$ denote the solution of $(\star)$ satisfying the initial condition $y_\circ(0) = y_\circ$ and suppose that the solution is defined for all $t \geq 0$.*

*(i) We say that $y_\circ(\cdot)$ is locally stable when for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\| y_\circ - y_1 \| < \delta$ then the solution $y_1(\cdot)$ of $(\star)$ satisfying the initial condition $y_1(0) = y_1$ exists for every $t \geq 0$ and $\| y_\circ(t) - y_1(t) \| < \varepsilon$ for $t \geq 0$.*

*(ii) We say that a solution $y_\circ(\cdot)$ is locally asymptotically stable (for short: LAS) if $y_\circ(\cdot)$ is locally stable and if there exists $r > 0$ such that for any solution $y_1(\cdot)$ of $(\star)$ satisfying the initial condition $y_1(0) = y_1$, $||y_1 - y_\circ|| < r$ we have the equality $\lim_{t\to\infty} \| y_\circ(t) - y_1(t) \| = 0$.*

*(iii) The solution $y_\circ(\cdot)$ is globally asymptotically stable (for short: GAS) when it is a LAS solution and any solution $y_1(\cdot)$ of $(\star)$ exists for all $t \geq 0$ and $\lim_{t\to\infty} \| y_\circ(t) - y_1(t) \| = 0$.*

*(iv) Let $y_\circ(\cdot) = 0$ be a LAS solution of the equation $(\star)$. The domain of attraction of the solution $y_\circ(\cdot) = 0$ (or the domain of attraction of the set $\{0\}$) is the subset $A$ of $E$ consisting of all points $a \in E$ such that the solution $y_1(\cdot)$ of the equation $(\star)$ starting at a point $a \in A$ exists for every $t \geq 0$ and satisfies the condition $y(t) \to 0$ as $t \to \infty$.*

Since $E$ is open and $y_\circ(\cdot)$ is LAS, then the domain of attraction is also open. In the sequel we assume (without loss of generality) that $F(0) = 0$ and $y_\circ = 0$, so $y_\circ(\cdot) = 0$ is a solution of $(*)$.

Let $F = (F_1, \cdots, F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map and let $F'(x)$ denote the Jacobian matrix (i. e. the matrix of the differential of the map $F$) at a point $x \in \mathbb{R}^n$. Let further $\text{Jac}\, F(x) = \det F'(x)$ be the jacobian of $F$ at a point $x \in \mathbb{R}^n$.

The following Global Asymptotic Stability Problem (GASP) was formulated by Markus and Yamabe (1960).

**(GASP)$_n$.** Assume that $F(0) = 0$ and all eigenvalues of $F'(x)$ have negative real parts at any point $x \in \mathbb{R}^n$. Then consider the associated autonomous system of differential equations

$$(\Diamond) \qquad \begin{aligned} \dot{x}_1(t) =& F_1(x_1(t), ..., x_n(t)), \\ \dot{x}_2(t) =& F_2(x_1(t), ..., x_n(t)), \\ &\vdots \\ \dot{x}_n(t) =& F_n(x_1(t), ..., x_n(t)). \end{aligned}$$

The question is whether the solution $x(t) = 0$ is globally asymptotically stable.

Since all eigenvalues of the Jacobian matrix $F'(0)$ have negative real parts, by the Lyapunov theorem each solution of $(\Diamond)$ is LAS. Therefore (GASP) is equivalent to the statement that every solution $x(t)$ of $(\Diamond)$ tends to the rest point $x = 0$ if $t$ tends to $\infty$.

### 1.2. Results in (GASP)

Obviously the assumptions of two dimensional (GASP)$_2$ can be written as follows.

(0) $\quad F \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2), \quad F(0,0) = (0,0),$

$(i)$ $\quad$ Jac $F(x) > 0 \quad$ for any $x = (x_1, x_2) \in \mathbb{R}^2$.

$(ii)$ $\quad$ *Trace* $F'(x) := \dfrac{\partial F_1}{\partial x_1}(x) + \dfrac{\partial F_2}{\partial x_2}(x) < 0 \quad$ for any $x \in \mathbb{R}^2$.

It has been showed that (GASP) has an affirmative solution under some additional conditions, see e.g. Markus and Yamabe (1960), Olech (1963), Hartman and Olech (1962), Parthasarathy (1983), Meisters and Olech (1988), Drużkowski and Tutaj (1992). In 1963 Olech proved the following:

Theorem 1.1 (Olech, 1963) *Assume (0), (i), (ii) of (GASP)$_2$ and additionally*

$(iii)$ $\quad \exists r > 0 \quad \exists R > 0 : \quad ||x|| \geq r \Rightarrow ||F(x)|| \geq R.$

*Then every solution curve $x(t)$ of $(\Diamond)$ approaches* $(0,0)$ *as $t \to \infty$.*

Olech (1963) formulated the global univalence problem: *Do the inequalities (i) and (ii) imply that the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ is globally one-to-one?* and showed the following:

Theorem 1.2 (Olech, 1963) *The two dimensional (GASP) is equivalent to injectivity of $F$ provided that the assumptions of (GASP)$_2$ are fulfilled.*

The implication that the positive answer to two dimensional (GASP) follows from the global univalence of $F$ is essential, the converse is not difficult. For pointing out how subtle the global univalence problem is we recall an example, where changing the sign of $\operatorname{Jac} F$ destroys the global univalence of $F$.

EXAMPLE 1.1 (Parthasarathy (1983)) *Define an analytic map $F:\mathbb{R}^2\to\mathbb{R}^2$ by the formula*

$$F(x,y) = (-2e^x + 3y^2 - 1, ye^x - y^3).$$

*Then one can check that $Jac\,F(x,y) = -2e^{2x} < 0$ and $Trace\,F'(x,y) = -e^x - 3y^2 < 0$ for every $(x,y)\in\mathbb{R}^2$, but $F$ is not injective because $F(0,1) = (0,0) = F(0,-1)$.*

Although $Trace\,F' < 0$, the condition $\operatorname{Jac} F < 0$ does not imply injectivity of $F$.

Meisters and Olech (1988) proved (using Theorem 1.1) that the answer is positive if $F$ is a polynomial mapping of $\mathbb{R}^2$; therefore (by Theorem 1.2) they obtained that a polynomial mapping $F:\mathbb{R}^2\to\mathbb{R}^2$ is injective (provided that the jacobian of $F$ is positive and the trace of differential $F'$ is negative). In 1995 $(\mathrm{GASP})_2$ (for $\mathcal{C}^1$ maps) was affirmatively solved by R. Fessler and C. Gutierrez (Fessler, 1995; Gutierrez, 1995) – they presented theirs proofs at the conference in Trento (Italy) in September 1993. Both authors proved injectivity of the mapping $F$ and used Olech's Theorem 1.2 that $(\mathrm{GASP})_2$ is equivalent to injectivity of the mapping $F$.

In 1988 the idea of the counterexample to $(\mathrm{GASP})_n$ for $n \geq 4$ was sketched in Barabanov (1988), but $(\mathrm{GASP})_3$ was still an open problem. In 1997 Dutch and Spanish mathematicians gave an explicit polynomial (even polynomial automorphism) counterexample to $(\mathrm{GASP})_n$ if $n > 2$.

EXAMPLE 1.2 (Cima et al., 1997) *Let $n \geq 3$, $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and let $F = (F_1, ..., F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be given by the formulas*

$$\begin{aligned} F_1(x) &= -x_1 + x_3(x_1 + x_2x_3)^2 \\ F_2(x) &= -x_2 - (x_1 + x_2x_3)^2 \\ F_3(x) &= -x_3 \\ &\vdots \\ F_n(x) &= -x_n\,. \end{aligned}$$

*Then $F$ is a counterexample to the Markus-Yamabe Conjecture, namely there exists a solution $x = x(t)$ of the equation $\dot{x} = F(x)$ such that $x(t)\to\infty$ if $t\to\infty$.*

*Proof.* One easily verifies that for all $x\in\mathbb{R}^n$ all eigenvalues of $F'(x)$ are equal to $-1$. Finally one checks that

$$x_1(t) = 18e^t, x_2(t) = -12e^{2t}, x_3(t) = e^{-t}, ..., x_n(t) = e^{-t}$$

is a solution of $\dot{x} = F(x)$ which obviously tends to infinity as $t$ tends to infinity. ■

### 1.3. The discrete analogue of the Markus-Yamabe problem

Let $F =: \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map, $F(0) = 0$ and let the absolute values of all eigenvalues of $F'(x)$ be less than $r$, $r < 1$, at any point $x \in \mathbb{R}^n$. Is the sequence $\left(\overset{k+1}{x} = F(\overset{k}{x}) : k \in \mathbb{N}\right)$, starting with $\overset{\circ}{x}$, bounded for any $\overset{\circ}{x} \in \mathbb{R}^n$?

Cima et al. (1997) also give a counterexample to the discrete analogue of Markus-Yamabe problem when $n \geq 3$.

EXAMPLE 1.3 *Let $n \geq 3$, $x \in \mathbb{R}^n$ and let $F = (F_1, ..., F_n) : \mathbb{R}^n \to \mathbb{R}^n$ be given by formulas*

$$\begin{aligned} F_1(t) &= \tfrac{1}{2}x_1 + x_3(x_1 + x_2x_3)^2 \\ F_2(t) &= \tfrac{1}{2}x_2 - (x_1 + x_2x_3)^2 \\ F_3(t) &= \tfrac{1}{2}x_3 \\ &\vdots \\ F_n(t) &= \tfrac{1}{2}x_n \end{aligned}$$

*Then there exists an initial condition $\overset{\circ}{x}$ such that the sequence $\overset{k+1}{x} = F(\overset{k}{x})$, tends to infinity when $k$ tends to infinity.*

*Proof.* One can check that for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to $\frac{1}{2} < 1$. Taking $\overset{\circ}{x} = (\frac{147}{32}, \frac{-63}{32}, 1, 0, ..., 0)$, $\overset{n+1}{x} = F(\overset{n}{x})$, it is easy to verify by induction that

$$\overset{n}{x} = (\frac{147 \cdot 2^n}{32}, \frac{-63 \cdot 4^n}{32}, \frac{1}{2^n}, 0, ..., 0)$$

which obviously tends to infinity as $n \to \infty$. ■

REMARK 1.1 *Note that the mappings on the right hand sides in both examples are invertible with polynomial inverse, so Olech's equivalence given in Theorem 1.2 does not hold in higher dimension.*

## 2. The Jacobian Conjecture

### 2.1. Formulation of the problem and basic facts about polynomial mappings.

Everyone knows Cramer's theorem that a linear mapping $T : \mathbb{K}^n \to \mathbb{K}^n$ is injective (bijective) if and only if $\operatorname{Jac} T(x) = \det T \neq 0$. If the inverse of a map $f : \mathbb{K}^n \to \mathbb{K}^n$ exists and is differentiable, then (by the chain rule) the jacobian of $f$ is different from 0 everywhere. This raises a natural question about the class of mappings $f : \mathbb{K}^n \to \mathbb{K}^n$, $n > 1$, such that the condition $\operatorname{Jac} f(x) = \text{constant} \neq 0$ (or the condition $\operatorname{Jac} f(x)$ vanishes nowhere) guarantees injectivity of $f$.

The answer is negative even for real or complex analytic mappings. As an example one can take the mapping $f(x, y) = (e^x \cos y, e^x \sin y)$, $(x, y) \in \mathbb{K}^2$. We have $\operatorname{Jac} f(x, y) = e^{2x} \neq 0$ for any $(x, y) \in \mathbb{K}^2$.

Because of the above facts we focus our considerations on polynomial mappings of $\mathbb{R}^n$ or $\mathbb{C}^n$. Remember that the jacobians of polynomial mappings are polynomials and have complex roots unless they are nonzero constants. Thus in the complex case of our problem we have to consider only polynomial mappings with the constant jacobian. This raises the question for analytic mappings again because now the assumptions are stronger. The answer for analytic mappings, however, is at once negative since the holomorphic (i.e. complex analytic) mapping $f(x, y) = (xe^{-y}, e^y)$ has $\operatorname{Jac} f = 1$ for any $(x, y) \in \mathbb{C}^2$ and $f$ is not injective.

Therefore, it is evident that we should concentrate our attention on polynomial mappings of $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $\mathbb{K}$ denote either $\mathbb{C}$ or $\mathbb{R}$. If $F_j \in \mathbb{K}[X_1, ..., X_n]$ (i. e. $F_j$ is a polynomial in $n$ variables), $j = 1, ..., n$, then we put $F = (F_1, \ldots, F_n)$, $\operatorname{Jac} F(x) := \det[\frac{\partial F_i}{\partial x_j}(x) : i, j = 1, ..., n]$, $\mathcal{P}(\mathbb{K}^n) := \{F : \mathbb{K}^n \to \mathbb{K}^n\}$, i.e. $\mathcal{P}(\mathbb{K}^n)$ is the set of polynomial transformations of $\mathbb{K}^n$. Now we recall the formulation of the n-dimensional Jacobian Conjecture for $n \geq 2$ (briefly $(JC)_n$)

$(JC)_n$ $[F \in \mathcal{P}(\mathbb{K}^n),\ \operatorname{Jac} F = \text{const} \neq 0] \Rightarrow [F \text{ is injective}]$

and the so called Generalized Jacobian Conjecture, for short $(GJC)$, namely

$(GJC)$ $(\mathrm{JC})_n$ holds for every $n \geq 2$.

If $\mathbb{K} = \mathbb{R}$ and $\operatorname{Jac} F(x) \neq 0$ for any $x \in \mathbb{R}^n$, then we can also ask about the injectivity of a polynomial map $F$ and we have the so called **Real Jacobian Problem**. The answer to this problem was unknown until May '94 when Pinchuk (1994) gave an example showing that the Real Jacobian Problem has a negative answer even in the case of $\mathbb{R}^2$ (so also in the case of $\mathbb{R}^n$, $n \geq 2$).

The two dimensional Jacobian Conjecture $(JC)_2$ (with integer coefficients of polynomials) was formulated by Keller (1939). Note that the Jacobian Conjecture is on Smale's list of "Mathematical Problems for the Next Century" as Problem 16 among 18 problems (Smale, 1998).

In the sequel we recall some important properties of polynomial maps.

THEOREM 2.1 (Białynicki-Birula and Rosenlicht, 1962; Kurdyka and Rusek, 1988) *Every injective polynomial map of* $\mathbb{K}^n$ *is bijective.*

THEOREM 2.2 (Bass, Connell and Wright, 1982; Winiarski, 1979; Yagzhev, 1980) *Every injective polynomial map* $F$ *of* $\mathbb{C}^n$ *is a polynomial automorphism, i.e. the inverse* $F^{-1}$ *exists and is a polynomial mapping.*

Remember that the above theorem is not true in the real case even if $n$=1 and the jacobian of a polynomial mapping $F$ is everywhere different from zero, e. g. $F(x) = x + x^3 : \mathbb{R} \to \mathbb{R}$ is bijective, but $F$ is not a polynomial automorphism. If $F$ is a polynomial automorphism, then it is possible to give a sharp estimate for the degree of its inverse, namely

Theorem 2.3 (Bass, Connell and Wright, 1982; Rusek and Winiarski, 1984) *If $F$ is a polynomial automorphism of $\mathbb{K}^n$, then*

$$\deg F^{-1} \leq (\deg F)^{n-1}$$

*and the above estimation is sharp.*

Finally we recall a theorem about the number of points in the fibre of a polynomial mapping whose jacobian is different from zero everywhere.

Proposition 2.1 (Drużkowski, 1991; Drużkowski and Tutaj, 1992) *Let $F = (F_1, ..., F_n) : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map such that $Jac\, F(x) \neq 0$ for every $x \in \mathbb{K}^n$. Then for every $b \in \mathbb{K}^n$ the equation $F(x) = b$ has only isolated solutions and*

$$\#\{x \in \mathbb{K}^n : F(x) = b\} \leq \deg F_1 \cdot ... \cdot \deg F_n.$$

If $\mathbb{K} = \mathbb{C}$, then this inequality is the well known Bezout inequality. Note that the assumption $\mathrm{Jac}\, F(x) \neq 0$ for every $x \in \mathbb{K}^n$ is essential because of the following

Example 2.1 *Let a dominating polynomial mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the formula*

$$\begin{aligned} F(x, y, z) = [\quad & (x-1)^2(x-2)^2(x-3)^2(x-4)^2(x-5)^2 + \\ & (y+1)^2(y+2)^2(y+3)^2(y+4)^2(y+5)^2, z(y+1), z]. \end{aligned}$$

*Then the equation $F(x, y, z) = (0, 0, 0)$ has only isolated solutions in $\mathbb{R}^3$, but*

$$\begin{aligned} & \#\{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = (0, 0, 0)\} = 25 \\ & > \deg F_1 \cdot \deg F_2 \cdot \deg F_3 = 20. \end{aligned}$$

Because of Pinchuk's example it is interesting to recall certain partial results on the injectivity of real maps. Besides Meisters' and Olech's result just mentioned in connection with $(\mathrm{GASP})_2$, we recall another result of theirs on global univalence in two dimensions.

Theorem 2.4 (Meisters and Olech, 1990) *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ map with non-vanishing jacobian, let $w \in \mathbb{R}^2$ and let $A_w$ denote the convex hull of the set$\{d_x F(w) : x \in \mathbb{R}^2\}$. The map $F$ is injective provided that there exist two linearly independent vectors $u, v \in \mathbb{R}^2$ such that neither $0 \in A_u$ nor $0 \in A_v$.*

N. V. Chau gave an elegant improvement of the above result in the polynomial case.

THEOREM 2.5 (Chau, 1993) *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map with non-vanishing jacobian. If there exists a vector $v \in \mathbb{R}^2$ and $C > 0$ such that*

$$(0,0) \notin \textit{convex hull of } \{d_x F(v) : x \in \mathbb{R}^2, ||x|| > C\},$$

*then $F$ is injective.*

As an immediate consequence of the above theorem we have the following

COROLLARY 2.1 *If a polynomial map $F=(f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ has the property that $\mathrm{Jac}\, F$ and at least one of the four partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ never vanishes on $\mathbb{R}^2$, then $F$ is injective.*

## 2.2. Equivalent formulations of the Jacobian Conjecture

One can check that due to Lefschetz Principle our formulation of the Jacobian Conjecture for $\mathbb{C}$ covers the case of the Jacobian Conjecture formulated for any field $k$ of characteristic zero, see Bass, Connell and Wright (1982), Drużkowski, (1991), van den Essen (2000). Note that up to this time the Jacobian Conjecture remains unsolved even if $n = 2$.

Since $F \in \mathcal{P}(\mathbb{C}^n)$ can be treated as $\widehat{F} \in \mathcal{P}(\mathbb{R}^{2n})$ and $\mathrm{Jac}\, \widehat{F}(x,y) = |\mathrm{Jac}\, F(x+ \boldsymbol{i}\, y)|^2$, it is evident that

$$(JC)_{2n} \ \text{ for } \ \mathbb{R}[X_1, ..., X_{2n}] \Longrightarrow (\mathrm{JC})_n \ \text{ for } \ \mathbb{C}[X_1, ..., X_n],$$

so "the real (GJC)" implies "the complex (GJC)". But we even do not know if

$$\boxed{?} \quad \text{real}\,(JC)_n \quad \Longrightarrow \text{complex}\,(JC)_n\,.$$

In 1987 Meisters and Olech gave an equivalent differential formulation of the Jacobian Conjecture and their result began a series of papers with other conditions of differential type (Stein, 1989; Krasiński and Spodzieja, 1991; Tutaj-Gasińska, 1996).

THEOREM 2.6 (Meisters and Olech, 1987) *Let $F$ be a polynomial map of $\mathbb{R}^n$ and $Jac\ F = const \neq 0$. Consider the following autonomous system of differential equations (the associated Ważewski equation)*

$$\dot{x}(t) = [F'(x)]^{-1}(a), \qquad x(0) = x_0$$

*with an arbitrary initial value $x_0 \in \mathbb{R}^n$ and an arbitrary vector parameter $a \in \mathbb{R}^n$ and denote by $\phi(\cdot, x_0, a)$ the solution of the above differential equation. Then $F$ is a polynomial automorphism if and only if the solution $\phi(\cdot, x_0, a)$ is a poly-flow, i.e. $\phi(t, x_0, a)$ is polynomial in both $x_0$ and $t$.*

Now we present another equivalent formulation of the Jacobian Conjecture. Let $E_n$ denote the ring of entire functions on $\mathbb{C}^n$ (i.e. holomorphic and defined on the whole $\mathbb{C}^n$), let $F_j \in E_n$ for $j = 1, ..., n$ and let $F = (F_1, ..., F_n)$ be a fixed entire mapping of $\mathbb{C}^n$ (we write $F \in E_n$). We endow the space $E_n$ with the standard topology of uniform convergence on compact subsets of $\mathbb{C}^n$. We define the linear differential operators

$$\frac{\partial}{\partial F_i} : E_n \ni g \to \mathrm{Jac}\,(F_1, ..., F_{i-1}, g, F_{i+1}, ..., F_n) \in E_n,\ i = 1, ..., n.$$

If we take $F \in \mathcal{P}(\mathbb{C}^n)$, then $\frac{\partial}{\partial F_i}$ are derivations of the ring $\mathbb{C}[X_1, ..., X_n]$, $j = 1, ..., n$. Stein (1989) formulated the following two dimensional differential Analytic Jacobian Conjecture

$$(AJC)_2\ \ [F \in E_2 \text{ and } \mathrm{Jac}\, F = 1] \quad \Rightarrow \quad [\frac{\partial}{\partial F_1}(E_2) \text{ is dense in } E_2]$$

and proved that $(AJC)_2$ is equivalent to $(\mathrm{JC})_2$ provided that $F$ is a polynomial mapping. Krasiński and Spodzieja (1991) formulated a natural generalization of $(AJC)_2$ to the n-dimensional case

$$[F \in E_n,\ \mathrm{Jac}\, F = 1] \ \Rightarrow\ [\frac{\partial}{\partial F_i}(E_n) \text{ is dense in } E_n,\ i = 1, ..., n-1]$$

and showed that it is equivalent to $(\mathrm{JC})_n$.

We end this section with a folk result, namely a topological formulation of the Jacobian Conjecture.

Remark 2.1 *Let* $F \in \mathcal{P}(\mathbb{C}^n)$, $Jac\, F = 1$ *and* $\delta(F) := \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : F(x) = F(y)\}$. *Then*

$$F \text{ is injective} \iff \delta(F) \text{ is connected.}$$

*Proof.* Assume that $\delta(F)$ is connected (in Zariski or Euclidean topology of $\mathbb{C}^n$). Then $\delta(F)$ is a smooth algebraic manifold and, due to a classical theorem, $\delta(F)$ is a smooth irreducible algebraic set. Evidently the diagonal $\Delta = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : x = y\} \subset \delta(F)$ and $\dim \Delta = \dim \delta(F) = n$. Since $\delta(F)$ is irreducible, we get the equality: $\delta(F) = \Delta$, i. e. $F$ is injective. The converse is obvious. ∎

## 2.3. The Jacobian Conjecture and the Dixmier Conjecture

The derivations $\frac{\partial}{\partial F_i}$ are used to relate the Jacobian Conjecture to the Dixmier Conjecture about the Weyl algebra. Let $k$ denote a field of characteristic 0.

Definition 2.1 *The n-th Weyl algebra over a field k is the k-subalgebra* $W_n = W_n(X_1, ..., X_n)$ *of k-linear endomorphisms of the ring of polynomials* $k[X_1, ..., X_n]$ *generated by the multiplication maps* $f_.$

$$f_. : k[X_1, ..., X_n] \ni g \to fg \in k[X_1, ..., X_n], \quad f \in k[X_1, ..., X_n]$$

*and the k-derivations* $\frac{\partial}{\partial X_i}$ *on* $k[X_1, ..., X_n]$, $i = 1, ..., n$.

We also write $W_n = k[X_1, ..., X_n, \partial_1, ..., \partial_n]$, where $\partial_i := \frac{\partial}{\partial X_i}$, $i = 1, ..., n$. One easily verifies the commutator relations

$$[\partial_i, X_j] = \delta_{i,j}, \quad [\partial_i, \partial_j] = 0, \quad [X_i, X_j] = 0 \text{ for all } i, j.$$

Due to the above equations every element $P \in W_n$ can be written uniquely as a finite sum of the form

$$P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$$

where $\partial^\alpha := \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}$, $a_\alpha = a_{\alpha_1,...,\alpha_n}(X_1, ..., X_n) \in k[X_1, ..., X_n]$, $m \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$. One can easily show the following

PROPOSITION 2.2 *$W_n$ is a simple ring, i.e. every two-sided ideal in $W_n$ is either zero or the whole ring.*

COROLLARY 2.2 *Any non-zero $k$-endomorphism of $W_n$ is injective.*

*Proof.* If $h$ be a nontrivial $k$-endomorphism of $W_n$, then $\ker h \ne W_n$. Since $W_n$ is a simple ring it follows that $\ker h = 0$, so every $k$-endomorphism of $W_n$ is injective. ■

The question what about epimorphism is the subject of the following Dixmier Conjecture (for short (DC) ):

$(\mathrm{DC})_n$ *Every $k$-endomorphism of $W_n$ is an epimorphism (i. e. is an isomorphism).*

In fact only $(\mathrm{DC})_1$ was formulated by Dixmier (1968) and $(\mathrm{DC})_n$ is still unsolved for $n \ge 1$. We show that the Jacobian Conjecture follows from the Dixmier Conjecture.

THEOREM 2.7 *$(DC)_n$ implies $(JC)_n$.*

*Proof.* Let $F := (F_1, ..., F_n) \in k[X_1, ..., X_n]^n$ satisfy $\det JF = 1$ and let

$$\frac{\partial}{\partial F_i} : k[X_1, ..., X_n] \ni g \to \mathrm{Jac}\,(F_1, ..., F_{i-1}, g, F_{i+1}, ..., F_n) \in k[X_1, ..., X_n]$$

for $i = 1, ..., n$. Then it is known (and not difficult to verify) that

(1) $\dfrac{\partial F_j}{\partial F_i} = \delta_{ij}$ for $i, j = 1, ..., n$,

(2) $\left[\dfrac{\partial}{\partial F_i}, \dfrac{\partial}{\partial F_j}\right] = 0$ for $i, j = 1, ..., n$,

(3) $\dfrac{\partial}{\partial F_1}, ..., \dfrac{\partial}{\partial F_n}$ is a commutative $k[X_1, ..., X_n]$-basis of the $k[X_1, ..., X_n]$-module $\mathrm{Der}_k k[X_1, ..., X_n]$ of $k$-derivations of $k[X_1, ..., X_n]$.

We define a $k$-endomorphism $h$ of $W_n$ by

$$h(X_i) := F_i \quad \text{and} \quad h(\partial_i) := \frac{\partial}{\partial F_i}, \qquad i = 1, ..., n.$$

Evidently $\dfrac{\partial}{\partial F_i} = \sum_{j=1}^{n} b_i^j(X_1, ..., X_n)\partial_j$, $i = 1, ..., n$. Since, by our hypothesis, the endomorphism $h$ is surjective, then there exists $P_j \in W_n$ such that $X_j = h(P_j)$ for some $P_j = \sum a_\alpha^j(X_1, ..., X_n)\partial^\alpha \in W_n$, thus

$$X_j = \sum a_\alpha^j(h(X_1), ..., h(X_n))h(\partial)^\alpha = \sum a_\alpha^j(F_1, ..., F_n)\left(\tfrac{\partial}{\partial F}\right)^\alpha.$$

Applying the operator $X_j\,.$ to the element $1 \in k[X_1, ..., X_n]$ we get $X_j = X_j\,.1 = a_0^j(F) \in k[F]$, since $\frac{\partial}{\partial F_i}(1) = 0$ for all $i = 1, ..., n$. Consequently $k[X_1, ..., X_n] \subset k[F] \subset k[X_1, ..., X_n]$, hence $F$ is invertible and $(JC)_n$ holds. ■

In March 2004 Belov (2004) announced that he and Kontsevich proved the implication $(JC)_{2n} \Rightarrow (DC)_n$. If it is true, then we have the following

REMARK 2.2 *(GJC) is equivalent to the Generalized Dixmier Conjecture, i.e.* $(DC)_n$ *holds for any* $n \geq 1$.

### 2.4. Reduction of the degree in (GJC)

We recall the reduction theorems that are used in the investigation of (GJC).

THEOREM 2.8 (Yagzhev, 1980; Bass, Connell and Wright, 1982; Drużkowski, 1983) *If we consider the Generalized Jacobian Conjecture, then it is sufficient to consider, for every* $n > 1$, *only polynomial mappings of the so-called cubic homogeneous form* $F = I + H$, *where* $I$ *denotes the identity,* $H = (H_1, ..., H_n)$ *and* $H_j : \mathbb{K}^n \to \mathbb{K}$ *is a cubic homogeneous polynomial of the degree 3 or* $H_j = 0$, $j = 1, ..., n$.

If the degree of $F$ is less than 3, then the injectivity follows easily, see, e.g., Drużkowski (1991).

THEOREM 2.9 *If* $F = (F_1, ..., F_n) \in \mathcal{P}(\mathbb{K}^n)$ *is a quadratic map, i. e.* $deg\, F := \max\{\deg F_j : j = 1, ..., n\} \leq 2$ *and* $Jac\, F(x) \neq 0$ *for any* $x \in \mathbb{K}^n$, *then* $F$ *is injective.*

As a consequence of Pinchuk's example and Theorem 2.9 we get

REMARK 2.3 *It is impossible to reduce every* $F \in \mathcal{P}(\mathbb{K}^n)$ *to a quadratic map* $\hat{F} \in \mathcal{P}(\mathbb{K}^N)$, $N \geq n$, *preserving injectivity and everywhere non-vanishing jacobian.*

It is easy to check that the cubic homogeneous form (Yagzhev's form) is invariant under the action of the full linear group $GL_n(\mathbb{K})$, i. e. if $F$ has a cubic homogeneous form and $L \in GL_n(\mathbb{K})$, then $L \circ F \circ L^{-1}$ has also a cubic homogeneous form. We have the following

Proposition 2.3 ((Bass, Connell and Wright, 1982; Drużkowski, 1983) *Let* $F = I + H$ *have a cubic homogeneous form. Then*

$$Jac\,F = 1 \Leftrightarrow \ \forall x \in \mathbb{K}^n \quad H'(x) \text{ is a nilpotent matrix.}$$

Note that $H'(x) = 3\tilde{H}(x,x,\cdot)$, where $\tilde{H}$ denotes the unique symmetric three-linear mapping such that $\tilde{H}(x,x,x) = H(x)$. Hence, if $\mathrm{Jac}\,(I+H) = 1$, then by Proposition 2.3 the matrix

$$H_x := \tilde{H}(x,x,\cdot) = \frac{1}{3}H'(x) \quad \text{is nilpotent.}$$

Therefore, for every $x \in \mathbb{K}^n$, there exists the index of nilpotency of the matrix $H_x$, i.e. there exists a natural number $p(x)$ such that ${H_x}^{p(x)} = 0$ and ${H_x}^{p(x)-1} \neq 0$. It is evident, that $1 \leq p(x) \leq 1 + \mathrm{rank}\,H_x \leq n$ for every $x \in \mathbb{K}^n$. We define the index of nilpotency of the mapping $F = I + H$ to be the number

$$\mathrm{ind}\,F := \sup\{p(x) \in \mathbb{N} : {H_x}^{p(x)} = 0,\ {H_x}^{p(x)-1} \neq 0, x \in \mathbb{K}^n\}.$$

Now we present a theorem which allows us to reduce the verification of the Generalized Jacobian Conjecture to the investigation of polynomial mappings of the so called cubic linear form.

Theorem 2.10 ((Drużkowski, 1983, 1993, 2001)) *In order to verify (GJC) it is sufficient to check it only for polynomial mappings* $F = (F_1, ..., F_n)$ *of the cubic linear form, i.e.*

$$\text{(CLF)}\quad F(x) = \begin{pmatrix} x_1 + (a_1x)^3 \\ x_2 + (a_2x)^3 \\ \vdots \\ x_n + (a_nx)^3 \end{pmatrix},$$

*where* $a_j = (a_j^1, ..., a_j^n) \in \mathbb{K}^n$, $a_jx := a_j^1x_1 + ... + a_j^nx_n$, $j = 1, ..., n$, $x \in \mathbb{K}^n$.

*Without loss of generality we can assume that the matrix* $A := [a_j^i : i,j = 1, ..., n]$ *of (CLF) has one of the following properties*

$$\text{(i)}\qquad \exists c \in \mathbb{K}^n,\ A = A_c := \begin{bmatrix} (a_1c)^2a_1^1 & ... & (a_1c)^2a_1^n \\ ... & ... & ... \\ (a_nc)^2a_n^1 & ... & (a_nc)^2a_n^n \end{bmatrix},\ ind\,A = ind\,F$$

*or*

(ii) $A^2 = 0$.

Now we recall a theorem which summarizes a few partial results on the Generalized Jacobian Conjecture.

Theorem 2.11 ((Drużkowski, 1983, 1993; Drużkowski and Rusek, 1985)) *For arbitrary $n > 1$ the following holds:*

*If a polynomial map $F = (F_1, ..., F_n) : \mathbb{K}^n \to \mathbb{K}^n$ with $Jac\, F = 1$ has a cubic linear form and if*

$$rank\, A < 3 \quad or \quad corank\, A < 3 \quad or \quad ind\, F = 1, 2, 3, n,$$

*then $F$ is a polynomial automorphism.*

Meisters (1994) has begun classifying matrices which define the cubic linear polynomial mapping with constant jacobian and Hubbers (1998) has continued this research. Using the result of Hubbers (1998) we get that (JC) is true for $F$ of a cubic linear form if $\operatorname{rank} A < 5$. Hence for polynomial mappings having a cubic linear form (JC) is true if $n < 8$.

### 2.5. Symmetric reduction.

It is possible to reduce (GJC) to Yagzhev's form with symmetric Jacobian matrix $F'(x)$ for any $x \in \mathbb{C}^n$. Let $x = (x_1, ..., x_n)$, $v = (v_1, ..., v_n)$. Let $F$ be a polynomial mapping of the cubic homogeneous form

$$F(X) = (X_1 + H_1, ..., X_n + H_n) : \mathbb{K}^n \to \mathbb{K}^n,$$

where $H_j$ is a cubic homogeneous polynomial of degree 3 or $H_j = 0$, $j = 1, ..., n$ and $\operatorname{Jac} F = 1$. Take

$$g(v, x) = v_1 F_1(x) + ... + v_n F_n(x)$$

and define $G \in \mathcal{P}(\mathbb{K}^{2n})$ by the formula

$$G(v, x) := \nabla g(v, x) := (\frac{\partial g}{\partial v_1}, ..., \frac{\partial g}{\partial v_n}, \frac{\partial g}{\partial x_1}, ..., \frac{\partial g}{\partial x_n}).$$

One easily verifies that

$$G(v, x) = (F_1(x), ..., F_n(x), \sum_{k=1}^{n} v_k \frac{\partial F_k}{\partial x_1}, ..., \sum_{k=1}^{n} v_k \frac{\partial F_k}{\partial x_n})$$

$$= (F(x), (F'(x))^T v\,).$$

Obviously $G$ is injective if and only if $F$ is. We calculate

$$G'(v, x) = \begin{bmatrix} 0 & F'(x) \\ [F'(x))]^T & [F''(x)]^T(v, \cdot) \end{bmatrix}.$$

By Laplace theorem $\operatorname{Jac} G = (\operatorname{Jac} F)^2$. Note that $[F''(x)]^T(v, \cdot)$ is symmetric $n \times n$ matrix, hence $G'(v, x)$ is symmetric $2n \times 2n$ matrix and

$$G'(0, 0) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $I$ is the $n \times n$ identity matrix. Thus there exists an orthogonal matrix $M$ such that

$$M \circ G'(0,0) \circ M^t = E,$$

where

$$E = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}.$$

The mapping $P(y) := M \circ G \circ M^T(y)$ is of the form $Ey + \breve{H}(y)$, where $\breve{H} \in \mathcal{P}(\mathbb{K}^{2n})$ is a cubic homogenous with symmetric Jacobian matrix. If we take complex dilation

$$J = \begin{bmatrix} \boldsymbol{i}I & 0 \\ 0 & I \end{bmatrix}$$

and put $Q(y) := J \circ P \circ J(y)$, then $Q(y) = y + \hat{H}(y)$ where $\hat{H}$ is a complex cubic homogenous with symmetric Jacobian matrix. It is obvious that $\operatorname{Jac} F = \operatorname{Jac} Q$ and $F$ is injective if and only if $Q$ is injective. The trick with a polynomial $g(v, x)$ was probably a folk result, we adopted it from Meng (2003) where the sketch of the reduction to the complex symmetric case is also given.

In this way we show the following

THEOREM 2.12 *It is sufficient in (GJC) to consider only polynomial mappings of the form* $F(x) = x + \hat{H}(x) : \mathbb{C}^n \to \mathbb{C}^n$, *where* $\hat{H} : \mathbb{C}^n \to \mathbb{C}^n$ *is a cubic homogeneous polynomial mapping of the degree 3 and* $\hat{H}'(x)$ *is a complex symmetric nilpotent matrix for any* $x \in \mathbb{C}^n$ $(n > 1)$.

*Added in proof.* I would like to thank the referees for careful reading the manuscript, pointing out misprints and language mistakes and bringing my attention to de Bondt and van den Essen (2003) where the reduction of (GJC) to the complex symmetric case has also been proved.

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