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Two-dimensional stable Lavrentiev phenomenon with and without boundary conditions

by

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Abstract: This work contains examples of regular 2D problems of the Calculus of Variations which exhibit stable Lavrentiev phenomenon, under different types of boundary conditions.

Keywords: multidimensional calculus of variations, Lavrentiev phenomenon.

1. Introduction

Given an open set $\Omega \subset \mathbb{R}^n$ and a function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm} \longrightarrow \mathbb{R}$, we consider the problem of finding

$$\inf_{u \in \mathcal{A}} \left[\int_\Omega f(x, u(x), \nabla u(x))\, dx \right]. \tag{1}$$

Here $\mathcal{A} \subset W^{1,p}(\Omega)$ is a class of admissible mappings from Ω to $\mathbb{R}^m$. For each $p \in [1, +\infty]$, $W^{1,p}(\Omega)$ are the usual Sobolev spaces.

Roughly speaking, the minimization problem is said to exhibit the *Lavrentiev phenomenon* when the above infimum depends on p. In 1926, Lavrentiev (1927) proved that the infimum of the functional $\displaystyle\int_0^1 e^{-\frac{2}{(u-\sqrt{x})^2}} f(u')\, dx$, over the class of absolutely continuous functions, subject to boundary conditions $u(0) = 0$ and $u(1) = 1$, is strictly less than the infimum over the class of continuously differentiable functions meeting the same boundary conditions, with the function f satisfying certain conditions (see Lavrentiev, 1927, pp.23-25). More one-dimensional examples have been constructed after the Lavrentiev's one. The most frequently referred to is the one of Manià from 1934 (see Dacorogna, 1989; Manià, 1934). The integrand for Manià's example is the polynomial $f(x, u, u') = \left(u^3 - x\right)^2 u'^6$ and the associated functional exhibits the $W^{1,1} - W^{1,\infty}$ Lavrentiev gap.

In the 1980's Ball and Mizel (1985) constructed an example, in 1D, of a *regular* functional of the Calculus of Variations which exhibits the Lavrentiev phenomenon. We recall that the problem (1) is called regular when the integrand $f(x, u(x), \nabla u(x))$ is coercive and convex with respect to gradient. There has been an extensive study of this and Manià-type examples in 1D; among publications we mention Loewen (1987). The later class of examples has been extended by constructing in Sarychev (1985) regular autonomous second-order 1D functionals which possess the Lavrentiev gap.

There are several other examples of functionals exhibiting the Lavrentiev phenomenon as well as results of Lipschitzian regularity of minimizers which, obviously, preclude the occurrence of the Lavrentiev phenomenon. Clarke and Vinter (1985) establish the Lipschitzian regularity for a class of autonomous integrands. More recent results on the subject, extending the results presented in Clarke and Vinter (1985), may be found in Torres (2003) and Ornelas (2004).

In comparison with the one-dimensional case, less is known for multidimensional problems. We would like to mention a work by Foss (2003), who constructed functionals, in 2D, whose infimum depends continuously upon the exponent of the Sobolev space from which the competing functions are taken. Some other examples in more than one dimension can be found in Alberti and Majer (1994), Zhikov (1995) and Foss, Hrusa and Mizel (2003). For a survey on this subject, see Buttazzo and Belloni (1995) and references therein.

Here we extend the example of Manià to two dimensions, thus obtaining 2D examples where the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap occurs. Furthermore, using the procedure of Ball and Mizel (1985) and Loewen (1987), we establish that the phenomenon persists for certain perturbations on the integrands, i.e., the occurrence of the phenomenon is stable. Thus we arrive at a class of functionals whose integrands are coercive and convex with respect to gradient and which exhibit the Lavrentiev phenomenon.

We also construct examples for the problems of the calculus of variations with different types of boundary conditions.

2. Main result: 2D example of the Lavrentiev phenomenon

Throughout the paper we will only consider $\Omega =]0,1[\times]0,1[$ and

$$\mathcal{A}^p := \{u \in W^{1,p}(\Omega) : u(s,0) = 0 \wedge u(s,1) = 1, \forall s \in [0,1]\}, \tag{2}$$

for each $p \in]2, +\infty]$.

As is well-known, for the values of p we are considering, $W^{1,p}(\Omega) \subset C(\overline{\Omega})$. Thus, given any $u \in W^{1,p}(\Omega)$, there is always one continuous representative that extends continuously to the boundary. When dealing with pointwise properties of such an u in this paper, we always assume that we are using its continuous representative. In particular, in this sense the consideration of the boundary values in (2) makes sense.

The main result of this paper is the following:

Theorem 2.1 *Let the constants l, k, m, r be positive and satisfy the conditions*

$$k < l < 2k,\ 2m \geq \frac{2k+1}{l-k} l,\ 2 < r < \frac{l}{(l-k)},\ l, m \in \mathbb{N}.$$

Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, the variational problem

$$J_\varepsilon[u] = \int_\Omega \left(u^l - t^k\right)^2 |\nabla u|^{2m} + \varepsilon \, |\nabla u|^r \, ds\, dt \ \longrightarrow \inf,\ \ u(s,0) = 0,\ \ u(s,1) = 1 \tag{3}$$

exhibits the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap for any $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

The integrand in (3) is coercive. Besides, if, in addition, $l < \frac{4}{3}k$, $r \geq 4$, then the integrand is convex with respect to the gradient.

Example 2.1 *We may choose, for example, $k = 4$, $l = 5$, $m = 23$ and $r = 4$; the corresponding variational problem*

$$J[u] = \int_\Omega \left(u^5(s,t) - t^4\right)^2 |\nabla u(s,t)|^{46} + \varepsilon \, |\nabla u|^4 \, ds\, dt \longrightarrow \inf,$$
$$u(s,0) = 0,\ \ u(s,1) = 1$$

exhibits the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap for any $p \in (2,5)$, provided that $\varepsilon > 0$ is sufficiently small.

3. Extension of the Manià's example to 2D

For l, k, m as in Theorem 2.1 we start by studying the (auxiliary) variational problem

$$J[u] = \int_\Omega \left(u^l(s,t) - t^k\right)^2 |\nabla u(s,t)|^{2m} \, ds\, dt \longrightarrow \inf,\ u(s,0) = 0,\ u(s,1) = 1. \tag{4}$$

Obviously, $J[u] \geq 0$ for all $u \in \mathcal{A}^p$, while for $u_0(s,t) = t^{\frac{k}{l}}$, $J[u_0] = 0$, so that by the computation made earlier one gets

$$\inf_{u \in \mathcal{A}^p} J[u] = 0,$$

for $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

We show that the infimum of J is strictly positive over the class $\mathcal{A}^\infty$. Let $u \in \mathcal{A}^\infty$ and let $l > k > 0$ (l, $k \in \mathbb{R}$).

Consider the function β defined by

$$\beta(s) = \inf\left\{t \in]0,1] : \; u(s,t) = \frac{1}{2} t^{\frac{k}{l}}\right\}, \;\; s \in [0,1]. \tag{5}$$

Recall that if $u \in \mathcal{A}^\infty$, then $u(s,0) = 0$ for $s \in [0,1]$ and u is Lipschitzian. Thus $|u(s,t)| = |u(s,t) - u(s,0)| \leq Lt$ for all $s, t \in [0,1]$ and some constant L.

Pick $\underline{t} \in]0,1]$ such that $L\underline{t}^{1-\frac{k}{l}} < \frac{1}{4}$. Since for any $(s,t) \in [0,1] \times]0, \underline{t}]$

$$u(s,t) - \frac{1}{2} t^{\frac{k}{l}} \leq Lt - \frac{1}{2} t^{\frac{k}{l}} \leq \left(Lt^{1-\frac{k}{l}} - \frac{1}{2}\right) t^{\frac{k}{l}},$$

then

$$\forall (s,t) \in [0,1] \times]0, \underline{t}], \; u(s,t) < \frac{1}{2} t^{\frac{k}{l}}. \tag{6}$$

As long as for $t = 1$, $u(s,1) = 1 > 1/2$ and u is continuous, the definition of β in (5) makes sense.

Let $u \in \mathcal{A}^\infty$, $l > k > 0$ and β be the function defined by (5). Consider the function $\alpha(s)$ defined by

$$\alpha(s) = \sup\left\{t \in [0, \beta(s)[: \; u(s,t) = 0\right\}, \;\; s \in [0,1]. \tag{7}$$

Finally for $u \in \mathcal{A}^\infty$ and $\alpha(s), \beta(s)$ as above we define the set

$$A = \left\{(s,t) : s \in [0,1] \, \wedge \, \alpha(s) \leq t \leq \beta(s)\right\}. \tag{8}$$

The set A is crucial for our construction and we call it *pertinent set*.

We would like to integrate over the pertinent set A and for this purpose we have to verify its measurability. This will be accomplished in the next subsection. Now we establish the occurrence of the Lavrentiev phenomenon for the auxiliary problem (4).

To do this we recall the Partial Integration Lemma which can be proved by using standard arguments of Measure and Integration Theory together with the fact that, in our context, the classical derivatives and weak derivatives coincide a. e. (see, for example, Evans and Gariepy, 1992, p.235).

Lemma 3.1 *(Partial Integration Lemma) If $v \in W^{1,\infty}(\Omega)$, $0 \le a \le b \le 1$ and v_t is weak partial derivative, then*

$$\int_a^b v_t(s,t)dt = v(s,b) - v(s,a)$$

for almost all $s \in [0,1]$.

Proposition 3.1 *Let the constants l, k, m be positive and satisfy the conditions*

$$k < l < 2k,\ 2m \ge \frac{2k+1}{l-k} l,\ l,\, m \in \mathbb{N}.$$

Then, the variational problem (4) exhibits the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap for any $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

Proof. The problem is to find the infimum for J over the class $\mathcal{A}^\infty$. Let $u \in \mathcal{A}^\infty$. Let the set A be defined by (8). For each $(s,t) \in A$, there holds $0 \le u(s,t) \le \frac{1}{2} t^{\frac{k}{l}}$. Hence, for $(s,t) \in A$,

$$\left(u^l(s,t) - t^k\right)^2 \ge \left(2^l u^l(s,t) - u^l(s,t)\right)^2 = \left(2^l - 1\right)^2 u^{2l}(s,t), \tag{9}$$

and therefore

$$\begin{aligned} J[u] &\ge \left(2^l - 1\right)^2 \int_A u^{2l}(s,t)\, |\nabla u(s,t)|^{2m}\, dt\, ds \\ &\ge \left(2^l - 1\right)^2 \int_A \left(u^{\frac{l}{m}}(s,t)\, u_t(s,t)\right)^{2m} dt\, ds. \end{aligned}$$

Since the weak partial derivative $\left(u^{\frac{l}{m}+1}\right)_t$ equals $\dfrac{l+m}{m} u^{\frac{l}{m}} u_t$ a. e. in A, by applying Fubini's Theorem we conclude that

$$\begin{aligned} \int_A \left(u^{\frac{l}{m}}(s,t)\, u_t(s,t)\right)^{2m} dt\, ds &= \left(\frac{m}{m+l}\right)^{2m} \int_A \left(\left(u^{\frac{l}{m}+1}\right)_t (s,t)\right)^{2m} dt\, ds \\ &= \left(\frac{m}{m+l}\right)^{2m} \int_\Omega \chi_A \left(\left(u^{\frac{l}{m}+1}\right)_t (s,t)\right)^{2m} dt\, ds. \\ &= \left(\frac{m}{m+l}\right)^{2m} \int_0^1 \left(\int_0^1 \chi_A \left(\left(u^{\frac{l}{m}+1}\right)_t (s,t)\right)^{2m} dt\right) ds \end{aligned}$$

and hence

$$J[u] \ge \left(2^l - 1\right)^2 \left(\frac{m}{m+l}\right)^{2m} \int_0^1 \left(\int_{\alpha(s)}^{\beta(s)} \left(\left(u^{\frac{l}{m}+1}\right)_t (s,t)\right)^{2m} dt\right) ds. \tag{10}$$

For each $s \in [0,1]$, the application of Jensen's inequality yields

$$\int_{\alpha(s)}^{\beta(s)} \left(\left(u^{\frac{l}{m}+1}\right)_t (s,t)\right)^{2m} dt \geq \frac{1}{(\beta(s)-\alpha(s))^{2m-1}} \left(\int_{\alpha(s)}^{\beta(s)} \left(u^{\frac{l}{m}+1}\right)_t (s,t)\, dt\right)^{2m}. \tag{11}$$

Using Lemma 3.1 with $v = u^{\frac{l}{m}+1}$ we obtain

$$\int_{\alpha(s)}^{\beta(s)} \left(u^{\frac{l}{m}+1}\right)_t (s,t)\, dt = u^{\frac{l}{m}+1}(s,\beta(s)) - u^{\frac{l}{m}+1}(s,\alpha(s)) \text{ for a. e. } s \in [0,1].$$

Since $u(s,\alpha(s)) = 0$ and $u(s,\beta(s)) = \frac{1}{2}\beta(s)^{\frac{k}{l}}$, then

$$\left(\int_{\alpha(s)}^{\beta(s)} \left(u^{\frac{l}{m}+1}\right)_t (s,t) dt\right)^{2m} = \left(\frac{1}{2}\right)^{2m\left(\frac{l}{m}+1\right)} \beta^{2m\frac{k}{l}\left(\frac{l}{m}+1\right)}(s) \text{ for a. e. } s \in [0,1]. \tag{12}$$

Suppose first that $2m = \frac{2k+1}{l-k} l$. Then $2m-1 = 2m\frac{k}{l}\left(\frac{l}{m}+1\right)$ and since $\frac{\beta(s)}{\beta(s)-\alpha(s)} \geq 1$ there holds

$$\frac{\beta(s)^{2m\frac{k}{l}\left(\frac{l}{m}+1\right)}}{(\beta(s)-\alpha(s))^{2m-1}} \geq 1. \tag{13}$$

If $2m > \frac{2k+1}{l-k} l$, then $2m\frac{k}{l}\left(\frac{l}{m}+1\right) - 2m + 1 < 0$ [1], and therefore

$$\frac{\beta(s)^{2m\frac{k}{l}\left(\frac{l}{m}+1\right)}}{(\beta(s)-\alpha(s))^{2m-1}} \geq \frac{\beta(s)^{2m\frac{k}{l}\left(\frac{l}{m}+1\right)}}{(\beta(s))^{2m-1}} = \beta(s)^{2m\frac{k}{l}\left(\frac{l}{m}+1\right)-2m+1} > 1. \tag{14}$$

Thus for $2m \geq \frac{2k+1}{l-k} l$, from (10)-(14) one obtains

$$J[u] \geq \frac{\left(2^l-1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}} \left(\frac{m}{l+m}\right)^{2m}.$$

This being valid for all $u \in \mathcal{A}^\infty$, we finally have that

$$\inf_{u\in\mathcal{A}^\infty} J[u] \geq \frac{\left(2^l-1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}} \left(\frac{m}{l+m}\right)^{2m} > 0. \tag{15}$$

■

[1] If $g(x) = \frac{k}{l}\left(\frac{2l}{x}+1\right)x - x + 1$, then g is strictly decreasing and $g\left(\frac{2k+1}{l-k}l\right) = 0$. Thus, for $x > \frac{2k+1}{l-k}l$, one gets $g(x) < 0$.

Remark 3.1 *We would like to draw attention to the fact that the proof allows for the estimation of the Lavrentiev gap. So, we can assert that, for the variational problem*

$$J[u]=\int_\Omega \left(u^l(s,t)-t^k\right)^2 |\nabla u(s,t)|^{2m}\,ds\,dt \longrightarrow \inf,\ u(s,0)=0,\ u(s,1)=1,$$

the Lavrentiev gap $\inf\limits_{u\in\mathcal{A}^\infty} J[u] - \inf\limits_{u\in\mathcal{A}^p} J[u]$ *is not less than* $\dfrac{\left(2^l-1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}}\left(\dfrac{m}{l+m}\right)^{2m}$,
for $p\in\left(2,\frac{1}{1-\frac{k}{l}}\right)$.

Remark 3.2 *In the case* $2m<\dfrac{2k+1}{l-k}l$ *the functional* J *does not exhibit the Lavrentiev phenomenon, since the sequence* $(u_n)_{n\in\mathbb{N}}\subset\mathcal{A}^\infty$ *defined by*

$$u_n(s,t)=\begin{cases} n^{(1-\frac{k}{l})}\,t & , \quad 0\le t\le \frac{1}{n},\ s\in[0,1]\\ t^{\frac{k}{l}} & , \quad \frac{1}{n}\le t\le 1,\ s\in[0,1]\end{cases}$$

is such that $\lim\limits_{n\to+\infty} J[u_n]=0$.

4. Measurability of the pertinent set A

The proof is based on the following technical Lemmas.

Lemma 4.1 *Let* $u\in\mathcal{A}^\infty$ *and let* $l>k>0$ *(*$l,\ k\in\mathbb{R}$*). Then the function* β*, defined by (5), is lower semicontinuous.*

Proof. Recall the definition of $\underline{t}$ in (6). Suppose $s_0\in[0,1]$ and $\beta(s_0)=t_0$. Then $t_0>\underline{t}$ and

$$u(s_0,t_0)-\frac{1}{2}t_0^{\frac{k}{l}}=0\ \wedge\ u(s_0,t)-\frac{1}{2}t^{\frac{k}{l}}\neq 0 \text{ for } t\in]0,t_0[.$$

Due to (6) the last condition can be changed to

$$u(s_0,t)-\frac{1}{2}t^{\frac{k}{l}}<0 \text{ for } t\in]0,t_0[.$$

Let $\delta>0$.

First, let us consider δ such that $t_0-\delta>\underline{t}$. Obviously $u(s_0,t)-\frac{1}{2}t^{\frac{k}{l}}<0$, $\forall t\in[\underline{t},t_0-\delta]$, and

$$\max_{t\in[\underline{t},t_0-\delta]}\left(u(s_0,t)-\frac{1}{2}t^{\frac{k}{l}}\right)=\mu<0.$$

Since u is Lipschitzian,

$$\max_{t\in[\underline{t},t_0-\delta]}\left(u(s,t)-\frac{1}{2}t^{\frac{k}{l}}\right) \leq \max_{t\in[\underline{t},t_0-\delta]}(u(s,t)-u(s_0,t)) + \max_{t\in[\underline{t},t_0-\delta]}\left(u(s_0,t)-\frac{1}{2}t^{\frac{k}{l}}\right)$$
$$< L|s-s_0| + \mu.$$

Thus, there exists $\varepsilon > 0$ such that

$$\max_{t\in[\underline{t},t_0-\delta]}\left(u(s,t)-\frac{1}{2}t^{\frac{k}{l}}\right) < 0, \tag{16}$$

for s verifying $|s-s_0| < \varepsilon$.
From (6) and (16)

$$u(s,t)-\frac{1}{2}t^{\frac{k}{l}} < 0,$$

for $t \in]0,t_0-\delta]$ and $s \in [0,1]$ such that $|s-s_0| < \varepsilon$. Hence, for these s one gets $\beta(s) > t_0-\delta = \beta(s_0)-\delta$.
In the case when δ is such that $]0,t_0-\delta] \subset]0,\underline{t}]$, there holds $u(s,t)-\frac{1}{2}t^{\frac{k}{l}} < 0, \forall (s,t) \in [0,1]\times]0,t_0-\delta]$ and, consequently, we also have $\beta(s) > t_0-\delta = \beta(s_0)-\delta$.

Hence β is lower semicontinuous in $[0,1]$. ■

Lemma 4.2 *Let $u \in \mathcal{A}^\infty$, $l > k > 0$ and β be the function defined by (5). Then the function α, defined by (7), is measurable.*

Proof. It suffices to establish the measurability of the sets

$$M^\gamma = \big\{s \in [0,1] : \alpha(s) \geq \gamma\big\}$$

for every $\gamma \in \mathbb{R}$. It is enough to consider $\gamma \in]0,1[$, since for $\gamma \leq 0$, $M^\gamma = [0,1]$ and for $\gamma \geq 1$, $M^\gamma = \emptyset$, which are clearly measurable.
From the definition of α and the continuity of u, one has the following characterization:

$$M^\gamma = \big\{s \in [0,1] : \exists t \in [\gamma,\beta(s)[: u(s,t) = 0\big\}.$$

Consider the set

$$B^\gamma = \big\{(s,t) \in [0,1]\times[0,1] : \gamma < t < \beta(s)\big\}.$$

This is the intersection of the subgraph of function β with the open half-plane $t > \gamma$. Since the function β is lower semicontinuous, using lower semicontinuity properties (see, for example, Horst, Pardalos and Thoai, 2000, p.13) its subgraph is open in $[0,1]\times\mathbb{R}$ and hence the set B^γ is also open in $[0,1]\times[0,1]$.

For every $n \in \mathbb{N}$ consider

$$B_n = \{(s,t) \in [0,1] \times [0,1] : |u(s,t)| < \frac{1}{n}\},$$

and the intersection $B_n \cap B^\gamma$. Both sets are open in $[0,1] \times [0,1]$.
Consider the projection $\pi_s : (s,t) \to s$ and the set

$$M_n^\gamma = \pi_s(B_n \cap B^\gamma).$$

Since the projection is an open map, the set M_n^γ is open in $[0,1]$ and hence M_n^γ is measurable. By definition of M_n^γ, it follows that

$$M_n^\gamma = \{s \in [0,1] : \exists t \in]\gamma, \beta(s)[: |u(s,t)| < \frac{1}{n}\}. \tag{17}$$

The proof will be complete once we prove that

$$M^\gamma = \bigcap_{n=1}^{\infty} M_n^\gamma. \tag{18}$$

This can be fulfilled as follows.

For $s_0 \in M^\gamma$ either there holds $\alpha(s_0) > \gamma$ or $\alpha(s_0) = \gamma$.

If $\alpha(s_0) > \gamma$, then there exists $t_0 \in]\gamma, \beta(s_0)[$ such that $u(s_0, t_0) = 0$. Thus, $s_0 \in M_n^\gamma$, $\forall n \in \mathbb{N}$, and hence $s_0 \in \bigcap_{n=1}^{\infty} M_n^\gamma$.

If $\alpha(s_0) = \gamma$, then $u(s_0, \gamma) = 0$ and $u(s_0, t) > 0$, for all $t \in]\gamma, \beta(s_0)[$. Let us choose $n_0 \in \mathbb{N}$ such that $\gamma + \dfrac{1}{n_0(L+1)} \in]\gamma, \beta(s_0)[$, where L is the Lipschitz constant of u. Under these conditions,

$$\begin{aligned} 0 < u\left(s_0, \gamma + \frac{1}{n_0(L+1)}\right) &= \left|u\left(s_0, \gamma + \frac{1}{n_0(L+1)}\right) - u(s_0, \gamma)\right| \\ &\leq \frac{L}{n_0(L+1)} < \frac{1}{n_0}. \end{aligned}$$

Thus, for $n \leq n_0$,

$$\exists t \in]\gamma, \beta(s_0)[: |u(s_0, t)| < \frac{1}{n},$$

and, consequently, $s_0 \in M_n^\gamma$.
If $n > n_0$, then $\gamma + \dfrac{1}{n(L+1)} \in]\gamma, \beta(s_0)[$ and

$$0 < u(s_0, \gamma + \frac{1}{n(L+1)}) \leq \frac{L}{n(L+1)} < \frac{1}{n},$$

and hence $s_0 \in M_n^\gamma$. Thus $s_0 \in M_n^\gamma$ for all $n \in \mathbb{N}$ and therefore

$$M^\gamma \subset \bigcap_{n=1}^{\infty} M_n^\gamma. \tag{19}$$

Let now $s_0 \in \bigcap_{n=1}^{\infty} M_n^\gamma$. Hence for all $n \in \mathbb{N}$, $s_0 \in M_n^\gamma$ and, by (17), there exists $t_n \in]\gamma, \beta(s_0)[$ such that $|u(s_0, t_n)| < \frac{1}{n}$. Thus, there exists $t_0 \in [\gamma, \beta(s_0)]$ and a subsequence $(t_{n_k})_{k \in N}$ of $(t_n)_{n \in \mathbb{N}}$ such that $\lim_k t_{n_k} = t_0$. Since u is continuous, $u(s_0, t_0) = 0$ and, from the definition of β, $t_0 \neq \beta(s_0)$. Thus, $s_0 \in M^\gamma$ and hence

$$\bigcap_{n=1}^{\infty} M_n^\gamma \subset M^\gamma. \tag{20}$$

It follows from (19) and (20) that $M^\gamma = \bigcap_{n=1}^{\infty} M_n^\gamma$. The proof is complete. ■

LEMMA 4.3 *The pertinent set A, defined by (8), is measurable.*

Proof. By Lemma 4.1 and using lower semicontinuity properties the *level set* $\{s \in [0,1] : \beta(s) \leq c\}$ is closed for every $c \in \mathbb{R}$. Thus, the function β is measurable. Then from Lemma 4.2 it follows easily that the set $A = \{(s,t) : s \in [0,1] \wedge \alpha(s) \leq t \leq \beta(s)\}$ is measurable. ■

5. Persistence of the Lavrentiev phenomenon

PROPOSITION 5.1 *Let the constants k, l, m be as in the Proposition 3.1 and $2 < r < \dfrac{l}{(l-k)}$. Then there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$ the functional*

$$J_\varepsilon[u] = \int_\Omega \left(u^l(s,t) - t^k\right)^2 |\nabla u(s,t)|^{2m} + \varepsilon\, |\nabla u(s,t)|^r \, ds\, dt,$$

under the boundary conditions $u(s,0) = 0$ and $u(s,1) = 1$ exhibits the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap for any $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

Proof. Consider the functional

$$P[u] = \int_\Omega |\nabla u(s,t)|^r \, ds\, dt = \int_\Omega \left(u_s^2 + u_t^2\right)^{\frac{r}{2}} ds\, dt.$$

Since

$$P\left[t^{\frac{k}{l}}\right] = \int_\Omega \left(\frac{k}{l} t^{\frac{k}{l}-1}\right)^r ds\, dt = \left(\frac{k}{l}\right)^r \frac{1}{r\left(\frac{k}{l} - 1\right) + 1} =: c,$$

and $J_\varepsilon[u] = J[u] + \varepsilon P[u]$, there holds

$$\inf_{u\in\mathcal{A}^p} J_\varepsilon[u] \le \varepsilon P\left[t^{\frac{k}{l}}\right] = \varepsilon \cdot c,$$

for $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

Let $\varepsilon > 0$ be such that

$$\varepsilon < \varepsilon_0 = c^{-1}\frac{\left(2^l - 1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}}\left(\frac{m}{l+m}\right)^{2m}. \tag{21}$$

Since $\inf_{u\in\mathcal{A}^\infty} J[u] \ge \frac{\left(2^l - 1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}}\left(\frac{m}{l+m}\right)^{2m}$, for $0 < \varepsilon < \varepsilon_0$ there holds, for $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$,

$$\inf_{u\in\mathcal{A}^p} J_\varepsilon[u] < \varepsilon_0 \cdot c \le \inf_{u\in\mathcal{A}^\infty} J[u],$$

that is

$$\inf_{u\in\mathcal{A}^p} J_\varepsilon[u] < \inf_{u\in\mathcal{A}^\infty} J[u]. \tag{22}$$

Furthermore,

$$\inf_{u\in\mathcal{A}^\infty} J[u] \le \inf_{u\in\mathcal{A}^\infty} \left(J[u] + \varepsilon P[u]\right) = \inf_{u\in\mathcal{A}^\infty} J_\varepsilon[u]. \tag{23}$$

Thus, by (22) and (23) it follows that, for $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$,

$$\inf_{u\in\mathcal{A}^p} J_\varepsilon[u] < \inf_{u\in\mathcal{A}^\infty} J_\varepsilon[u],$$

and therefore the functional J_ε exhibits the Lavrentiev phenomenon. ■

Example 5.1 *For the variational problem of Example 2.1 we have persistence of the Lavrentiev phenomenon as long as we keep ε between zero and* $\frac{\left(2^5-1\right)^2 5^3}{2^{64}}\left(\frac{23}{28}\right)^{46}$.

The integrand $f_\varepsilon(s,t,u,u_s,u_t) = \left(u^l(s,t) - t^k\right)^2 \left|\nabla u(s,t)\right|^{2m} + \varepsilon\left|\nabla u\right|^r$ is coercive but might not be convex with respect to $\nabla u = (u_s, u_t)$. However, an appropriate choice for l ($k < l < \frac{4}{3}k$) and r ($4 \le r < \frac{l}{(l-k)}$) guarantees convexity of f_ε.

Proof of Theorem 2.1. Let l, k, r and m be constants under conditions of Theorem 2.1. By Proposition 5.1 there exists ε_0 such that for all $0 < \varepsilon < \varepsilon_0$

the problem (3) exhibits the Lavrentiev phenomenon. Moreover, with the more restrictive hypotheses $k < l < \frac{4}{3}k$ and $4 \leq r < \frac{l}{(l-k)}$, the integrand $f_\varepsilon(s,t,u,u_s,u_t) = \left(u^l(s,t) - t^k\right)^2 |\nabla u(s,t)|^{2m} + \varepsilon |\nabla u|^r$ is also convex with respect to gradient.

Thus we arrive at a class of functionals whose integrands are convex and coercive with respect to gradient and which exhibit the Lavrentiev phenomenon. ■

6. Lavrentiev phenomenon with free boundary

The functional J defined in (4) does not exhibit the Lavrentiev phenomenon when one of the boundary conditions, $u(s,0) = 0$ or $u(s,1) = 1$ is omitted. However, starting from this functional one can construct a functional without boundary conditions which possesses the Lavrentiev gap.

LEMMA 6.1 *Let* $t_0 \in]0,1[$, $u \in W^{1,\infty}(\Omega)$ *such that* $u(s,t) < \dfrac{1}{2}t^{\frac{k}{l}}$, $\forall (s,t) \in [0,1]\times]0,t_0]$, l *and* k *be as in Proposition 3.1 and* D *a closed subset of* $[0,1]$. *Then the sets*

$$B_1 = \left\{ s \in D : \exists t \in]t_0, 1] : u(s,t) \geq \frac{1}{2}t^{\frac{k}{l}} \right\} \tag{24}$$

and

$$B_2 = \left\{ s \in D : u(s,t) < \frac{1}{2}t^{\frac{k}{l}}, \forall t \in]0,1] \right\}, \tag{25}$$

are measurable.

Proof. Due to the continuity of u, the following sets

$$B = \left\{ (s,t) \in D \times [t_0, 1] : u(s,t) > \frac{1}{2}t^{\frac{k}{l}} \right\},$$
$$C_n = \left\{ (s,t) \in D \times [t_0, 1] : |u(s,t) - \frac{1}{2}t^{\frac{k}{l}}| < \frac{1}{n} \right\},\ n \in \mathbb{N}$$

are open in $D \times [t_0, 1]$.

Consider the projection π_s onto the first component, and the corresponding images

$$\pi_s(B) = B_1' = \left\{ s \in D : \exists t \in]t_0, 1] : u(s,t) - \frac{1}{2}t^{\frac{k}{l}} > 0 \right\}$$

and

$$\bigcap_{n=1}^{\infty} \pi_s(C_n) = B_1'' = \left\{ s \in D : \exists t \in]t_0, 1] : u(s,t) - \frac{1}{2}t^{\frac{k}{l}} = 0 \right\}.$$

Since

$$B_1 = B_1' \cup B_1''$$

and the sets B_1' and $\pi_s(C_n)$ are open in D, then the sets B_1' and B_1'' are measurable. Thus, B_1 is measurable and hence B_2 is measurable as far as $B_2 = D \backslash B_1$. ■

Proposition 6.1 *Let the constants l, k, m be as in Proposition 3.1. Then the variational problem*

$$\overline{J}[u] = \int_\Omega \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\, dt \longrightarrow \inf,\ u(s,0) = 0 \quad (26)$$

exhibits the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap for any $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

Proof. Since $\overline{J}$ is nonnegative and $\overline{J}\left[t^{\frac{k}{l}}\right] = 0$ it follows that

$$\inf_{u \in \mathcal{A}^p} \overline{J}[u] = 0,$$

for $p \in \left(2, \dfrac{1}{1-\frac{k}{l}}\right)$.

It follows from (15) that

$$\inf_{u \in \mathcal{A}^\infty} \overline{J}[u] \geq \inf_{u \in \mathcal{A}^\infty} J[u] \geq \frac{\left(2^l - 1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}} \left(\frac{m}{l+m}\right)^{2m} > 0,$$

and therefore the variational problem $\overline{J}[u] \longrightarrow \inf$, $u(s,0) = 0$, $u(s,1) = 1$ exhibits the Lavrentiev phenomenon.

For $p \in]2, +\infty]$ consider

$$\mathcal{A}_0^p := \left\{u \in W^{1,p}(\Omega) : u(s,0) = 0, \quad \forall s \in [0,1]\right\}. \quad (27)$$

Since $t^{\frac{k}{l}} \in \mathcal{A}_0^p$ when $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$ we have

$$\inf_{u \in \mathcal{A}_0^p} \overline{J}[u] = 0,$$

for same p.

We wish to show that

$$\inf_{u \in \mathcal{A}_0^\infty} \overline{J}[u] > 0.$$

Suppose first that l is odd. Recall that any $u \in \mathcal{A}_0^\infty$ is Lipschitzian and $u(s,0) = 0$. Then for some constant L

$$u(s,t) \leq |u(s,t) - u(s,0)| \leq Lt$$

for all $s, t \in [0,1]$.

Let $\underline{t} \in]0,1]$ be such that $L\underline{t}^{1-\frac{k}{l}} < \frac{1}{4}$. The argument used to prove Lemma 4.1 yields

$$\forall (s,t) \in [0,1]\times]0,\underline{t}],\ u(s,t) - \frac{1}{2}t^{\frac{k}{l}} < 0. \tag{28}$$

Consider $D = [0,1]$ in (24) and (25), and $t_0 = \underline{t}$ in (24) . By Lemma 6.1 both sets B_1 and B_2, defined by (24)-(25), are measurable and either $|B_1| \geq \frac{1}{2}$ or $|B_2| \geq \frac{1}{2}$, where $|.|$ is the Lebesgue's measure.

Suppose first that $|B_2| \geq \frac{1}{2}$. Since l is odd, then

$$u(s,t) - \frac{1}{2}t^{\frac{k}{l}} < 0 \Longleftrightarrow u^l(s,t) < \frac{1}{2^l}t^k \Longrightarrow \left(u^l - t^k\right)^2 > \left(\frac{1}{2^l} - 1\right)^2 t^{2k}.$$

Applying Fubini's Theorem we obtain for odd l and $u \in \mathcal{A}_0^\infty$ such that $|B_2| \geq \dfrac{1}{2}$:

$$\begin{aligned}\overline{J}[u] &= \int_\Omega \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\,dt \geq \int_\Omega \left(u^l(s,t) - t^k\right)^2 ds\,dt \\ &\geq \left(\frac{1}{2^l} - 1\right)^2 \int_{B_2} \left(\int_0^1 t^{2k} dt\right) ds = \frac{1}{2}\left(\frac{1}{2^l} - 1\right)^2 \frac{1}{2k+1} = c_1 > 0. \end{aligned} \tag{29}$$

Now if $|B_2| < \frac{1}{2}$, then $|B_1| \geq \frac{1}{2}$. Suppose $s \in B_1$ and let the function β be defined by

$$\beta(s) = \inf\left\{t \in]0,1] :\ u(s,t) = \frac{1}{2}t^{\frac{k}{l}}\right\}. \tag{30}$$

By the definition of B_1 and (28) $\exists t \in]\underline{t},1]$ such that $u(s,t) = \dfrac{1}{2}t^{\frac{k}{l}}$. Thus the definition of β makes sense.

Let

$$\alpha(s) = \sup\left\{t \in [0,\beta(s)[:\ u(s,t) = 0\right\}. \tag{31}$$

Arguments similar to those by which we proved Lemma 4.1 and 4.2 allow us to conclude that β is lower semicontinuous and α is measurable.

Consider the set $\Omega_2 = \left\{(s,t) : s \in B_1 \wedge \alpha(s) \leq t \leq \beta(s)\right\}$. For each $(s,t) \in \Omega_2$, there holds $0 \leq u(s,t) \leq \frac{1}{2}t^{\frac{k}{l}}$. Hence, for $(s,t) \in \Omega_2$

$$\left(u^l(s,t) - t^k\right)^2 \geq \left(2^l u^l(s,t) - u^l(s,t)\right)^2 = \left(2^l - 1\right)^2 u^{2l}(s,t)$$

and therefore,

$$\begin{aligned}\overline{J}[u] &= \int_\Omega \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\,dt \\ &\geq \int_{\Omega_2} \left(u^l(s,t) - t^k\right)^2 \left(u_t^{2m}(s,t)\right) ds\,dt \\ &\geq \left(1 - 2^l\right)^2 \int_{\Omega_2} u^{2l}(s,t) u_t^{2m}(s,t) dt\,ds.\end{aligned}$$

Then, as in Proposition 3.1, we conclude

$$\overline{J}[u] \geq \frac{1}{2}\frac{\left(2^l-1\right)^2}{2^{2m\left(\frac{l}{m}+1\right)}}\left(\frac{m}{l+m}\right)^{2m} = c_2 > 0, \tag{32}$$

for $u \in \mathcal{A}_0^\infty$ such that $|B_2| < \dfrac{1}{2}$.

Thus by (29) and (32), for odd l

$$\inf_{u\in\mathcal{A}_0^\infty} \overline{J}[u] \geq c > 0. \tag{33}$$

If l is even, then $|u(s,t)| = |u(s,t)-u(s,0)| \leq Lt$ and hence

$$|u(s,t)| - \frac{1}{2}t^{\frac{k}{l}} < 0,\ \forall (s,t) \in [0,1]\times]0,\underline{t}].$$

In this case substituting u by $|u|$ in (24),(25),(31),(30), we are able to accomplish the proof in the same way as in the previous case.

Therefore also for even l (33) holds. ■

Consider $\Omega_1 =]0,1[\times]-1,1[$. Reflecting with respect to the axis Os the example provided in Proposition 6.1 we can obtain (following Dani, Hrusa and Mizel, 2000) a problem without any boundary conditions which exhibits the Lavrentiev phenomenon.

PROPOSITION 6.2 *Let the constants l, k, m be strictly positive integers as in Proposition 3.1 and consider the variational problem*

$$\overline{\overline{J}}[u] = \int_{\Omega_1}\left(u^l(s,t)-t^k\right)^2\left(|\nabla u(s,t)|^{2m}+1\right)ds\,dt \longrightarrow \inf. \tag{34}$$

If both l and k are odd, then the functional $\overline{\overline{J}}$ exhibits the $W^{1,p}$–$W^{1,\infty}$ Lavrentiev gap for any $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$. Otherwise, $\overline{\overline{J}}$ does not exhibit the Lavrentiev gap.

Proof. We consider three cases:

- Assume that k is even.

Since $\overline{\overline{J}}\left[|t|^{\frac{k}{l}}\right] = 0$, we have

$$\inf_{u\in W^{1,p}(\Omega_1)} \overline{\overline{J}}[u] = 0, \quad \text{for} \quad p \in \left(2, \frac{1}{1-\frac{k}{l}}\right).$$

Consider the sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\infty}(\Omega_1)$ defined by

$$u_n(s,t) = \begin{cases} |t|^{\frac{k}{l}} & , \quad -1 \leq t \leq -\frac{1}{n},\ s\in[0,1] \\ \left(\frac{1}{n}\right)^{\frac{k}{l}} & , \quad -\frac{1}{n} \leq t \leq \frac{1}{n},\ s\in[0,1] \\ t^{\frac{k}{l}} & , \quad \frac{1}{n} \leq t \leq 1,\ s\in[0,1] \end{cases}.$$

By a straightforward computation we conclude that

$$\lim \overline{\overline{J}}[u_n] = 0,$$

and hence

$$\inf_{u \in W^{1,p}(\Omega_1)} \overline{\overline{J}}[u] = \inf_{u \in W^{1,\infty}(\Omega_1)} \overline{\overline{J}}[u] = 0.$$

- Assume now k to be odd and l even.

Observe that for all $u \in W^{1,p}(\Omega_1)$ and $p \in]2, \infty]$,

$$\begin{aligned} \overline{\overline{J}}[u] &= \int_{\Omega_1} \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\, dt \\ &\geq \int_{]0,1[\times]-1,0[} \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\, dt \end{aligned}$$

and since

$$\left(u^l(s,t) - t^k\right)^2 = u^{2l} + t^{2k} - 2t^k u^l \geq t^{2k},\ \forall (s,t) \in]0,1[\times]-1,0[,$$

we get

$$\overline{\overline{J}}[u] \geq \frac{1}{2k+1}, \forall u \in W^{1,p}(\Omega_1),\ \forall p \in]2, +\infty].$$

Consider now the sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\infty}(\Omega_1)$ defined by

$$u_n(s,t) = \begin{cases} 0 & , \quad -1 \leq t \leq -\frac{1}{n},\ s \in [0,1] \\ t + \frac{1}{n} & , \quad -\frac{1}{n} \leq t \leq 0,\ s \in [0,1] \\ \frac{1}{n} & , \quad 0 \leq t \leq \left(\frac{1}{n}\right)^{\frac{l}{k}},\ s \in [0,1] \\ t^{\frac{k}{l}} & , \quad \left(\frac{1}{n}\right)^{\frac{l}{k}} \leq t \leq 1,\ s \in [0,1] \end{cases}.$$

A straightforward computation shows that

$$\lim \overline{\overline{J}}[u_n] = \frac{1}{2k+1},$$

and hence

$$\inf_{u \in W^{1,p}(\Omega_1)} \overline{\overline{J}}[u] = \frac{1}{2k+1},\ \forall p \in]2, +\infty].$$

- Consider now k and l both odd.

Since $\overline{\overline{J}}\left[t^{\frac{k}{l}}\right] = 0$ and $\overline{\overline{J}}$ is nonnegative, we conclude that

$$\inf_{u \in W^{1,p}(\Omega_1)} \overline{\overline{J}}[u] = 0, \quad \text{for} \quad p \in \left(2, \frac{1}{1 - \frac{k}{l}}\right).$$

Consider $u \in W^{1,\infty}(\Omega_1)$ and the sets

$$D_1 = \{s \in [0,1] : u(s,0) \geq 0\}$$

and

$$D_2 = \{s \in [0,1] : u(s,0) \leq 0\}.$$

Both sets are closed in $[0,1]$ and either $|D_1| \geq \frac{1}{2}$ or $|D_2| \geq \frac{1}{2}$.
Suppose that $|D_2| \geq \frac{1}{2}$ and consider $D = D_2$ in (24) and (25).

Then

$$\begin{aligned}\overline{\overline{J}}[u] &= \int_{\Omega_1} \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\, dt \\ &\geq \int_{D_2\times]0,1[} \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) ds\, dt.\end{aligned}$$

As far as u is Lipschitzian, then for $(s,t) \in D_2 \times [0,1]$ we obtain

$$u(s,t) \leq u(s,t) - u(s,0) \leq |u(s,t) - u(s,0)| \leq Lt.$$

Choosing $\underline{t} \in]0,1]$ as in the proof of the Proposition 6.1, we get

$$u(s,t) - \frac{1}{2}t^{\frac{k}{l}} < 0,\ \forall (s,t) \in D_2\times]0,\underline{t}],$$

and the proof of this case is analogous to the proof of Proposition 6.1 in $D_2\times]0,1[$. Observe that since $|B_1|+|B_2| \geq \frac{1}{2}$, then either $|B_1| \geq \frac{1}{4}$ or $|B_2| \geq \frac{1}{4}$.
Now if $|D_2| < \frac{1}{2}$, then $|D_1| \geq \frac{1}{2}$.
Since

$$\overline{\overline{J}}[u] = \int_{\Omega_1} \left((-u)^l(s,t) - (-t)^k\right)^2 \left(|\nabla(-u)(s,t)|^{2m} + 1\right) ds\, dt$$

then by the change of variable $t \mapsto -t$, we get

$$\overline{\overline{J}}[u] = \int_{\Omega_1} \left((-u)^l(s,-t) - t^k\right)^2 \left(|\nabla(-u)(s,-t)|^{2m} + 1\right) ds\, dt.$$

Since $\forall s \in D_1,\ -u(s,-0) \leq 0$ this case is similar to the previous one.

Thus

$$\overline{\overline{J}}[u] \geq c_0 > 0,\ \forall u \in W^{1,\infty}(\Omega_1)$$

and therefore

$$\inf_{u\in W^{1,\infty}(\Omega_1)} \overline{\overline{J}}[u] \geq c_0. \tag{35}$$

■

PROPOSITION 6.3 *Let the constants l, k, m, r be as in Proposition 5.1 and assume that l and k are both odd. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the functional*

$$\overline{\overline{J}}_\varepsilon[u] = \int_{\Omega_1} \left(u^l(s,t) - t^k\right)^2 \left(|\nabla u(s,t)|^{2m} + 1\right) + \varepsilon |\nabla u|^r \, ds \, dt \tag{36}$$

exhibits the $W^{1,p} - W^{1,\infty}$ Lavrentiev gap for any $p \in \left(2, \frac{1}{1-\frac{k}{l}}\right)$.

Proof. The proof is analogous to the proof of Proposition 5.1. ∎

7. Concluding remark

As one can see, when one attempts to extend the examples of the Lavrentiev phenomenon onto multidimensional domains, additional technical difficulties appear. In addition to establishing the presence of the gap, one needs to verify the regularity (e.g. measurability) of sets and functions involved into the construction. In the present paper this kind of difficulties is dealt with in Section 4.

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