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Existence and uniqueness of solutions to wave equations with nonlinear degenerate damping and source terms

by

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Abstract: In this article we focus on the global well-posedness of the differential equation $u_{tt} - \Delta u + |u|^k j'(u_t) = |u|^{p-1} u$ in $\Omega \times (0,T)$, where j' denotes the derivative of a C^1 convex and real valued function j. The interaction between degenerate damping and a source term constitutes the main challenge of the problem. Problems with *non-degenerate damping* ($k = 0$) have been studied in the literature (Georgiev and Todorova, 1994; Levine and Serrin, 1997; Vitillaro, 2003). Thus the degeneracy of monotonicity is the main novelty of this work. Depending on the level of interaction between the source and the damping we characterize the domain of the parameters p, m, k, n (see below) for which one obtains existence, regularity or finite time blow up of solutions. More specifically, when $p \le m+k$ global existence of generalized solutions in $H^1 \times L_2$ is proved. For $p > m + k$, solutions blow up in a finite time. Higher energy solutions are studied as well. For $H^2 \times H^1$ initial data we obtain both local and global solutions with the same regularity. Higher energy solutions are also proved to be unique.

Keywords: wave equations, damping and source terms, weak solutions, sub-differential, blow-up of solutions, energy estimates.

1. Introduction

Let $j(s)$ be a C^1 convex, real valued function defined on $\mathbb{R}$ and let j' denotes the derivative of j. The following assumptions are imposed throughout the paper.

ASSUMPTION 1.1 *There exist positive constants* c, c_0, c_1, c_2 *such that* $\forall s, v \in R$:
1. $j(s) \geq c|s|^{m+1}, \quad |j'(s)| \leq c_0|s|^m + c_1$
2. $(j'(s) - j'(v))(s - v) \geq c_2|s - v|^{m+1}$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ. We consider the following initial-boundary value problem:

$$\begin{aligned} &u_{tt} - \Delta u + |u|^k j'(u_t) = |u|^{p-1}\, u, \quad \text{ in } \Omega \times (0, T) \equiv Q_T, \\ &u(x, 0) = u_0(x) \in H_0^1(\Omega), \quad u_t(x, 0) = u_1(x) \in L_2(\Omega), \\ &u = 0, \ \textit{on } \Gamma \times (0, T), \end{aligned} \tag{1}$$

where the problem is studied for positive k, m, p and such that: $k \leq \frac{n}{n-2}$, $p + 1 < \frac{2n}{n-2}$, if $n \geq 3$. This paper is concerned with the long-time behavior of solutions to the initial-boundary value problem (1). Of central interest is the relationship of the source and damping terms to the behavior of solutions. Interestingly, the partial differential equation in (1) is a special case of the prototype evolution equation

$$u_{tt} - \Delta u + \mathcal{R}(x, t, u, u_t) = \mathcal{F}(x, u), \tag{2}$$

where in (2) the nonlinearities satisfy the structural conditions:

$$v\mathcal{R}(x, t, u, v) \geq 0, \ \mathcal{R}(x, t, u, 0) = \mathcal{F}(x, 0) = 0, \ \text{and } \mathcal{F}(x, u) \sim |u|^{p-1}\, u \text{ for large } |u|\,.$$

Various special cases of (2) arise in quantum field theory and some important mechanical applications. See for example Jörgens (1961) and Segal (1963).

A benchmark equation, which is a special case of (1), is the following well-known polynomially damped wave equation studied extensively in the literature (see for instance Pitts and Rammaha, 2002; Rammaha and Strei, 2002):

$$\begin{aligned} &u_{tt} - \Delta u + |u|^k\, |u_t|^{m-1}\, u_t = |u|^{p-1}\, u, \quad \text{in } \Omega \times (0, T), \\ &u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad \text{in } \Omega, \\ &u(x, t) = 0 \text{ on } \Gamma \times (0, T)\,. \end{aligned} \tag{3}$$

Indeed, by taking $j(s) = \frac{1}{m+1}|s|^{m+1}$ we easily verify that Assumption 1.1 is satisfied. It is also easy to see in this case that problem (1) is equivalent to (3). It is worth noting here that when the damping term $|u|^k\, |u_t|^{m-1}\, u_t$ is absent, the source term $|u|^{p-1}\, u$ drives the solution of (3) to blow-up in finite time, Glassey (1973), Levine (1974), Payne and Sattinger (1981), Tsutsami (1972). In addition, if the source term $|u|^{p-1}\, u$ is removed from the equation, then damping terms of various forms are known to yield existence of global solutions, (see Agre and Rammaha, 2001; Barbu, 1976; Ang and Dinh, 1988; Haraux, 1981). However, when both damping and source terms are present in the equation, then the analysis of their interaction and their influence on the global behavior of solutions becomes more difficult. We refer the reader to Georgiev and Todorova (1994), Levine, Park and Serrin (1997), Pitts and Rammaha (2002), Rammaha and Strei (2002), Serrin, Todorova and Vitillaro (2003), Todorowa (1998), and the references therein.

It should be noted that if $k = 0$ and $p = 0$ then equation (1) can be treated via the theory of monotone operators and the full well-posedness of strong solutions (in the terminology of monotone operator theory) is now classical, Barbu (1976). In addition, with $k = 0$, the presence of a locally Lipschitz source term from $H^1(\Omega)$ into $L_2(\Omega)$ does not affect the arguments for establishing the existence of local solutions via perturbation theory of monotone operators. Moreover, if $p \leq m$ then one can derive the necessary a priori bounds that guarantee that every local solution is indeed global in time, Georgiev and Todorova (1994), Levine and Serrin (1997), Pitts and Rammaha (2002).

The situation, however is different when the damping term is degenerate. From the applications point of view degenerate problems of this type arise quite often in specific physical contexts: for example when the friction is modulated by the strains. However, from the mathematical point of view this leads to degeneracy of the monotonicity argument. In fact, when $k > 0$, (3) is no longer a locally Lipschitz perturbation of a monotone problem (even in the case when $p \leq \frac{n}{n-2}$, i.e., the source term is a locally Lipschitz function from $H^1(\Omega)$ into $L_2(\Omega)$). Thus, the monotone operator theory does not apply. This fact combined with a potential strong growth of the damping term (particularly acute when $m > 1$) makes the problem interesting and the analysis more subtle.

One of the fundamental issues that one has to deal with is a correct definition of a solution and its relation to the equation. The problem with degenerate damping has been first addressed in Levine and Serrin (1997), where global *nonexistence* of solutions was shown for the case $k + m < p$ under several other restrictions imposed on the parameters n, k, m, p. However, Levine and Serrin (1997) provide only negative results (blow up of solutions in a finite time) without any assurance that a relevant local solutions does indeed exist. In fact, proving existence of solutions in the degenerate case turned out to be the main issue. The techniques previously developed for monotone and non-degenerate models are no longer applicable. For this reason, the global non-existence result in Levine and Serrin (1997) has attracted considerable attention to the problem. It became clear that in order to justify fully the meaning of the global nonexistence, one must prove local existence of solutions. In fact, the first result in this direction is given in the recent paper, Pitts and Rammaha (2002), where the case of *sub-linear* damping, i.e., $m < 1$ is treated. For this case Pitts and Rammaha (2002) established local and global (when $m + k \geq p$) existence and uniqueness. In addition, the blow up of solutions (when $m + k < p$) is also proved in Pitts and Rammaha (2002) for the relevant class of solutions. While the case $m < 1$ has been fully understood, nothing was known until recently about the most challenging super-linear damping $m \geq 1$. Clearly, the techniques used for the sub-linear case in Pitts and Rammaha (2002) and based on Schauder's fixed point theorem can no longer be applied.

It is the main goal of this paper to address the issue of local and global existence for the *super-linear* case. It turns out that in order to deal with the problem one needs to introduce a suitable concept of solutions that is based on variational inequalities. As a consequence, a natural setting of the problem is within the realm of multi-valued analysis. The recent manuscript, Barbu, Lasiecka and Rammaha (2005), provides a detailed study of this problem in a more general framework of nonlinear damping given by a sub-differential ∂j of a convex function that is not necessarily smooth. In this paper we restrict our attention to differentiable convex functions $j(s)$. In that case a more direct analysis can be provided and additional regularity properties of solutions can be established. Thus, the main goal of the present short manuscript is to highlight some of the results given in Barbu, Lasiecka and Rammaha (2005) and to explain, in a more direct framework, the basic ideas and concepts used for the proof of the well-posedness for the degenerate model. In addition, we complement the study in Barbu, Lasiecka and Rammaha (2005) by addressing the issue of uniqueness of solutions.

In order to proceed with the presentation of our results we shall introduce the appropriate definitions of solutions. First, we give the definition of a *generalized solution*, which satisfies a certain variational inequality. In discussing finite energy solutions (i.e., $(u, u_t) \in H^1(\Omega)\times L_2(\Omega)$) we shall impose another restriction on the parameters p, m, k :

$$p \leq \max\{\frac{p^*}{2}, \frac{p^*m+k}{m+1}\}; \quad p^* \equiv \frac{2n}{n-2}. \tag{4}$$

Remark 1.1 *We note here that the range of values of the parameter* p *is beyond what is required for the source term to be a locally Lipschitz function from* $H^1(\Omega)$ *into* $L_2(\Omega)$*), as typically assumed in the literature, Pitts and Rammaha (2002), and even in the monotone non-degenerate case, Georgiev and Todorova (1994).*

Definition 1.1 *A function* $u \in C_w([0,T], H_0^1(\Omega)) \cap C_w^1([0,T], L_2(\Omega))$ *with* $|u|^k j(u_t) \in L_1(\Omega\times(0,T))$ *and under the condition (4) is said to be a generalized solution to (1) if and only if for all* $0 < t \leq T$ *the following inequality holds:*

$$\int_0^t\int_\Omega (u_tv_t - \nabla u\nabla v)dxdt + \frac{1}{2}\int_\Omega [u_t^2(t) + |\nabla u(t)|^2]dx + \int_0^t\int_\Omega |u|^k[j(u_t) - j(v)]dxdt$$
$$\leq \int_0^t\int_\Omega |u|^{p-1}u(u_t - v)dxdt + \frac{1}{2}\int_\Omega [u_1^2 + |\nabla u_0|^2 - 2u_1v(0)]dx \tag{5}$$

for all functions v *satisfying*

$$v \in H^1(0,T; L_2(\Omega)) \cap L_2(0,T; H_0^1(\Omega)) \cap L_\infty(\Omega\times(0,T)), \quad v(t) = 0.$$

- We note here that, if u is a generalized solution to (1), then u satisfies $|u|^k j'(u_t) \in L_r(Q_t)$ where $r = \frac{p^*(m+1)}{k+p^*m} > 1$ and $|u|^p|u_t| \in L_1(Q_t)$.

- It should be noted here that Definition 1.1 is a proper extension of the notion of classical solutions. Indeed, if u is a sufficiently smooth *generalized* solution, then u satisfies the classical definition of "weak" solution. To see this it suffices to take in Definition 1.1 the test function $v(t) = u_t(t)+\psi(t)$, where $\psi \in H^{1,1}(Q_t) \cap L_\infty(Q_t)$. Integration by parts and accounting for cancellation of terms yields classical variational definition of *weak* solutions given by *equality*:

$$\int_0^t \int_\Omega (-u_t v_t + \nabla u \nabla v) dxdt + \int_\Omega u_1 v(0) dx + \int_0^t \int_\Omega |u|^k j'(u_t) v dxdt$$
$$= \int_0^t \int_\Omega |u|^{p-1} uv dxdt, \tag{6}$$

for all v satisfying $v \in H^1(0,T; L_2(\Omega)) \cap L_2(0,T; H_0^1(\Omega)), \quad v(t) = 0.$
It should be noted here that the above definition is equivalent to

$$\Box u = -|u|^k j'(u_t) + |u|^{p-1} u, \text{ a.e. } (x,t) \in \Omega \times (0,T),$$

where $\Box \equiv \dfrac{d^2}{dt^2} - \Delta$ is understood in the sense of distributions.

The following notation will be used in the sequel:

$$|u|_{s,\Omega} \equiv |u|_{H^s(\Omega)} \text{ and } \|u\|_p \equiv \|u\|_{L_p(\Omega)},$$

where $H^s(\Omega)$ and $L_p(\Omega)$ stands for the classical Sobolev spaces and the Lebesgue spaces, respectively. Also, we let $A : L^2(\Omega) \to L^2(\Omega)$, where $A = -\Delta$ with its domain $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

Our main result, which establishes local and global existence of generalized solutions, reads as follows:

Theorem 1.1 **Generalized solutions.** *Under Assumption 1.1 and condition (4), there exists a local generalized solution to (1) defined on $(0, T_0)$ for some $T_0 > 0$. If, in addition, $p \le k + m$, then the said generalized solution is global and T_0 may be taken arbitrarily large.*

Remark 1.2 *If $k = 0$, then the variational inequality in (1.1) becomes equality and the solution u is unique and satisfies the equation in the sense of (1) with $j(u_t) \in L_1(Q_t), u_t \in L_{m+1}(Q_t)$. It should be pointed out that the main difficulty of the problem under consideration in Theorem 1.1 is the fact that the damping term is not monotone and degenerate ($k > 0$). This difficulty goes away when $k = 0$. As shown later (see Remark 3.1), the proof of Theorem 1.1 simplifies drastically when $k = 0$ and our arguments lead to stronger conclusions. In particular, the strong monotonicity allows us to replace inequalities by equalities. Thus, for $k = 0$ one obtains the existence theory which is consistent with the literature and provides an extension to a larger "supercritical" set for the parameter p, namely, $p \ge \frac{n}{n-2}$ (see Georgiev and Todorova (1994) for details).*

The next Theorem addresses the issue of propagation of regularity. This means that more regular data produce more regular solutions. In fact, the result below states that this is always the case locally (i.e., for sufficiently small times). However, in the special case when the parameter p is below the critical value $k+m$, then the propagation of regularity is a global phenomenon.

THEOREM 1.2 **Strong solutions.** *With the validity of Assumption 1.1, further assume that* $n<5$ *and*

$$k \geq 1,\ 2 \leq p < \frac{4}{n-2}+1,\ m+1 < \frac{n}{n-2},\ k+m < \frac{4}{n-2}+1. \qquad (7)$$

Then, for every initial data satisfying $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, *there exists* $T_0 > 0$ *such that (1) has a unique local solution* u *with the regularity that* $u \in C([0,T], H^2(\Omega)) \cap C^1([0,T], H^1(\Omega))$, *for some* $T \leq T_0$ *where* T_0 *may be finite. The said solution depends continuously on the initial data, with respect to finite energy norm. In addition, if we assume that* $p \leq k+m$, $p \leq \frac{p^*}{2}$, *and either* $k=0$ *or else* $\frac{k}{p^*}+\frac{m}{2} \leq \frac{1}{2}$, *then regular solutions are global and* T_0 *can be taken arbitrarily large.*

Finally, we address the issue of a *strong source* (large values of p when $p > k+m$) which may lead to a finite-time blow up of solutions. Here, our results are inspired by Georgiev and Todorova (1994), where the question of finite time blow up in the presence of damping in wave equations has been addressed first and solved optimally. The arguments of Georgiev and Todorova (1994) were later generalized to a larger class of damped hyperbolic like dynamics, Levine and Serrin (1997), and more recently adapted in Pitts and Rammaha (2002) in order to treat blow-up of solutions in the degenerate case with $k>0$, $j(s)=|s|^{m+1}$. An adaptation of the arguments in Pitts and Rammaha (2002) (for details see Barbu, Lasiecka and Rammaha, 2005) leads to the following blow up result:

THEOREM 1.3 *Assume the validity of Assumption 1.1 with* $c_1=0$ *and* $p>k+m$. *In addition, assume that* $E_0(0)<0$, *where* $E_0(0)$ *is the initial energy given by*

$$E_0(0) = \frac{1}{2}\left(|u_1|_{0,\Omega}^2 + \left|A^{1/2}u_0\right|_{0,\Omega}^2\right) - \frac{1}{p+1}\|u_0\|_{L^{p+1}(\Omega)}^{p+1}.$$

Then, weak solutions to (1) blow up in a finite time.

REMARK 1.3 *The global existence result obtained in Theorem 1.1 is optimal. Indeed, in view of Theorem 1.3 the range of parameters* $p \leq k+m$ *is the largest possible in order to obtain globality of solutions.*

We conclude the introduction with few words about the methods used for the proofs. Our method of the proof of the main result in Theorem 1.1 relies on the following:

We first establish the a priori bound for the damping-source problem under the assumption that $p \le k+m$. This a priori bound in Section 2 allows us to construct a multi-valued fixed point argument. In order to show an existence of a fixed point, one must establish two facts:

(i) First, the solvability of the problem for a fixed argument (see equation (24)). This is accomplished by applying an appropriate Faedo-Galerkin method.

(ii) Second, the upper semi-continuity of the nonlinear map F (see Section 3). For this part, our argument is based on subtle approximations by weakly lower semi-continuous functions. As usual in this type of problems, reconstruction of a weak limit is the main technical issue and core of the argument.

2. A priori bounds - local and global

We shall show that all generalized solutions admit an a priori bound in the topology specified by the Definition 1.1. In addition, this a priori bound is global (i.e., it holds on $[0,T]$ for any $T>0$) provided $p \le k+m$.

LEMMA 2.1 *Let u be a generalized solution of problem (1) with the assumption that $p \le k+m$. Then for all initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L_2(\Omega)$ and all $T>0$, we have the inequality*

$$|u(t)|_{1,\Omega} + |u_t(t)|_{0,\Omega} + \int_0^t \int_\Omega |u|^k j(u_t)dxd\tau \le C_T(|u_0|_{1,\Omega}, |u_1|_{0,\Omega}), \tag{8}$$

for all $t \in [0,T]$. If $p > k+m$ and condition (4) is valid, then the bound in (8) holds for $0 \le t \le T \le T_0$ for some $T_0 > 0$, where T_0 may be finite and depends on the $H^1 \times L_2$ norm of the initial data.

Proof. Define the following energy functions:

$$E(t) \equiv \frac{1}{2}\left(|\nabla u(t)|_{0,\Omega}^2 + |u_t(t)|_{0,\Omega}^2\right) \text{ and } E_1(t) \equiv E(t) + \frac{1}{p+1}\int_\Omega |u(t)|^{p+1}dx.$$

By the Sobolev embedding theorem along with restriction $p+1 \le p^*$, one has $\int_\Omega |u|^{p+1}dx \le C(|u|_{1,\Omega})$ along with the obvious bounds

$$E(t) \le E_1(t) \le C(E(t)), \tag{9}$$

where $C(s)$ denotes throughout the proof a real valued function which is bounded for bounded values of s.

By applying Definition 1.1 with $v=0$, we obtain

$$E(t) + \int_0^t \int_\Omega |u|^k j(u_t)dxd\tau \le \int_0^t \int_\Omega |u|^p |u_t| dxd\tau + E(0). \tag{10}$$

By adding the term $\int_0^t \int_\Omega |u|^{p-1} u u_t dx d\tau = \frac{1}{p+1} \int_\Omega \left(|u|^{p+1} - |u_0|^{p+1}\right) dx$ to both sides of inequality (10) we obtain

$$E_1(t) + \int_0^t \int_\Omega |u|^k j(u_t) dx d\tau \leq 2 \int_0^t \int_\Omega |u|^p |u_t| dx d\tau + E_1(0). \tag{11}$$

For $Q_t \equiv \Omega \times (0,t)$, we define

$$Q_A \equiv \{(x,s) \in Q_t, |u(x,s)| > 1\} \text{ and } Q_B \equiv \{(x,s) \in Q_t, |u(x,s)| \leq 1\}.$$

Then, it follows from (11) that

$$E_1(t) + \int_{Q_t} |u|^k j(u_t) dQ_t \leq E_1(0) + 2 \int_{Q_A} |u|^p |u_t| dQ_A + 2 \int_{Q_B} |u|^p |u_t| dQ_B. \tag{12}$$

We estimate the integrals on the right hand side of (12) as follows:

$$\begin{aligned} \int_{Q_B} |u|^p |u_t| dQ_B &\leq \rho |Q_B| + C_\rho \int_{Q_B} |u_t|^2 dQ_B \\ &\leq \rho |Q_t| + C_\rho \int_0^t E(s) ds, \end{aligned} \tag{13}$$

where $\rho > 0$ is a sufficiently small value that will be chosen later. Also, here and later $|Q_t|$ denotes the Lebesgue measure of Q_t. In order to estimate the other integral over Q_A we choose $r = \frac{p-m}{m+1}$, $q = \frac{m+1}{m}$, $\bar{q} = m+1$. If $r \leq 0$, the application of Young's inequality gives

$$\begin{aligned} \int_{Q_A} |u|^p |u_t| dQ_A &\leq C_\epsilon \int_{Q_t} |u|^{p+1} dQ_t + \epsilon \int_{Q_A} |u_t|^{m+1} dQ_A \\ &\leq C_\epsilon \int_{Q_t} |u|^{p+1} dQ_t + \epsilon \int_{Q_t} |u|^k |u_t|^{m+1} dQ_t, \end{aligned} \tag{14}$$

where $\epsilon > 0$ will be chosen later. On the other hand if $r > 0$, then we modify the argument as follows:

$$\begin{aligned} \int_{Q_A} |u|^p |u_t| dQ_A &\leq \left(\int_{Q_A} |u|^{q(p-r)} dQ_A \right)^{\frac{1}{q}} \left(\int_{Q_A} |u|^{r\bar{q}} |u_t|^{\bar{q}} dQ_A \right)^{\frac{1}{\bar{q}}} \\ &= \left(\int_{Q_A} |u|^{p+1} dQ_A \right)^{\frac{1}{q}} \left(\int_{Q_A} |u|^{p-m} |u_t|^{m+1} dQ_A \right)^{\frac{1}{m+1}}. \end{aligned} \tag{15}$$

We shall first deal with the case when $p \leq m+k$. By applying Young's inequality and exploiting the assumption that $p - m \leq k$, we obtain

$$\begin{aligned} \int_{Q_A} |u|^p |u_t| dQ_A &\leq C_\epsilon \int_{Q_A} |u|^{p+1} dQ_A + \epsilon \int_{Q_A} |u|^{p-m} |u_t|^{m+1} dQ_A \\ &\leq C_\epsilon \int_{Q_t} |u|^{p+1} dQ_t + \epsilon \int_{Q_t} |u|^k |u_t|^{m+1} dQ_t. \end{aligned} \tag{16}$$

Thus, in both cases we have

$$\begin{aligned}\int_{Q_t} |u|^p |u_t| dQ_t &\leq \rho\,|Q_t| + C_\rho \int_0^t E(s)ds \\ &+ \epsilon \int_{Q_t} |u|^k |u_t|^{m+1} dQ_t + C_\epsilon \int_{Q_t} |u|^{p+1} dQ_t, \qquad (17)\end{aligned}$$

where the constants $\rho, \epsilon > 0$ can be taken arbitrary small. By combining inequalities (12) and (17), we obtain

$$\begin{aligned}E_1(t) + \int_{Q_t} |u|^k j(u_t) dQ_t &\leq E_1(0) + \epsilon \int_{Q_t} |u|^k |u_t|^{m+1} dQ_t \\ &+ \rho\,|Q_t| + (C_\rho + C_\epsilon) \int_0^t E_1(s)ds. \qquad (18)\end{aligned}$$

By taking ϵ sufficiently small and keeping in mind (9) along with the coercivity in Assumption 1.1, we obtain

$$E_1(t) + c_\epsilon \int_{Q_t} |u|^k j(u_t) dQ_t \leq E_1(0) + \rho\,|Q_T| + C \int_0^t E_1(s)ds, \qquad (19)$$

for some $c_\epsilon > 0$. Now, by Gronwall's inequality it follows that

$$E_1(t) \leq (E_1(0) + \rho\,|Q_t|)\, e^{Ct}. \qquad (20)$$

Finally, (20) leads to $E_1(t) + \int_{Q_t} |u|^k j(u_t) \leq C_T\,(E_0(0) + \rho\,|Q_t|)$, where the last inequality is valid for all $t \leq T$ and T is being arbitrary as long as $p \leq k+m$. If $p > k+m$, then the above bound holds locally for sufficiently small T. Indeed, by using Hölder's and Young's inequalities, we have instead of (13)-(16) the following estimate:

$$\begin{aligned}\int_{Q_t} |u|^p |u_t| dQ_t &\leq \epsilon \int_{Q_t} |u|^k |u_t|^{m+1} dQ_t + C_\epsilon \int_{Q_t} |u|^{\frac{p(m+1)-k}{m}} dQ_t \\ &\leq \epsilon \int_{Q_t} |u|^k |u_t|^{m+1} dQ_t + C_\epsilon \int_0^t |u|_{1,\Omega}^{\frac{p(m+1)-k}{m}} dt, \qquad (21)\end{aligned}$$

whenever $\frac{p(m+1)-k}{m} \leq p^*$. If, instead, $p \leq \frac{p^*}{2}$ then the argument is direct. In both cases a standard continuity argument yields the bound in (8) for a sufficiently small $T > 0$. ■

3. Kakutani fixed point argument - the proof of Theorem 1.1

Let $w \in C(0,T; L_q(\Omega))$ be a given element where throughout this section the parameter q satisfies

$$\max\{2k, p+1, \min\{2p, \frac{p(m+1)-k}{m}\}\} < q < p^*. \qquad (22)$$

We shall consider the following *variational inequality:*

Find $u \in C_w([0,T], H_0^1(\Omega)) \cap C_w^1([0,T], L_2(\Omega))$, with $\int_0^T \int_\Omega |w|^k j(u_t) dxdt < \infty$, such that for all $0 < t \leq T$ the following inequality holds:

$$\begin{aligned} \int_0^t \int_\Omega (u_t\ v_t - \nabla u \nabla v) dxdt + \frac{1}{2} \int_\Omega [u_t^2(t) + |\nabla u(t)|^2] dx \\ + \int_0^t \int_\Omega |w|^k [j(u_t) - j(v)] dxdt \\ \leq \int_0^t \int_\Omega |w|^{p-1} w(u_t - v) dxdt + \frac{1}{2} \int_\Omega [u_1^2 + |\nabla u_0|^2 + 2u_1 v(0)] dx \end{aligned} \tag{23}$$

for all functions $v \in H^1(0,t; L_2(\Omega)) \cap L_2(0,t; H_0^1(\Omega)) \cap L_\infty(Q_t)$, $\quad v(t) = 0$. For a given argument $w \in C(0,T; L_q(\Omega))$ we consider the multi-valued mapping

$$F : C(0,T; L_q(\Omega)) \to C(0,T; L_q(\Omega)),$$

where the action of F is defined by $u \in Fw \quad$ iff $\quad u \quad$ is a solution to (3). In the next subsections we shall prove that the mapping F is a well defined multi-valued mapping on $C(0,T; L_q(\Omega))$, i.e., *Range* $F(w)$ is nonempty for each $w \in C(0,T; L_q(\Omega))$. In order to establish Theorem 1.1, it suffices to show that F has a fixed point. We accomplish this, by using Kakutani-type Theorem, Zeidler (1986), and the a priori bound established in Lemma 2.1. However, we first need to prove the following facts: $F(K)$ is convex and compact in $C(0,T; L_q(\Omega))$, where K is being a suitably chosen (large) ball in $C(0,T; L_q(\Omega))$ whose radius depends on the initial data. The second requirement of Kakutani's theorem is the upper semi-continuity of F. Due to the compactness of $F(K)$, proving the upper semi-continuity of F amounts to showing the following statement:

for a given sequence $w_n \to w$ in $C(0,T; L_q(\Omega))$, and $u_n \to u$ in $C(0,T; L_q(\Omega))$ where $u_n \in F(w_n)$, we have $u \in F(w)$. Indeed, this is equivalent to the fact that the graph of F is closed in $C([0,T]; L_q(\Omega)) \times C([0,T]; L_q(\Omega))$.

3.1. Well-posedness of the map F

For a given function $w \in C(0,T; L_q(\Omega))$, we consider the equation

$$u_{tt} - \Delta u + |w|^k j'(u_t) = |w|^{p-1} w, \tag{24}$$

whose variational formulation is the following:

find $u \in C([0,T], H_0^1(\Omega)) \cap C_w^1([0,T], L_2(\Omega))$, $u_{tt} \in L_2(0,T; H^{-1}(\Omega))$ such that the following identity holds

$$\int_0^t \int_\Omega [u_{tt} v + \nabla u \nabla v] dxdt + \int_0^t \int_\Omega |w|^k j'(u_t) v dxdt = \int_0^t \int_\Omega |w|^{p-1} w v dxdt \tag{25}$$

with $u(0) = u_0, u_t(0) = u_1$ and test functions

$$v \in C_w([0,T];\ H_0^1(\Omega) \cap L_{(m+1)\frac{q}{q-k}}(\Omega)).$$

The main results in this subsection are the following:

Lemma 3.1 *Assume the validity of Assumption 1.1 and condition (22). Then, there exists a unique solution u to the variational identity (25) such that $u \in C(0,T;H_0^1(\Omega)) \cap C^1(0,T;L_2(\Omega))$ and the following bound holds for all $t \le T$:*

$$\begin{aligned} |u(t)|_{1,\Omega}^2 \quad &+ \quad |u_t(t)|_{0,\Omega}^2 + \int_{Q_t} |w|^k |u_t|^{m+1} dQ_t \\ &\le \quad C_T \left(|u_0|_{1,\Omega}, |u_1|_{0,\Omega}, |w|_{C([0,T];L_q(\Omega))} \right). \end{aligned} \tag{26}$$

In addition, the following energy identity holds:

$$\begin{aligned} &\frac{1}{2}\left(|\nabla u(t)|_{0,\Omega}^2 + |u_t(t)|_{0,\Omega}^2 \right) + \int_{Q_t} |w|^k j'(u_t) u_t dQ_t = \\ &\quad \frac{1}{2}\left(|\nabla u(0)|_{0,\Omega}^2 + |u_t(0)|_{0,\Omega}^2 \right) + \int_{Q_t} |w|^{p-1} w u_t dQ_t. \end{aligned} \tag{27}$$

Corollary 3.1 *For each $w \in C([0,T], L_q(\Omega))$, $F(w) \neq \emptyset$. Moreover, $C([0,T], L_q(\Omega)) \subset Dom\ F$.*

Proof. We consider a standard Galerkin approximation scheme to the solution of (25) based on the eigenfunctions $\{e_k\}_{k=1}^{\infty}$ of the operator $A = -\triangle$ with zero boundary condition on $\partial\Omega$. That is, we let $u_n(t) = \sum_{k=1}^{n} u_{n,k}(t) e_k$ where $u_n(t)$ satisfies

$$\begin{aligned} (u_{ntt}, v)_\Omega + (\nabla u_n, \nabla v)_\Omega + (|w|^k j'(u_{nt}), v)_\Omega &= (|w|^{p-1} w, v)_\Omega \\ (u_n(0), v)_\Omega = (u_0, v)_\Omega, \quad (u_{nt}(0), v)_\Omega &= (u_1, v)_\Omega \end{aligned} \tag{28}$$

for all $v \in V_n :=$ the linear span of $\{e_1, ..., e_n\}$, and for convenience, we use $(.,.)_\Omega$ to denote the standard $L_2(\Omega)$–inner product.

By standard nonlinear ordinary differential equations theory one obtains the existence of a global solution to (28) with the following a priori bounds which are *uniform* in n:

$$\begin{aligned} &\frac{1}{2}\left(|\nabla u_n(t)|_{0,\Omega}^2 + |u_{nt}(t)|_{0,\Omega}^2 \right) + \int_{Q_t} |w|^k j'(u_{nt}) u_{nt} dQ_t = \\ &\quad \frac{1}{2}\left(|\nabla u_n(0)|_{0,\Omega}^2 + |u_{nt}(0)|_{0,\Omega}^2 \right) + \int_{Q_t} |w|^{p-1} w u_{nt} dQ_t. \end{aligned} \tag{29}$$

By using the restrictions imposed on the parameter q, we obtain the estimates:

$$\int_{Q_t} |w|^p |u_{nt}| dQ_t \le \epsilon \int_{Q_t} |u_{nt}|^{m+1} |w|^k dQ_t + C_\epsilon \int_0^t |w|_{L_q(\Omega)}^{\frac{p(m+1)-k}{m}} dt \tag{30}$$

when $\frac{p(m+1)-k}{m} \leq q$, and

$$\int_{Q_t} |w|^p |u_{nt}| dQ_t \leq \epsilon \int_{Q_t} |u_{nt}|^2 dQ_t + C_\epsilon \int_0^t |w|^{2p}_{L_q(\Omega)} dt \tag{31}$$

when $2p \leq q$. Thus, it follows from (30), (31) and (3.1) that

$$\begin{aligned} &|u_n(t)|^2_{1,\Omega} + |u_{nt}(t)|^2_{0,\Omega} + \int_{Q_t} |w|^k j(u_{nt}) dQ_t \\ &\quad \leq C_T \left(|u_0|_{1,\Omega}, |u_1|_{0,\Omega}, |w|_{C([0,T];L_q(\Omega))} \right). \end{aligned} \tag{32}$$

By using the coercivity condition in Assumption 1.1, we obtain

$$\begin{aligned} &|u_n(t)|^2_{1,\Omega} + |u_{nt}(t)|^2_{0,\Omega} + \int_{Q_T} |w|^k |u_{nt}|^{m+1} dQ_T \\ &\quad \leq C_T \left(|u_0|_{1,\Omega}, |u_1|_{0,\Omega}, |w|_{C([0,T];L_q(\Omega))} \right). \end{aligned} \tag{33}$$

Hence, there exists a subsequence of $\{u_n\}$, which we still denote by $\{u_n\}$, that satisfies

$$(u_n, u_{nt}) \to (u, u_t) \ \textit{weakly}^* \text{ in } L_\infty(0,T; H^1(\Omega) \times L_2(\Omega)) \tag{34}$$

Now, consider two solutions u_n and u_l, where without loss of generality we assume $l \geq n$. Denote $U_{nl} \equiv u_n - u_l$. Then, it follows from (25) that U_{nl} satisfies variational equality

$$\begin{aligned} &(U_{nltt}, v)_\Omega + (\nabla U_{nl}, \nabla v)_\Omega + (|w|^k j'(u_{nt}) - j'(u_{lt}), v)_\Omega = 0 \\ &(U_{nl}(0), v)_\Omega = (u_{n0} - u_{l0}, v)_\Omega, (u_{nlt}(0), v)_\Omega = (u_{1n} - u_{1l}, v)_\Omega, \end{aligned} \tag{35}$$

for all $v \in V_n$. By setting $v = U_{nlt}$ in (3.1) and by using the strong convergence of the approximations to the initial data, one easily obtains the following convergence result

$$|U_{nlt}(t)|^2_{0,\Omega} + |U_{nl}(t)|^2_{1,\Omega} + \int_0^t |w|^k \left(j'(u_{nt}) - j'(u_{lt}), U_{nlt} \right)_\Omega ds \to 0, \tag{36}$$

as $n, l \to \infty$. Now from (36) and the strong coercivity assumption, we conclude

$$|U_{nlt}(t)|^2_{0,\Omega} + |U_{nl}(t)|^2_{1,\Omega} + \int_0^t \int_\Omega |w|^k |U_{nlt}|^{m+1} dx ds \to 0, \tag{37}$$

as $n, l \to \infty$. From (37) we infer the strong convergence

$$|w|^{\frac{k}{m+1}} u_{nt} \to \eta \ \textit{in} \ L_{m+1}(Q_T), \ \textit{as} \ n \to \infty, \tag{38}$$

for some $\eta \in L_{m+1}(Q_T)$. Moreover, we have

$$(u_n, u_{tn}) \to (u, u_t) \ \textit{strongly in} \ L_\infty(0,T; H^1_0(\Omega) \times L_2(\Omega)). \tag{39}$$

We remark here that the above strong convergence allows us to reconstruct the limit function η. Indeed, $\eta = |w|^{\frac{k}{m+1}} u_t$. Thus,

$$|w|^{\frac{k}{m+1}} u_{nt} \to |w|^{\frac{k}{m+1}} u_t \textit{ strongly in } L_{m+1}(Q_T), \text{ as } n \to \infty. \tag{40}$$

Equivalently,

$$\int_{Q_T} |w|^k |u_{tn} - u_t|^{m+1} dQ_T \to 0, \text{ as } n \to \infty.$$

In particular, as $n \to \infty$ (passing to a subsequence if necessary)

$$|w|^{\frac{k}{m+1}} u_{nt} \to |w|^{\frac{k}{m+1}} u_t \text{ point-wise almost everywhere } x, t \in Q_T. \tag{41}$$

Now in order to pass to the limit in the nonlinear term, we shall prove

$$|w|^{\frac{km}{m+1}} j'(u_{nt}) \to |w|^{\frac{km}{m+1}} j'(u_t) \text{ weakly in } L_{\frac{m+1}{m}}(Q_T), \text{ as } n \to \infty. \tag{42}$$

To see this, we recall the a priori bound in (3.1), which implies

$$\int_{Q_T} \left[|w|^{\frac{km}{m+1}} |u_{nt}|^m \right]^{\frac{m+1}{m}} dQ_T = \int_{Q_T} |w|^k |u_{nt}|^{m+1} dQ_T \le C_T.$$

Also, the growth condition imposed on $j'(s)$ yields

$$\int_{Q_T} \left[|w|^{\frac{km}{m+1}} |j'(u_{nt})| \right]^{\frac{m+1}{m}} dQ_T \le M_T,$$

for some constant $M_T > 0$. Hence, by passing to a subsequence if necessary, one has

$$|w|^{\frac{km}{m+1}} |u_{nt}|^m \to l \text{ weakly in } L_{\frac{m+1}{m}}(Q_T)$$

and

$$|w|^{\frac{km}{m+1}} j'(u_{nt}) \to J \text{ weakly in } L_{\frac{m+1}{m}}(Q_T).$$

By appealing to the almost everywhere point-wise convergence in (41) and continuity of $j'(s)$, we can identify the limits l and J. Indeed,

$$l = |w|^{\frac{km}{m+1}} u_t^m, \quad J = |w|^{\frac{km}{m+1}} j'(u_t).$$

Therefore, we have

$$|w|^{\frac{km}{m+1}} |u_{nt}|^m \to |w|^{\frac{km}{m+1}} |u_t|^m \text{ weakly in } L_{\frac{m+1}{m}}(Q_T) \tag{43}$$

and

$$|w|^{\frac{km}{m+1}} j'(u_{nt}) \to |w|^{\frac{km}{m+1}} j'(u_t) \text{ weakly in } L_{\frac{m+1}{m}}(Q_T), \tag{44}$$

as desired in (42).

From the weak convergence in (42) and the strong convergence in (40) we infer that

$$\int_{Q_t} |w|^k j'(u_{nt}) u_{nt} dQ_t \to \int_{Q_t} |w|^k j'(u_t) u_t dQ_t. \tag{45}$$

Indeed, to see (45) we write

$$\int_{Q_t} |w|^k j'(u_{tn}) u_{nt} dQ_t = \int_{Q_t} |w|^{\frac{km}{m+1}} j'(u_{nt}) |w|^{\frac{k}{m+1}} u_{nt} dQ_t,$$

and thus (45) follows easily from (42), (40) and duality.

Our next step is to establish the following convergence:

$$\int_{Q_t} |w|^k |u_{nt}|^{m+1} dxds \to \int_{Q_t} |w|^k |u_t|^{m+1} dxds. \tag{46}$$

Indeed, (46) becomes clear after writing

$$|w|^k |u_{nt}|^{m+1} = |w|^{\frac{km}{m+1}} |u_{nt}|^{m-1} u_{nt} |w|^{\frac{k}{m+1}} u_{nt}, := g(z_n) z_n \tag{47}$$

where we have used the notation

$$z_n \equiv |w|^{\frac{k}{m+1}} u_{nt}, \quad g(z_n) \equiv |z_n|^{m-1} z_n.$$

Let $z = |w|^{\frac{k}{m+1}} u_t$. Then, we note that (40) and (43) yield

$$z_n \to z \text{ strongly in } L_{m+1}(Q_T); \quad g(z_n) \to g(z) \text{ weakly in } L_{\frac{m+1}{m}}(Q_T). \tag{48}$$

Therefore, (46) follows easily from (47), the convergence in (48) and duality. By applying the convergence in (46) to inequality (3.1) and keeping in mind weak lower semi-continuity of the norm, we obtain

$$|u(t)|^2_{1,\Omega} + |u_t(t)|^2_{0,\Omega} + \int_{Q_t} |w|^k |u_t|^{m+1} dQ_t \le C_T \left(|u_0|_{1,\Omega}, |u_1|_{0,\Omega}, |w|_{C(0,T;L_q(\Omega))} \right), \tag{49}$$

which establishes (26).

We shall prove next that

$$\int_{Q_t} |w|^{p-1} w u_{nt} dQ_t \to \int_{Q_t} |w|^{p-1} w u_t dQ_t. \tag{50}$$

When $2p \leq q$ the above is just a consequence of the L_2–weak convergence of u_{nt} and the fact that $|w|^p \in L_2(Q_t)$. Otherwise, if $\frac{p(m+1)-k}{m} \leq q$, then

$$\int_{Q_t} ||w|^{p-1} w(u_{nt} - u_t)| dQ_t \leq \int_{Q_t} |w|^p |u_{nt} - u_t| dQ_t$$
$$\leq \epsilon \int_0^t \|w\|_{L_q(\Omega)}^{\frac{p(m+1)-k}{m}} dt + \epsilon |Q_t| + C_\epsilon \int_{Q_t} |w|^k |u_{nt} - u_t|^{m+1} dQ_t. \tag{51}$$

By (40) we conclude that the last term above converges to zero as $n \to \infty$. Thus letting $\epsilon \to 0$, we obtain the desired conclusion in (50). By using (39), (45), (50) and the energy identity in (3.1), we obtain the energy identity (3.1).

Our final step is the passage to the limit in the variational form of the equation. By taking first a test function v as smooth as necessary we obtain

$$\left(|w|^k j'(u_{nt}), v\right)_{Q_T} = \left(|w|^{\frac{km}{m+1}} j'(u_{nt}), |w|^{\frac{k}{m+1}} v\right)_{Q_T} \to \left(|w|^{\frac{km}{m+1}} j'(u_t), |w|^{\frac{k}{m+1}} v\right)_{Q_T}, \tag{52}$$

as long as $v|w|^{\frac{k}{m+1}} \in L_{m+1}(Q_T)$. Indeed, the latter holds for $v \in L_{q(m+1)/(q-k)}$, as desired. The passage to the limit in the linear terms is standard, and thus it is omitted. ■

As to compactness and convexity of F, these properties are now straightforward and follow from variational inequality (3) applied with $v = 0$. In fact, the following result is easily obtained (see Barbu, Lasiecka and Rammaha, 2005, for full details).

Lemma 3.2 *$F(K)$ is compact and $F(w)$ is convex for every $w \in K$.*

3.2. Upper semi-continuity of F

Before proving the upper semi-continuity of the mapping F, we shall prove the following Proposition which is central to the argument.

Proposition 3.1 *Let u_{nt} be any sequence which converges weakly in $L_2(Q_T)$ to a function u_t. Let $w_n \to w$ in $C([0,T]; L_q(\Omega))$, where q satisfies (22). Further assume that $\left\| |w_n|^k |u_{nt}|^{m+1} \right\|_{L_1(Q_T)} \leq M$ uniformly in n. Then, we have the following:*

$$\int_{Q_t} |w|^k j(u_t) dQ_t \leq \liminf_{n\to\infty} \int_{Q_t} |w_n|^k j(u_{nt}) dQ_t,$$
$$\int_{Q_t} |w_n|^{p-1} w_n (u_{nt} - v) dQ_t \to \int_{Q_t} |w|^{p-1} w (u_t - v) dQ_t, \text{ as } n \to \infty, \tag{53}$$
$$\int_{Q_t} |w_n|^k j(v) dQ_t \to \int_{Q_t} |w|^k j(v) dQ_t, \text{ as } n \to \infty,$$

for all $v \in L_\infty(Q_t)$.

Proof. The second part of Proposition 3.1, in the case when $p \le \frac{q}{2}$, follows directly from the strong convergence

$$|w_n|^{p-1}w_n \to |w|^{p-1}w \text{ in } L_2(Q_t)$$

and the weak convergence

$$u_{nt} \to u_t \text{ weakly in } L_2(Q_t).$$

If, instead, $p(m+1)-k < qm$ (see condition (22)) we have:

$$\begin{aligned} &w_n^{\frac{k}{m+1}}u_{nt} \to w^{\frac{k}{m+1}}u_t \textit{ weakly in } L_{m+1}(Q_T),\\ &|w_n|^{p-1-\frac{k}{m+1}}w_n \to |w|^{p-1-\frac{k}{m+1}}w \textit{ strongly in } L_{\frac{m+1}{m}}(Q_T). \end{aligned} \tag{54}$$

Indeed, the second assertion follows from the strong convergence of w_n in $L_q(Q_T)$ and the restriction $(p-\frac{k}{m+1})\frac{m+1}{m} \le q$, implied by (22). As for the second statement in (3.2), we notice first that by the assumption imposed in Proposition 3.1, then $|w_n|^{\frac{k}{m+1}}u_{nt}$ is uniformly bounded in $L_{m+1}(Q_T)$. Hence, $|w_n|^{\frac{k}{m+1}}u_{nt} \to \eta$ weakly in $L_{m+1}(Q_T)$. On the other hand, by using the weak convergence of u_{nt} in $L_2(Q_T)$ and the strong convergence $w_n^{\frac{k}{m+1}} \to w^{\frac{k}{m+1}}$ in $L_2(Q_T)$ (note that by (22) $\frac{k}{m+1} < \frac{q}{2}$) we obtain

$$|w_n|^{\frac{k}{m+1}}u_{nt} \to |w|^{\frac{k}{m+1}}u_t \text{ weakly in } L_1(Q_T).$$

This allows us to identify η with $\eta = w^{\frac{k}{m+1}}u_t$, as desired. Having established (3.2) the rest of the argument is straightforward. It suffices to write

$$\int_{Q_t}|w_n|^{p-1}w_n u_{nt}dQ_t = \int_{Q_t}\left(|w_n|^{\frac{k}{m+1}}u_{nt}\right)\left(|w_n|^{p-1-\frac{k}{m+1}}w_n\right)dQ_t, \tag{55}$$

where the first bracket in the right hand side of (55) converges weakly in $L_{m+1}(Q_t)$, and the second bracket converges strongly in $L_{\frac{m+1}{m}}(Q_t)$. This completes the proof of convergence

$$\int_{Q_t}|w_n|^{p-1}w_n u_{nt}dQ_t \to \int_{Q_t}|w|^{p-1}wu_t dQ_t,$$

and hence the second convergence in (3.1) follows. *The third part* in the Proposition is straightforward and it follows from the strong convergence of w_n in $L_{\frac{2}{k}}(Q_T)$, which is implied by the assumption $k \le q$. To complete the proof of Proposition 3.1 we need to *prove the first part.* To accomplish this, we introduce the following approximation (truncation) of j:

$$j_N(s) \equiv \begin{cases} j(s), & |s| \le N\\ j(N)+\partial j(N)(s-N), & s > N\\ j(-N)+\partial j(-N)(s+N), & s < -N. \end{cases}$$

It is easy to see that for each N, J_N is convex, continuous, and satisfies

$$\begin{aligned} j_N(s) &\leq j(s) \\ j_N(s) &\rightarrow j(s),\ \text{as } N \rightarrow \infty,\ \text{for all } s \in \mathbb{R}. \end{aligned} \tag{56}$$

Moreover, $j'_N(s) = j'(s)$, for all $s \in [-N, N]$, $j'_N(s) = j'(N)$, for all $s \geq N$ and $j'_N(s) = j'(-N)$, for all $s \leq -N$. Hence, for all $v \in L_2(Q_t)$, we have

$$j_N(v) \in L_2(Q_t),\ j'_N(v) \in L_\infty(Q_t). \tag{57}$$

In what follows we shall assume, without loss of generality, that $j \geq 0$. Then, from (56) we infer that for each fixed N

$$\liminf_{n\to\infty} \int_{Q_t} |w_n|^k j(u_{nt}) dxdt \geq \liminf_{n\to\infty} \int_{Q_t} |w_n|^k j_N(u_{nt}) dxdt. \tag{58}$$

From convexity of j it follows that

$$j_N(v) \leq j_N(u_{nt}(t,x)) + j'_N(v)(v - u_{nt}(t,x));\ \text{for all } v \in \mathbb{R}. \tag{59}$$

By recalling (57), it follows from (59) that

$$\int_{Q_t} |w_n|^k \left[j_N(v) - j'_N(v)(v - u_{nt}) \right] dxdt \leq \int_{Q_t} |w_n|^k j_N(u_{nt}) dxdt, \tag{60}$$

for all $v \in L_2(Q_t)$, and from (56) one has

$$\int_{Q_t} |w_n|^k \left[j_N(v) - j'_N(v)(v - u_{nt}) \right] dxdt \leq \int_{Q_t} |w_n|^k j(u_{nt}) dxdt, \tag{61}$$

for all $v \in L_2(Q_t)$. By noting that

$$\begin{aligned} &|w_n|^k u_{nt} \rightarrow |w|^k u_t \ \textit{weakly in } L_r(Q_t),\ \textit{for some } r > 1,\ \textit{as } n \rightarrow \infty;\ \text{and} \\ &\int_{Q_t} |w_n|^k j_N(v) dxdt \rightarrow \int_{Q_t} |w|^k j_N(v) dxdt,\ \textit{as } n \rightarrow \infty, \end{aligned} \tag{62}$$

and by recalling the fact that $\partial j_N(v) \in L_\infty(Q_t)$, we obtain

$$\int_{Q_t} |w|^k \left[j_N(v) - j'_N(v)(v - u_t) \right] dxdt \leq \liminf_{n\to\infty} \int_{Q_t} |w_n|^k j(u_{nt}) dxdt, \tag{63}$$

for all $v \in L_2(Q_t)$. By taking $v = u_t \in L_2(Q_t)$, (63) yields

$$\int_{Q_t} |w|^k j_N(u_t) dxdt \leq \liminf_{n\to\infty} \int_{Q_t} |w_n|^k j(u_{nt}) dxdt. \tag{64}$$

Since $j_N(u_t) \rightarrow j(u_t)$ almost everywhere in Q_t as $N \rightarrow \infty$, and $j_N(s)$ is non-negative for each N, we are in a position to apply Fatou's Lemma and able to conclude that

$$\int_{Q_t} |w|^k j(u_t) dxdt \leq \liminf_{n\to\infty} \int_{Q_t} |w_n|^k j(u_{nt}) dxdt. \tag{65}$$

Hence, the first part in Proposition 3.1 follows immediately, which completes the proof. ∎

We are now in a position to prove the upper semi-continuity of the mapping F. Specifically, we have the following Lemma.

LEMMA 3.3 *Let $w_n \to w$ in $C([0,T];L_q(\Omega))$. Let $u_n \in F(w_n)$ be such that $u_n \to u$ in $C([0,T];L_q(\Omega))$. Then, $u \in F(w)$.*

Proof. Since $u_n \in F(w_n)$, then from the definition of the mapping F we have the following a priori bounds:

$$|u_n(t)|_{1,\Omega} + |u_{nt}(t)|_{0,\Omega} \leq C(|w|_{C([0,T];L_q(\Omega))}, |u_0|_{1.\Omega}, |u_1|_{0,\Omega})$$
$$\int_{Q_T} |w_n|^k j(u_{nt}) dQ_T \leq C(|w|_{C([0,T];L_q(\Omega))}, |u_0|_{1.\Omega}, |u_1|_{0,\Omega}). \tag{66}$$

Therefore, by passing to a subsequence if necessary we have

$$(u_n, u_{nt}) \to (u, u_t) \text{ weakly}^* \text{ in } L_\infty(0,T;L_2(\Omega)\times H^1(\Omega)). \tag{67}$$

By Simon's compactness criterion and recalling that $q < \frac{2n}{n-2}$ we conclude that

$$u_n \to u, \text{ strongly in } C([0,T];L_q(\Omega)). \tag{68}$$

Therefore, the proof of the Lemma will be completed if we show that $u \in F(w)$. In order to do so, we recall the variational definition of the mapping F given in (3). Since $u_n \in F(w_n)$, we have

$$\int_0^t \int_\Omega (u_{nt}\ v_t - \nabla u_n \nabla v) dx dt + \frac{1}{2}\int_\Omega [|u_{nt}(t)|^2 + |\nabla u_n(t)|^2] dx$$
$$+ \int_0^t \int_\Omega |w_n|^k [j(u_{nt}) - j(v)] dx dt$$
$$\leq \int_0^t \int_\Omega |w_n|^{p-1} w_n (u_{nt} - v) dx dt + \frac{1}{2}\int_\Omega [u_1^2 + |\nabla u_0|^2 + 2u_1 v(0)] dx, \tag{69}$$

for all test functions $v \in H^1(0,t;L_2(\Omega)) \cap L_2(0,t;H_0^1(\Omega)) \cap L_\infty(Q_t)$, $v(t) = 0$. Our goal is to pass to the limit in inequality (3.2). Indeed, by using the results of Proposition 3.1 and the weak lower semi-continuity of the energy function $E(t)$, we can easily pass to the limit in inequality (3.2) to obtain that u satisfies the variational inequality (3). Moreover, since we also have the a priori regularity (see (67))

$$u \in C_w([0,T];H^1(\Omega)) \cap C_w^1([0,T];L_2(\Omega))$$

we may apply (3) with $v = 0$ to obtain $\int_{Q_T} |w|^k j(u_t) dQ_T < \infty$. Therefore, $u \in F(w)$ as desired. ∎

3.3. Proof of Theorem 1.1

Proof. Since $F(K)$ is compact, F is upper semi-continuous (Lemma 3.3), $F(w)$ is convex, and the a priori bound holds in Lemma 2.1 (in the case $p > k+m$ the time T may be finite), then by applying standard truncation device for the mapping F we are in a position to apply Kakutani's Theorem. Indeed, let R be large enough so that for any $u \in \gamma F(u)$, where $0 < \gamma < 1$, we have

$$|u|_{C([0,T];L_q(\Omega))} < R. \tag{70}$$

Indeed, R can be determined by using the a priori bound Lemma 2.1 and the Sobolev embedding $H^1(\Omega) \hookrightarrow L_q(\Omega)$. We choose K to be a ball of radius R in $C([0,T];L_q(\Omega))$ centered at the origin. Specifically, we set $K \equiv B_{C(L_q)}(0,R)$, where $C(L_q) \equiv C([0,T];L_q(\Omega))$. Next, we define the truncated mapping F_R as follows:

$$y_R \in F_R(w) \text{ iff } \begin{cases} y_R = y, y \in F(w) \cap B_{C(L_q)}(0,R) \\ \\ \frac{R}{|y|_{C(L_q)}} y, y \in F(w), \quad |y|_{C(L_q)} > R. \end{cases} \tag{71}$$

Thus, $F_R(C(L_q)) \subset K$ and F_R satisfies all assumptions of Kakutani's Theorem (see Zeidler, 1986, Theorem 9B, page 452). Therefore, F_R has a fixed point, i.e., there exists $u \in C([0,T];L_q(\Omega))$ such that $u \in F_R(u)$. At this end, we note that we have two possibilities. Either $u \in F(u)$ or else $u \in \gamma F(u)$, where $\gamma = \frac{R}{|y|_{C(L_q)}} < 1$ for some $y \in F(u)$, $|y|_{C(L_q)} > R$. However, the latter case cannot occur since if it did, then we would have $|u|_{C(L_q)} = R$. But this contradicts the a priori bound $|u|_{C(L_q)} < R$. Thus, we would have $u \in F(u)$ as desired. ■

Remark 3.1 *In the special case when $k = 0$ the argument is much simpler and the conclusions obtained are stronger than what has been stated in Proposition 3.1. Indeed, if $k = 0$, then the strong monotonicity condition imposed on j' allows us to prove the strong convergence: $u_{nt} \to u_t$ in $L_{m+1}(Q_t)$, where u_n satisfies equation (28). This follows from (37) after setting $k = 0$. Having obtained the strong convergence $u_{nt} \to u_t$ in $L_{m+1}(Q_t)$, we likewise obtain the strong convergence: $j(u_{nt}) \to j(u_t)$ in $L_1(Q_t)$. Based on the strong convergence of $j(u_{nt})$ we can pass to the limit in equation (28) proving that $u = F(w)$, where $F(w)$ is defined by the variational equality and not inequality. In addition, the uniqueness of solutions is a direct consequence of monotonicity.*

3.4. Theorem 1.2 and uniqueness of solutions

Theorem 1.1 provides existence of generalized solutions under very general assumptions assumed in the parameters p, m, k, n. However, this theorem does not provide any uniqueness statement (in the degenerate case $k > 0$) for this large class of solutions. There are two classes of degenerate problems where uniqueness of solutions is available: the sub-linear case $m \leq 1$ and the case of higher energy solutions. The proof of uniqueness of solutions along with continuous dependence on initial conditions - for the strictly sub-linear case ($m < 1$) - is given in Pitts and Rammaha (2002). The critical sub-linear case $m = 1$ has been treated in Barbu, Lasiecka and Rammaha (2005). In this latter case, uniqueness of solutions has been proved in Barbu, Lasiecka and Rammaha (2005) but without the continuous dependence on initial conditions (in the finite energy norm). As for higher energy solutions, the existence of strong solutions requires additional restrictions imposed on the parameters p, m. Indeed, under the assumption (7) Barbu, Lasiecka and Rammaha (2005) provide a proof of local existence of higher energy $H^2(\Omega) \times H^1(\Omega)$ solutions. When $p \leq k + m$ it is also shown there that solutions are global. The proof of existence of higher energy solutions in Barbu, Lasiecka and Rammaha (2005) is technical and involves rather special fixed point argument along with a barrier method used for quasilinear hyperbolic equations in Lasiecka and Ong (1999). It turns out that higher energy solutions are also unique. In fact, the uniqueness result holds for the same domain of parameters as local existence. This result is proved below.

LEMMA 3.4 *Local solutions given in Theorem 1.2 are unique. Moreover, the solutions depend continuously on the initial data in the topology of finite energy space i.e.* $H^1(\Omega) \times L_2(\Omega)$.

Proof. Let u_1 and u_2 denote two possible local solutions given by Theorem 1.2 and originating at the same initial condition. Our aim is to show that $\tilde{u} \equiv u_1 - u_2$ is equal identically to zero. Since the solutions are a-priori in $H^2(\Omega) \times H^1(\Omega)$ one can justify application of energy method applied to the equation obtained for $\tilde{u}$. By exploiting the convexity of j and denoting $\tilde{E}(t) = \int_\Omega [|\tilde{u}_t|^2 + |\nabla \tilde{u}|^2] dx$, one obtains the following inequality satisfied for $0 \leq t < T_m$ where T_m is maximal time of existence:

$$\tilde{E}(t) \leq C \int_0^t \int_\Omega \tilde{u}_t [(|u_1|^k - |u_2|^k) j'(u_{2t}) + (|u_1|^p - |u_2|^p)] dx dt. \tag{72}$$

The main task is in estimating the two nonlinear terms on the right hand side of (72). This is done by exploiting growth condition in Assumption 1.1 along with multiple applications of Hölder's and interpolation inequalities.

$$\begin{aligned} &\int_\Omega \tilde{u}_t [(|u_1|^k - |u_2|^k) \partial j(u_{2t}) dx \\ &\qquad \leq C |\tilde{u}_t|_{0,\Omega} |\tilde{u}|_{L_{\frac{2n}{n-2}}} [\int_\Omega [|u_1|^{n(k-1)} + |u_2|^{n(k-1)}] |u_{2t}|^{mn} dx]^{\frac{1}{n}}. \end{aligned} \tag{73}$$

Since $n < 5$, we have $H^2(\Omega) \subset L_r(\Omega), 1 \leq r < \infty$, we are in a position to apply

Hölder's inequality to the second term in (73) with $r \to \infty$. This gives

$$\begin{aligned}
&\int_\Omega [|u_1|^{n(k-1)} + |u_2|^{n(k-1)}]|u_{2t}|^{mn} dx \\
&\quad \le C[\int_\Omega [|u_1|^{n(k-1)r} + |u_2|^{n(k-1)r}]dx]^{\frac{1}{r}} [\int_\Omega |u_{2t}|^{mn\bar{r}} dx]^{\frac{1}{\bar{r}}} \\
&\quad \le C[|u_1|_{2,\Omega}^{n(k-1)} + |u_2|_{2,\Omega}^{n(k-1)}]|u_{2t}|_{L_{mn\bar{r}}}^{mn}. \qquad (74)
\end{aligned}$$

Since $\bar{r} \to 1$ and condition (7) implies

$$m + 1 < \frac{n}{n-2} \Longrightarrow m < \frac{2}{n-2} \Longrightarrow mn < \frac{2n}{n-2} = p^*;$$

then by selecting an appropriate large r we obtain $mn\bar{r} \le p^*$. The Sobolev embedding at the critical level along with (74) then imply

$$\int_\Omega [|u_1|^{n(k-1)} + |u_2|^{n(k-1)}]|u_{2t}|^{mn} dx \le C[|u_1|_{2,\Omega}^{n(k-1)} + |u_2|_{2,\Omega}^{n(k-1)}]|u_{2t}|_{1,\Omega}^{mn}. \qquad (75)$$

As for the second nonlinear term in (72) we have

$$\begin{aligned}
&\int_\Omega |\tilde{u}_t|[|u_1|^p - |u_2|^p]dx \\
&\quad \le C|\tilde{u}_t|_{0,\Omega}[\int_\Omega |\tilde{u}|^{\frac{2n}{n-2}} dx]^{\frac{n-2}{2n}} (\int_\Omega [|u_1|^{n(p-1)} + |u_2|^{n(p-1)}]dx)^{\frac{1}{n}} \\
&\quad \le C|\tilde{u}_t|_{0,\Omega}|\tilde{u}|_{L_{p^*}}(|u_1|_{L_{n(p-1)}}^{p-1} + |u_2|_{L_{n(p-1)}}^{p-1}) \\
&\quad \le C|\tilde{u}_t|_{0,\Omega}|\tilde{u}|_{1,\Omega}(|u_1|_{2,\Omega}^{p-1} + |u_2|_{2,\Omega}^{p-1}), \qquad (76)
\end{aligned}$$

where in the last step we have used the Sobolev embeddings:

$$H^1(\Omega) \subset L_{p^*}(\Omega) = L_{\frac{2n}{n-2}}(\Omega), \text{ and } H^2(\Omega) \subset L_{n(p-1)}(\Omega).$$

Combining (72), (73, (75) and (76) one has for $0 \le t < T_m$

$$\tilde{E}(t) \le C \int_0^t |\tilde{u_t}|_{0,\Omega}|\tilde{u}|_{1,\Omega}[|u_1|_{2,\Omega}^{n(k-1)} + |u_2|_{2,\Omega}^{n(k-1)}]|u_{2t}|_{1,\Omega}^{m} dt. \qquad (77)$$

Since $u_i \in H^2(\Omega), u_{it} \in H^1(\Omega)$ for $i = 1, 2$, the above inequality implies the desired conclusion via standard Gronwall's inequality. ■

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