Dedicated to
Professor Czesław Olech

Control and Cybernetics

vol. **34** (2005) No. 3

Bang-bang controls in the singular perturbations limit

by

Zvi Artstein[1]

Department of Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel
e-mail: zvi.artstein@weizmann.ac.il

Abstract: A general form of the dynamics obtained as a limit of trajectories of singularly perturbed linear control systems is presented. The limit trajectories are described in terms of probability measure-valued maps. This allows to determine the extent to which the bang-bang principle for linear control systems is carried over to the singular limit.

Keywords: bang-bang, singular perturbations, Young measures.

1. Introduction

The paper examines linear control systems of the form

$$\begin{aligned} \frac{dx}{dt} &= A_1(t)x + F_1(t)y + B_1(t)u \\ \varepsilon\frac{dy}{dt} &= A_2(t)x + F_2(t)y + B_2(t)u \\ x(t_0) &= \bar{x},\ y(t_0) = \bar{y}, \quad u \in K. \end{aligned} \tag{1}$$

Here $x \in R^n$, $y \in R^m$ and the set K is a fixed set in R^k (the set K is possibly, but not necessarily, compact). The matrix-valued maps in (1) have the apparent dimensions; they are assumed to be continuous in the t variable. The coefficient $\varepsilon > 0$ is thought of as a small parameter. It is therefore clear why the states x and y are referred to as the slow and, respectively, the fast states. In particular,

[1]Incumbent of the Hettie H. Heineman Professorial Chair in Mathematics. Research supported by a grant from the Israel Science Foundation.

we are interested in the characteristics of the limit behavior of solutions of the system as $\varepsilon \to 0$.

Singularly perturbed control systems in general, and singularly perturbed linear control systems in particular, have attracted the attention of investigators for both the mathematical challenges they offer and for the applications. An account of the theory, examples and applications can be found in Kokotovic, Khalil and O'Reilly (1999). A detailed analysis of linear systems was carried out in a series of papers by A.L. Dontchev and V.M. Veliov (1983, 1985a,b). The approach followed in these references is to analyze the limit characteristics of the solutions of (1) via the solutions of the, so called, system of order reduction; it is based on Tikhonov's approach to singularly perturbed equations. It suggests that the limit behavior of (1) as $\varepsilon \to 0$ is captured when the value $\varepsilon = 0$ is plugged in (1), namely,

$$\begin{aligned} \frac{dx}{dt} &= A_1(t)x + F_1(t)y + B_1(t)u \\ 0 &= A_2(t)x + F_2(t)y + B_2(t)u \\ x(t_0) &= \bar{x}, \quad u \in K. \end{aligned} \tag{2}$$

Thus, the coupled system of differential and algebraic equations should reveal the limit behavior of the singularly perturbed differential system. The initial condition for the fast dynamics is absent from (2) since it is assumed that it can be steered via a boundary layer to any of the solutions of the algebraic equation. The method is very effective in describing the limit behavior, in particular when applied to linear systems; but the solutions of (2) do not reveal the whole limit structure.

Recent studies allow for an analysis of cases where the order reduction method does not apply. The limit behavior then is captured by a system based on the notion of limit distributions of control and fast state on the fast time scale. See Artstein (2000, 2002, 2004c), Artstein and Gaitsgory (1997), Vigodner (1997). The method is an extension of the order reduction method. It is needed when the fast dynamics does not converge to an equilibrium. In the linear setting (1) one can also employ weak convergence of solutions in order to compensate for the lack of convergence on the fast time scale. This is the approach taken in Dontchev and Veliov (1983, 1985a,b). A more detailed description is provided by an analysis which incorporates the mentioned limit distributions. This is what we carry out in this paper.

Expressing the limit dynamics in terms of limit distributions allows us to get an extension of the bang-bang principle in the singular limit. The bang-bang principle guarantees that any attainable point can be reached with controls in the extreme points of the constraint set K. See, e.g., Hermes and LaSalle (1969), and Olech (1966, 1967) for a refined structure of the extreme trajectories. A straightforward extension is not captured by the order reduction method; an extension is, however, possible within the limit distributions framework.

The paper is organized as follows. The setting is displayed in the next section, where we also introduce a uniform integrability assumption under which the results are obtained, and show why it is needed. The general structure of the limit dynamics is presented in Section 3, followed, in Section 4, by conditions which guarantee that any trajectory that meets the criteria of a limit dynamics is indeed generated via such a limit process. The closing section displays our findings in regard to the attainable set and the bang-bang principle.

2. An underlying assumption

After indicating a pathological behavior of solutions of (1) if arbitrary sequences of solutions are allowed, we display in this section a uniform integrability assumption on the solutions. The results throughout the paper are obtained under this assumption.

As customary, a trajectory $(x_\varepsilon(\cdot), y_\varepsilon(\cdot), u_\varepsilon(\cdot))$ defined on an interval $[t_0, t_1]$ with values in $R^n \times R^m \times R^k$ is called an *admissible trajectory* of the system if it constitutes a solution to the differential equations in (1) (notice: in this terminology the initial conditions in (1) do not play a role). At times we shall be interested only in the slow part $x_\varepsilon(\cdot)$ of the admissible trajectory. We note that for prescribed initial conditions, say the conditions $x(t_0) = \bar{x}$ and $y(t_0) = \bar{y}$ in (1), a control $u_\varepsilon(\cdot)$ which is integrable on bounded intervals determines, for a fixed ε, a unique admissible trajectory. The following example indicates a possible pathological behavior in the limit.

EXAMPLE 2.1 Let x, y and u be scalars where $u \in [-1, 1]$, and consider the system

$$\frac{dx}{dt} = y$$
$$\varepsilon \frac{dy}{dt} = u. \tag{3}$$

Consider also the initial conditions $x(0) = 0$ and $y(0) = 0$. A straightforward calculation reveals that for a fixed ε and a fixed $t > 0$ any point x in the interval $[-\frac{t^2}{2\varepsilon}, \frac{t^2}{2\varepsilon}]$ is such that $x = x_\varepsilon(t)$ for some admissible trajectory which satisfies the initial conditions. The same set is, in fact, reachable when the controls are subject to be in the set $\{-1, 1\}$. In particular, as $\varepsilon \to 0$ the limit of the sets reachable by slow trajectories exhibits a discontinuity, and an instantaneous jump of the slow state at $t = 0+$ may occur; this happens in spite of the boundedness of the control variable.

The instantaneous jumps of the slow variable generated by the singular perturbations may be of interest (for an initial study of these see Artstein, 2004a). In the present paper, however, we restrict our attention to a situation where such jumps cannot occur. Rather than placing on (1) an assumption, which guarantees the continuity of a limit of the slow variable, we place an assumption on the choice of controls.

Assumption 2.1 Unless stated otherwise we assume: The family of admissible trajectories is such that $(y_\varepsilon(\cdot), u_\varepsilon(\cdot))$, when restricted to a bounded interval $[t_0, t_1]$, are uniformly integrable (as functions from $[t_0, t_1]$ into $R^m \times R^k$).

Proposition 2.1 *Let $(x_\varepsilon(\cdot), y_\varepsilon(\cdot), u_\varepsilon(\cdot))$ be a family of admissible trajectories which satisfy Assumption 2.1 and such that at the initial time t_0 the points $x_\varepsilon(t_0)$ belong to a bounded set in R^n. Assume that the trajectories are defined on an interval $[t_0, t_1]$. Then the slow solutions $x_\varepsilon(\cdot)$ are equi-continuous on the interval. In particular, for any sequence $\varepsilon_i \to \infty$ there is a subsequence, say ε_j, such that $x_{\varepsilon_j}(\cdot)$ converges uniformly to a continuous function.*

Proof. The proof follows easily from the variation of parameters formula. ■

Remark 2.1 Note that our underlying assumption is on the trajectories we examine rather than being an assumption on the system (1). This is a reasonable restriction a controller who faces a singularly perturbed system with small yet unknown perturbation would follow, in order to avoid a blow up effect. For instance, it is easy to see that within the framework of Example 2.1, any continuous function $x(\cdot)$ which is differentiable almost everywhere, satisfying $x(0) = 0$ and with an integrable derivative, is a uniform limit of trajectories $x_\varepsilon(\cdot)$ satisfying Assumption 2.1; while, as pointed out earlier, the system produces other limits, even discontinuous ones, of sequences not satisfying the assumption. In the sequel we recall conditions which guarantee that the assumption is satisfied whenever the initial conditions $(x_\varepsilon(t_0), y_\varepsilon(t_0))$ are in a bounded set.

3. The form of the limit trajectories

In this section we display the general form of a limit, as $\varepsilon \to \infty$, of admissible trajectories of the system (1) under Assumption 2.1. In the next section we comment on when a trajectory of such a form is indeed a limit of solutions of the perturbed system.

To this end we recall two notions, as follows.

A *Young measure*, in general, is a probability measure-valued map from a domain space into the family of probability measures on another space. In this paper we utilize Young measures defined on a time interval, say $[t_0, t_1]$, with values being probability measures on $R^m \times R^k$, namely the product space of the fast state and the control space. We denote these Young measures either by $\mu(\cdot)$ or by the bold face character $\boldsymbol{\mu}$. An ordinary trajectory $(y(\cdot), u(\cdot))$ defined on the interval can be regarded as a particular case of such Young measure where the value $(y(t), u(t))$ is interpreted as a Dirac measure, namely, a measure supported on singleton. Measurability of Young measures and convergence among Young measures are determined by the weak convergence of probability measures on the image space $R^m \times R^k$. A criterion for the convergence of $\mu_i(\cdot)$ to $\mu_0(\cdot)$ is the convergence

$$\int_{t_0}^{t_1}\int_{R^m\times R^k} h(t,y,u)\mu_i(t)(dy\times du)dt \to \int_{t_0}^{t_1}\int_{R^m\times R^k} h(t,y,u)\mu_0(t)(dy\times du)dt \tag{4}$$

for every bounded and continuous real-valued function $h(t, y, u)$. Useful introductions to Young measures theory are Valadier (1994) and Balder (2000). Integrability of a Young measure $\boldsymbol{\mu}$ over the interval $[t_0, t_1]$ is determined by the integrability of the t-dependent expression

$$|\mu(t)| = \int_{R^m\times R^k} (|y| + |u|)\mu(t)(dy \times du) \tag{5}$$

over $[t_0, t_1]$. Uniform integrability of a family $\boldsymbol{\mu}_i$ of Young measures is determined by the uniform integrability of the respective expressions $|\mu_i(\cdot)|$ induced by (5). It is clear that the integrability and uniform integrability notions for Young measures extend the corresponding notions for functions. We shall also use convex sets of probability measures, i.e., referring to the affine structure on probability measures given by $(\alpha\mu + (1-\alpha)\nu)(B) = \alpha\mu(B) + (1-\alpha)\nu(B)$.

The *distribution* of a function, say of $\gamma(\cdot)$ defined on a time interval $[s_0, s_1]$, is the probability measure on the image space determined by

$$D(\gamma(\cdot), [s_0, s_1])(B) = \frac{1}{s_1 - s_0}\lambda(\{s : \gamma(s) \in B\}), \tag{6}$$

where λ is the Lebesgue measure on the line. Let $\gamma_j(\cdot)$ be a sequence of functions defined, respectively, on intervals $[s_0, s_j]$ with $s_j \to \infty$. A *limit distribution* of the sequence is a cluster point in the space of probability measures of a sequence $D(\gamma_j(\cdot), [s_0, s_j])$ as $j \to \infty$. A particular case is *the individual limit distribution* of a function $\gamma(\cdot)$ defined on an infinite half line $[s_0, \infty)$. This is the limit point, as $s \to \infty$, of $D(\gamma(\cdot), [s_0, s])$ (if it exists), in the space of probability measures.

In the sequel we utilize limit distributions and individual limit distributions of functions of the form

$$(y(\cdot), u(\cdot)) : [0, \infty) \to R^m \times R^k \tag{7}$$

which solve the linear differential equation

$$\frac{dy}{ds} = A_2(t)x + F_2(t)y(s) + B_2(t)u(s), \tag{8}$$

where the slow time variable t and the slow state variable x are held fixed; notice that the time variable in (8) is denoted by s. For fixed x and t and a fixed initial condition y_0 we denote by $D(x, t, y_0)$ the set of individual limit distributions of pairs $(y(\cdot), u(\cdot))$ which solve (8), and for which the limit distribution exists. We denote by $D(x,t)$ the union of sets $D(x,t,y_0)$ of all individual limit distributions, with any initial condition y_0.

Finally, we introduce the following notations: Let μ be a probability distribution over $R^m \times R^k$. We denote by $M_y(\mu)$ and $M_u(\mu)$ the marginals of μ on the fast state space R^m and, respectively, on the control space R^k. We denote by $E(\mu)$ the expectation of μ (if exists), namely

$$E(\mu) = \int_{R^m \times R^k} (y,u)\mu(dy \times du), \tag{9}$$

and denote by $E_y(\mu)$ and $E_u(\mu)$ the projections of $E(\mu)$ on R^m and, respectively, R^k.

PROPOSITION 3.1 *Let $(x_\varepsilon(\cdot), y_\varepsilon(\cdot), u_\varepsilon(\cdot))$ be a family of admissible trajectories of (1) which satisfies Assumption 2.1 and such that the points $x_\varepsilon(t_0)$ belong to a bounded set in R^n. Assume also that the trajectories are defined on an interval $[t_0, t_1]$. Then for any sequence $\varepsilon_i \to 0$ there is a subsequence, say ε_j, such that:*

(i) The sequence $x_{\varepsilon_j}(\cdot)$ converges uniformly to a continuous function, say to $x_0(\cdot)$,

(ii) The sequence $(y_{\varepsilon_j}(\cdot), u_{\varepsilon_j}(\cdot))$ converges in the sense of Young measures, say to the Young measure $\mu_0(\cdot)$.

For any sequence which satisfies (i) and (ii) the following holds:

(iii) For almost every t in $[t_0, t_1]$ the value $\mu_0(t)$ is in the convex hull of the individual limit distributions in $D(x_0(t), t)$,

(iv) The expectation functions $E_y(\mu_0(\cdot))$ and $E_u(\mu_0(\cdot))$ are actually the weak-L_1 limits of $y_{\varepsilon_j}(\cdot)$ and, respectively, $u_{\varepsilon_j}(\cdot)$; denote these functions by $\overline{y}_0(\cdot)$ and $\overline{u}_0(\cdot)$. For almost every t the linear equation

$$0 = A_2(t)x_0(t) + F_2(t)\overline{y}_0(t) + B_2(t)\overline{u}_0(t) \tag{10}$$

is then satisfied, and,

(v) The limit trajectory $x_0(\cdot)$ solves the differential equation (here x is a variable)

$$\frac{dx}{dt} = A_1(t)x + F_1(t)\overline{y}_0(t) + B_1(t)\overline{u}_0(t). \tag{11}$$

Proof. Many of the elements of the proof have been established elsewhere, at times in more generality. Some comparisons and references are given in Remark 3.1 below.

Item (i) is covered by Proposition 2.1. Item (ii) follows from the compactness, in the space of Young measures, of the family $(y_\varepsilon(\cdot), u_\varepsilon(\cdot))$, parameterized by ε. Indeed, the uniform integrability estimate in Assumption 2.1 implies the tightness of this family when interpreted as a family of Young measure.

We now verify that any sequence satisfying (i) and (ii) satisfies property (iii). Denote by $\gamma_{\varepsilon_j}(\cdot) = (y_{\varepsilon_j}(\cdot), u_{\varepsilon_j}(\cdot))$ the sequence which satisfies (i) and (ii).

Recall that the convergence in the sense of Young measures implies that for almost every point τ, for large j the distribution of $\gamma_{\varepsilon_j}(\cdot)$ over an interval $[\tau-\delta, \tau+\delta]$, namely, the distribution $D(\gamma_{\varepsilon_j}(\cdot), [\tau-\delta, \tau+\delta])$, is close to $\mu_0(\tau)$ if δ is small enough. To quantize the claim: We fix such a point τ and for a given $\eta > 0$ we choose a $\delta > 0$ such that for j large enough the distance between $D(\gamma_{\varepsilon_j}(\cdot), [\tau-\delta, \tau+\delta])$ and $\mu_0(\tau)$ is smaller that η (the distance being measured by a prescribed metric on the space of probability measures which is compatible with weak convergence, say, the Prohorov metric).

For the chosen δ we denote the interval $[\tau-\delta, \tau+\delta]$ by I_δ. When I_δ is partitioned into a number of smaller intervals $I_{\delta,l}$, say $l = 1, \ldots, r$, then for each ε_j the distribution $D(\gamma_{\varepsilon_j}(\cdot), I_\delta)$ is the average over the index l (weighted by the lengths of the intervals) of the distributions $D(\gamma_{\varepsilon_j}(\cdot), I_{\delta,l})$.

Our aim now is to show that if η and further, δ, are small enough, then for large enough j and appropriately chosen $I_{\delta,l}$, most (according to the mentioned weights) of the distributions $D(\gamma_{\varepsilon_j}(\cdot), I_{\delta,l})$ are all close (again, in the chosen metric) to the set $D(x_0(\tau), \tau)$. Then their average is close to the convex hull of $D(x_0(\tau), \tau)$. By letting $\eta \to 0$, the proof is completed.

To this end we write $I_{\delta,l} = [\tau_1, \tau_2]$ and consider the change of variables $s = \varepsilon_j^{-1}(t - \tau_1)$ on $I_{\delta,l}$. In the new time scale the interval is, say, $[0, s_2]$. Now we fix j and compare the fast dynamics $\gamma_{\varepsilon_j}(\cdot)$ given above with the dynamics obtained by solving the fast equation (8) with the t-parameter fixed at τ and with the original control function $u_{\varepsilon_j}(\cdot)$. Thus, the control variable coordinate in the two versions is the same. The original fast trajectory is given by

$$y_{\varepsilon_j}(\sigma) = \Phi_{\varepsilon_j}(\sigma, 0) y_{\varepsilon_j}(0) + \int_0^\sigma (\Phi_{\varepsilon_j}(\sigma, s)(A_2(s) x_{\varepsilon_j}(s) + B_2(s) u_{\varepsilon_j}(s))) ds, \quad (12)$$

while in the (8) version the fast trajectory is

$$y_0(s) = e^{F_2(\tau)\sigma} y_{\varepsilon_j}(0) + \int_0^\sigma e^{F_2(\tau)(\sigma - s)} (A_2(\tau) x_0(\tau) + B_2(\tau) u_{\varepsilon_j}(s)) ds \quad (13)$$

where $\Phi_{\varepsilon_j}(\sigma, s)$ is the transition matrix associated with the homogeneous part of the linear equation (notice that the variables are expressed in the fast scale and that we use the same initial condition in the two versions).

The uniform convergence of the slow trajectories (item (i)) implies that for δ small enough the values $x_{\varepsilon_j}(s)$ are uniformly close to $x_0(\tau)$. Together with the continuity of the functions $F_2(\cdot), A_2(\cdot)$ and $B_2(\cdot)$ it follows that given a fixed σ_0, if δ is small enough, the functions $y_{\varepsilon_j}(\cdot)$ are uniformly close to $y_0(\cdot)$, on intervals $I_{\delta,l}$ of length less than σ_0, and for a uniformly integrable family of control functions $u_{\varepsilon_j}(\cdot)$. This estimate can be applied now to the majority (according to length) of control functions $u_{\varepsilon_j}(\cdot)$ in any partition $I_{\delta,l}$ of I_δ. This follows since the fast trajectories $\gamma_{\varepsilon_j}(\cdot)$ are assumed (Assumption 2.1) uniformly integrable on I_δ. To sum up, for a given σ_0 if δ is small enough and for a given j the partition $I_{\delta,l}$ is chosen such that the lengths of its intervals (in the s-variable) are all less than σ_0, then for most of the elements in the partition the distributions of $\gamma_{\varepsilon_j}(\cdot)$ as generated by (12) will be close to those generated by the equation with fixed coefficients (13).

Since the length of the fixed σ_0 is arbitrarily large, it follows (see Artstein, 1999, 2004b) that these distributions are close to the convex hull of $\mathrm{D}(x_0(\tau), \tau)$. As mentioned, this completes the proof (since their average is also close to the convex hull of $\mathrm{D}(x_0(\tau), \tau)$ and so is the average of the approximations $(y_{\varepsilon_j}(\cdot), u_{\varepsilon_j}(\cdot)))$. This verifies claim (iii).

The first part of claim (iv) follows from the definition of convergence in the sense of Young measures and Assumption 2.1. The equality almost everywhere in (10) follows easily when the measure is an individual limit measure. Indeed, the spatial average with respect to the limit measure of the right hand side of (8) amounts then to the right hand side of (11), while, since the trajectory does not converge to infinity, the time average of the dynamics clearly converges to zero (see, e.g., Artstein, 1999). Claim (v) follows now from standard averaging arguments. This completes the proof. ■

Remark 3.1 For the compactness arguments needed in the verification of (i) and (ii) see Balder (2000), Valadier (1994), and references therein. Property (ii) was stated and proved in Artstein and Vigodner (1996) within a dynamical systems setting and under a boundedness assumption. The extension to a general control system is straightforward, see, e.g., Artstein (1999, 2004b). The novelty in the present proof is that the result is verified under a uniform integrability, rather than boundedness, condition. Property (iv) implies, in particular, that the weak-L_1 limits of $y_{\varepsilon_j}(\cdot)$ and, respectively, $u_{\varepsilon_j}(\cdot)$ satisfy (10). This property has been established directly (for weak-L_2 convergence, but the arguments for L_1 are essentially the same) by Dontchev and Veliov (1983, 1985a,b). Property (v) has also been established by Dontchev and Veliov directly for the weak-L_2 limit.

The previous result provides a necessary condition for the limit dynamics, namely, it identifies candidates for being limits, as $\varepsilon \to 0$, of admissible (under Assumption 2.1) triplets $(x_\varepsilon(\cdot), y_\varepsilon(\cdot), u_\varepsilon(\cdot))$ which solve (1). The analog of our derivations within the classical order reduction approach would be trajectories $(x_0(\cdot), y_0(\cdot), u_0(\cdot))$ which solve equations (1) where 0 replaces ε, namely, satisfy (2). The latter is then the limit system, namely the system which (supposedly) determines the limit trajectories (see Kokotovic, Khalil and O'Reilly, 1999). Motivated by this approach we introduce the following terminology.

Definition 3.1 A pair $(x_0(\cdot), \mu_0(\cdot))$ is a trajectory of the limit system of (1) if:

(i) $x_0(\cdot) : [t_1, t_2] \to R^n$,

(ii) $\mu_0(\cdot)$ is an integrable Young measure, mapping $[t_1, t_2]$ to probability measures on $R^m \times R^k$,

(iii) For almost every t in $[t_0, t_1]$ the value $\mu_0(t)$ is in the convex hull of the individual limit distributions in $\mathrm{D}(x_0(t), t)$, in particular,

(iv) The point-wise expectation function $(\overline{y}_0(\cdot), \overline{u}_0(\cdot)) = E(\mu_0(\cdot))$, satisfies (10), and,

(v) The trajectory $x_0(\cdot)$ solves (11).

Notice that we define a trajectory of the limit system without actually defining the limit system itself. One may regard properties (i) - (v) in the previous definition as defining the limit system. It is easy to see that a trajectory of the order reduction (2) is then a particular case.

MOTIVATION 3.1 It is easy to see that in the case the constraint set K is convex, if $(x_0(\cdot), \mu_0(\cdot))$ is a trajectory of the limit system of (1) then the map $(x_0(\cdot), (\overline{y}_0(\cdot), \overline{u}_0(\cdot)))$ (namely, where the Young measure is the Dirac measure-valued map of the expectation functions) is also a trajectory of the limit system. The latter is then a solution of the coupled differential-algebraic equations (2), namely, the solution of the order reduction model. Our interest in the general measure-valued limit (even when K is convex) stems from applications, some of which are beyond the scope of this paper. Indeed, when solving an optimal control problem with a linear plant but with a nonlinear cost function, very often the order reduction equation (2) is not capable of providing a solution, and the measure-valued limit is the appropriate one. One such example was analyzed in detail in Artstein (2002).

4. Generation of the limit trajectories

The natural question arises, namely, under what conditions is a trajectory of the limit system (as given in Definition 3.1) indeed a limit (as $\varepsilon_j \to 0$) of trajectories of the perturbed equation? The problem arises already within the order reduction framework. The examples displayed now demonstrate what may go wrong; they motivate the positive results that follow.

EXAMPLE 4.1 Lack of stability and controllability gives rise to the following example:

$$\begin{aligned} \varepsilon\frac{dy_1}{dt} &= -y_1 + u \\ \varepsilon\frac{dy_2}{dt} &= -y_1 + u \\ y_1(0) &= 0, \quad y_2(0) = 0, \end{aligned} \tag{14}$$

where (14) is considered on the time interval $[0, 1]$ and the controls are scalars. The trajectory $(y_1(t), y_2(t), u(t)) = (t, 0, t)$ is a solution of the order reduction system. Yet it is not a limit of solutions of the perturbed equation; indeed, in the perturbed system the equality $y_1(t) = y_2(t)$ is satisfied. In (14) the initial condition of the system is compatible with the desired limit. A different, yet similar, problem arises when lack of controllability implies that the initial condition cannot be driven to the desired limit on the fast time scale.

Example 4.2 Controllability of the unrestricted system would not help if the controls are restricted, as follows:

$$\begin{aligned} &\varepsilon\frac{dy_1}{dt} = -y_1 + u \\ &\varepsilon\frac{dy_2}{dt} = -y_1 + y_2 + u \\ &y_1(0) = 0, \quad y_2(0) = 0, \end{aligned} \tag{15}$$

where (15) is considered again on $[0,1]$ and the control is restricted to, say, the positive half line $[0,\infty)$. The trajectory $(y_1(t), y_2(t), u(t)) = (t, 0, t)$ is, again, a solution of the order reduction system. Yet it cannot be materialized as a limit trajectory of solutions of the perturbed equation, this although the unrestricted system is controllable. Indeed, the restriction on u and a simple comparison argument would show that $y_2(t) \geq y_1(t)$.

The preceding examples motivate the following terminology:

Definition 4.1 The trajectory $(x_0(\cdot), \mu_0(\cdot))$ of the limit system of (1) is said to be generated by the perturbed system (1) if, for a sequence $\varepsilon_j \to 0$, the trajectory is the limit in the Young measures sense of admissible trajectories $(x_{\varepsilon_j}(\cdot), y_{\varepsilon_j}(\cdot), u_{\varepsilon_j}(\cdot))$ of (1) which satisfy Assumption 2.1.

We now display conditions under which any trajectory of the limit system is generated by the perturbed system. Within the order reduction approach the issue is, traditionally, resolved by assuming that the system can be stabilized around the desired limit point, a property which follows, typically, from controllability. Then each feasible trajectory of the limit system (which consists of functions solving (10) and (11)) is generated by the perturbed system. See Kokotovic, Khalil and O'Reilly (1999). The following results extend the arguments to the general setting (but notice that the property of steering which we assume does not imply the standard notion of controllability).

Proposition 4.1 *Let the pair $(x_0(\cdot), \mu_0(\cdot))$ be a trajectory of the limit system of (1) on the time interval $[t_0, t_1]$. Let $M_y(\mu_0(t))$ be the y-marginal of $\mu_0(t)$. Let $S_y(\mu_0(t))$ be the support of $M_y(\mu_0(t))$. Suppose that for every $\bar{t} \in [t_0, t_1]$ there is an open set $O(\bar{t})$ such that $S_y(\mu_0(t))$ is included in $O(\bar{t})$ for all t close enough to $\bar{t}$. Furthermore, any point y_1 in $O(\bar{t})$ can be steered to any other point, say y_2, in $O(\bar{t})$ employing the control system*

$$\frac{dy}{ds} = A_2(\bar{t})x_0(\bar{t}) + F_2(\bar{t})y(s) + B_2(\bar{t})u(s) \tag{16}$$

along a finite time interval (in the fast time scale) whose length depends only on the set $O(\bar{t})$ and the norms $|y_1|$ and $|y_2|$. Finally, assume that the initial condition y_0 in (1) can be steered on the fast scale to the set $O(t_0)$. Then the trajectory $(x_0(\cdot), \mu_0(\cdot))$ of the limit system is generated by the perturbed system (1).

Proof. The proof is constructive. We verify the claim under the assumption that $\mu_0(\cdot)$ is continuous in the weak topology. Otherwise we can either employ the Lusin's Theorem and make a reduction to the case of continuity, or, alternatively, choose the points $\bar{t}_i$ below to be appropriate Lebesgue points of the map; we leave out these details.

Consider now a fixed $\bar{t}$. The probability measure $\mu_0(\bar{t})$ is in the convex hull of the individual limit distributions in $D(x_0(\bar{t}), \bar{t})$. The properties of $O(\bar{t})$, in particular the steering property and the inclusion of $S_y(\mu_0(t))$ in $O(\bar{t})$, imply that for any $y_1 \in O(\bar{t})$ there exists a control function $u(\cdot)$ on $[0, \infty)$ which generates (via (16), i.e, on the fast time scale) $\mu_0(\bar{t})$ as an individual limit distributions in $D(x_0(\bar{t}), \bar{t}, y_1)$. Moreover, the possibility of steering any initial point within a bounded set to y_1 from any initial point within a finite interval, implies that except for an initial interval, say $[0, s(\bar{t})]$, the control function may be independent of the initial condition y_1 provided that the latter is within a prescribed bounded set, say $B(\bar{t})$. We choose $B(\bar{t})$ such that it includes part of $S_y(\mu_0(t))$ in its interior. No confusion should then arise if we ignore the initial interval, suppress the dependence on y_1, and denote the control function by $u(\bar{t}, \cdot)$.

Now, for a prescribed estimate $\eta > 0$ there is a bound $s(\eta) = s(\eta, \bar{t})$ such that the distance (say in the Prohorov metric) between $\mu_0(\bar{t})$ and the distribution resulting from applying $u(\bar{t}, \cdot)$ to (16) along $s(\eta)$ is less than η. The continuity of the coefficients in (1) implies that for $\delta(\eta) = \delta(\eta, \bar{t})$ small enough, if rather than at (16) the control is applied to

$$\frac{dy}{ds} = A_2(t(s))x(t(s)) + F_2(t(s))y(s) + B_2(t(s))u(s) \tag{17}$$

where $t(s)$ is within a $\delta(\eta)$ neighborhood of $\bar{t}$, and $x(t(s))$ is close to $x_0(\bar{t})$, then the resulting distribution will be, say, 2η-close to $\mu_0(\bar{t})$. Equation (17) is the one generated by applying the change of variables $t = \tau_1 + \varepsilon_j s$ for an appropriate choice of τ_1 near $\bar{t}$. We conclude therefore that, given $\eta > 0$ and given $s(\eta)$, if an interval $[\tau_1, \tau_2]$ within the $\delta(\eta, \bar{t})$ neighborhood of $\bar{t}$ is identified such that the length of the interval is related to $s(\eta)$ by $s(\eta) = \varepsilon_j(\tau_2 - \tau_1)$, then with an initial condition in the prescribed set $B(\bar{t})$ in $O(\bar{t})$, an appropriate control function will generate on this interval a distribution which is close up to 2η from $\mu_0(\bar{t})$.

Given $\eta > 0$ we choose now a finite partition of $[t_0, t_1]$, determined by points $\bar{t}_i$, such that $\bar{t}_{i+1} - \bar{t}_{i-1}$ is less that $\delta(\eta, \bar{t})$ and also that for $\bar{t}_{i-1} \le t \le \bar{t}_{i+1}$ the set $S_y(\mu_0(t))$ is included in $O(\bar{t}_i)$. Such a partition is possible due to the compactness of the interval. For ε_j small enough we partition the interval $[t_0, t_1]$

to subintervals such that if the point $\bar{t}_i$ is the one closest to a subinterval, then the length of this subinterval is $\varepsilon_j s(\eta, \bar{t}_i)$. On each such interval if the initial condition of the fast dynamics is within $B(\bar{t}_i)$ and if the slow variable is within a small neighborhood of $x_0(\bar{t}_i)$ then a distribution close to $\mu_0(\bar{t}_i)$ can be generated. The possibility to have indeed the initial condition within $B(\bar{t})$ follows from the continuity assumption on $\mu_0(\cdot)$. When this is done then a Young measure close to $\mu_0(\cdot)$ is obtained. Indeed, for fixed $\eta > 0$ when $\varepsilon_j \to 0$ the approximation of the Young measure by the resulting distributions is of order η. As $\eta \to 0$ the desired limit is generated, provided that the process is feasible, namely, provided that the resulting slow trajectory converges to $x_0(\cdot)$.

The feasibility of the process for $\eta \to 0$ and for $\varepsilon_j(\eta) \to 0$ follows from the linearity of the equation of the slow dynamics, which, in turn, implies the uniqueness of the solution. Indeed (compare with Artstein, 2004c), the Peano type approximations of the slow dynamics that are obtained in this manner must converge to a solution of (11). Since for the given initial condition the only solution of (11) is $x_0(\cdot)$, the convergence implies that the slow dynamics parameters in (17) satisfy the desired estimates. This completes the proof. ■

Extension 4.1 A generalization to the previous result can be achieved as follows. Notice that the role of the steering property stated in the proof was to allow to generate the approximation to the distribution $\mu_0(\bar{t})$ starting from any initial point y_1 in a bounded subset of $O(\bar{t})$. The steering property allows to do that by steering y_1 to a common point from which $\mu_0(\bar{t})$ is generated. The time the steering itself consumes does not affect the approximation in the limit, this since the steering time is finite, and in particular independent of η, and hence of $s(\eta)$, on the fast scale. A generalization can be formulated such that rather than exact steering within $O(\bar{t})$ one assumes the possibility to steer y_1 to a small neighborhood (small enough to maintain the approximations) of the aforementioned common point, within a time which may depend on η, and may not be bounded, as long as the ratio of this steering time to $s(\eta)$ tends to zero as $\eta \to 0$. To work out the details may be tedious, yet the building blocks of the construction are similar to those presented in the proof of Proposition 4.1.

Example 4.3 We provide an example of the previous generalization, where the proof can be verified directly. Consider the system (1) with the additional assumption that for each t the matrix $F_2(t)$ is stable, namely, has eigenvalues with negative real part. As mentioned, this is a common assumption within the order reduction approach and was employed in various generalizations. A thorough analysis of well posedness under this assumption is provided in Dontchev and Veliov (1983). Furthermore, assume that the set K of controls u is compact. It is easy to see then that Assumption 2.1 is satisfied by any family with initial conditions in a bounded set.

Under these conditions Dontchev and Veliov (1983) have determined the limit attainable set. In particular, given the fixed slow state x and a fixed time t, the limit attainable set of the fast variable y is given by

$$F_2(t)^{-1}A_2(t)x + \int_0^\infty e^{F_2(s)}B_2(t)K ds \tag{18}$$

where the integral is interpreted as the Aumann integral of a set-valued map. For details see Dontchev and Veliov (1983). The displayed formula determines the closure of the points y in the attainable set in large fast time intervals. It is easy now to determine the limit distributions generated by the fast equation. Indeed, these are the distributions generated by trajectories of the form

$$(y(s),\ u(s)) = \left(F_2(t)^{-1}A_2(t)x + \int_0^s e^{F_2(\sigma)}B_2(t)K d\sigma,\ u(s)\right) \tag{19}$$

with $u(\cdot)$ any measurable function with values in K. Since the contribution of the initial condition y_1 is decaying, the consequence of the previous results holds, although the steering within finite fast time may be possible.

EXAMPLE 4.4 A concrete example of the previous argument is the system (with scalar variables)

$$\begin{aligned} \frac{dx}{dt} &= y_1 - y_2 \\ \varepsilon\frac{dy_1}{dt} &= -y_1 + u \\ \varepsilon\frac{dy_2}{dt} &= -2y_2 + u \\ u &\in [-1, 1], \end{aligned} \tag{20}$$

analyzed in Dontchev and Veliov (1983); the latter reference displays the attainable set of the system. The need to examine limit distributions of (20) arose in Artstein (2002) in connection with an optimization problem, namely, an optimal distribution is detected, and a scheme for generating it is displayed.

REMARK 4.1 As mentioned, within the classical order reduction method the measure $\mu_0(t)$ is assumed to be a singleton, around which the fast dynamics can be stabilized by a linear feedback. The analog of this method in the framework of this paper would be the ability to stabilize the fast dynamics around a trajectory $(y_0(\cdot), u_0(\cdot))$ (on the fast time scale $s \in [0, \infty)$) which generates $\mu_0(\bar{t})$. Under such an assumption the proof of Proposition 4.1 can be made simpler. But notice that even controllability of the system (16) does not imply the possibility to stabilize around a given trajectory.

5. The attainable set and bang-bang controls

The celebrated bang-bang principle for linear control systems (verified first in LaSalle, 1959) asserts, roughly, that any state that can be reached employing controls in a compact convex set K, can also be reached using controls restricted to the extreme points of K. See, e.g., Hermes and LaSalle (1969). A further major development was carried out by Olech (1966, 1967) where the structure of these extreme point-valued functions has been revealed. In particular, Olech showed that an extreme point in the attainable set of the linear system is reached by a unique, so called, extremal solution; and any other point can be reached via concatenating at most n (the dimension of the space) extremal solutions. These principles apply, of course, to the the linear system (1) for each fixed ε. In this section we examine the structure of the attainable set in the singular limit of (1) and the extent to which the bang-bang principle is carried over to the limit.

In what follows we consider the system (1) with the controls constrained to K, a compact and convex set. (The derivations go through when $K = K(t)$ is compact-valued, integrally bounded and varies measurably in time. We do not pursue this possibility here in detail.) We consider the system along a finite time interval $[t_0, t_1]$.

In general, one is interested in the attainable set of the coupled slow and fast dynamics. As was demonstrated in Dontchev and Veliov (1983, 1985a,b), in a quite general situation the two dynamics can be treated separately. We comment on this aspect toward the end of the section and treat now the slow variables in the attainable set. The following is a capturing of the notion of the attainable set of the slow flow within the limit dynamics displayed in this paper.

Definition 5.1 Given the initial condition $x(t_0) = x_0$ we denote by $\mathcal{A}(x_0, t_0, \bar{t})$ the set of points x such that $x = x(\bar{t})$ for some pair $(x_0(\cdot), \mu_0(\cdot))$ which is a trajectory of the limit system of (1) satisfying $x(t_0) = x_0$ (see Definition 3.1). Given an integrable real-valued function $\beta = \beta(\cdot)$, we denote by $\mathcal{A}_\beta(x_0, t_0, \bar{t})$ the subset of $\mathcal{A}(x_0, t_0, \bar{t})$ of points $x = x(\bar{t})$ obtained as above with the Young measure satisfying $|E_y(\mu_0(\cdot))| \leq \beta(\cdot)$.

Notice that in the generation of $\mathcal{A}(x_0, t_0, \bar{t})$ we assume that $\mu_0(\cdot)$ is integrable (see (5)), but we do not place a bound on its integral. Furthermore, in the definition of $\mathcal{A}_\beta(x_0, t_0, \bar{t})$ the integrability bound is assumed on the expectation of the measure and not on the measure itself.

Proposition 5.1 *Any point in $A_0(x_0, t_0, \bar{t})$ is also generated by a trajectory of the limit system of the form $(x_0(\cdot), y(\cdot), u(\cdot))$, namely with a Dirac measure-valued Young measure on the fast state and control spaces.*

Proof. As was already noted earlier, in case the constraint set K is convex, if $(x_0(\cdot), \mu_0(\cdot))$ is a trajectory of the limit system of (1) then the triplet of point-valued functions $(x_0(\cdot), (E_y(\mu_0(\cdot)), E_u(\mu_0(\cdot)))$ is also a trajectory of the limit system. ■

Remark 5.1 In the special case where $F_2(t)$ is invertible for each t (this case is the most common one in the available literature, see Kokotovic, Khalil and O'Reilly, 1999; Dontchev and Veliov, 1983, 1985a,b), the fast variable in the triplet is determined by the control. Thus, the attainable set is determined by the control function in the triplet.

Proposition 5.2 *For each x_0 and $\bar{t}$ the set $A_0(x_0, t_0, \bar{t})$ is convex but may not be compact. For an integrable $\beta = \beta(\cdot)$ the set $\mathcal{A}_\beta(x_0, t_0, \bar{t})$ is convex and compact.*

Proof. As pointed out in Proposition 5.1, since K is convex, any point in $A_0(x_0, t_0, \bar{t})$ is reached by invoking triplets $(x(\cdot), y(\cdot), u(\cdot))$ which solve (10) and (11), and satisfying $x(t_0) = x_0$. When applying the argument to a point in $\mathcal{A}_\beta(x_0, t_0, \bar{t})$ it follows from the definition that the resulting triplet also satisfies the integral bound $\beta(\cdot)$. To the resulting triplets we may apply the convexity arguments; the linearity of the two equations implies that the set of such triplets is convex. In particular, the attainable set $\mathcal{A}(x_0, t_0, \bar{t})$, and for any $\beta = \beta(\cdot)$ the set $\mathcal{A}_\beta(x_0, t_0, \bar{t})$, are convex. This verifies the first claim. The second one is verified by Example 2.1, which is one for which the attainable set, say $A_0(0, 0, \bar{t})$, is unbounded for any $\bar{t} > 0$. Indeed, $(x(\cdot), y(\cdot), 0)$ with $x(t) = \int_0^t y(s)ds$ is a trajectory of the limit system of (3). (Moreover, the conditions of Proposition 4.1 hold and therefore any such trajectory of the limit system is generated as a limit of solutions of the perturbed system.) It is clear then that any point x can be attained as $x = x(\bar{t})$ when $\bar{t} > 0$. Finally, for a given integrable real-valued function $\beta(\cdot)$ the trajectories of the limit system of the form $(x(\cdot), y(\cdot), u(\cdot))$, and such that $|y(t)| \leq \beta(t)$, form a compact set in the space of Young measures. In particular, since the map which maps such a triplet to the value $x(\bar{t})$ is continuous, the compactness of $\mathcal{A}_\beta(x_0, t_0, \bar{t})$ follows. This completes the proof. ■

In view of the preceding considerations, especially Proposition 5.1, the natural extension of the bang-bang principle to the singularly perturbed framework would be that any point in the attainable set of the limit system is reached by the triplet $(x_0(\cdot), y(\cdot), u(\cdot))$ with $u(\cdot)$ taking values in the extreme points of K, which we denote by $extK$. Such a result does not hold. Indeed, equation (10) imposes a state-dependent constraint on the controls which, in turn, may prohibit bang-bang controls, as the following example demonstrates.

Example 5.1 Consider the system with scalar variables given by

$$\begin{aligned} &\frac{dx}{dt} = y \\ &\varepsilon \frac{dy}{dt} = x + u \\ &x(0) = 0, \ y(0) = 0, \end{aligned} \tag{21}$$

where $u \in K = [-1, 1]$. It is clear that $x(\bar{t}) = 0$ is in the attainable set of the system; indeed, it is reached when the control $u(t) \equiv 0$ is chosen. However, as long as $|x(t_0)| < 1$, the extreme points of the constraint set do not satisfy (10).

The following is the bang-bang result in the singular limit. Notice that we state it for trajectories of the limit systems obtained under Assumption 2.1, but we do not assume the steering type assumptions displayed in Section 4.

PROPOSITION 5.3 *Let* $x_1 \in \mathcal{A}(x_0, t_0, \bar{t})$ *be obtained as* $x_1 = x_0(\bar{t})$ *with* $(x_0(\cdot), \mu_0(\cdot))$ *being a trajectory of the limit system of (1) which is generated by the perturbed system and satisfying* $x(t_0) = x_0$. *Then there exists a trajectory, say* $(x_1(\cdot), \mu_1(\cdot))$, *of the limit system of (1) generated by the perturbed system and satisfying* $x(t_1) = x_0$, *such that* $x_1(\bar{t}) = x_1$ *and such that for every* t *the marginal* $M_u(\mu_1(t))$ *is supported on the closure of extreme points of* K.

Proof. Let $(x_{\varepsilon_j}(\cdot), y_{\varepsilon_j}(\cdot), u_{\varepsilon_j}(\cdot))$ be the sequence of admissible trajectories which generates $(x_0(\cdot), \mu_0(\cdot))$ (according to Definition 4.1). We use now the bang-bang principle in linear ordinary control systems and deduce that for every ε_j there is a trajectory of (1), say $(\bar{x}_{\varepsilon_j}(\cdot), \bar{y}_{\varepsilon_j}(\cdot), \bar{u}_{\varepsilon_j}(\cdot))$, for which $\bar{x}_{\varepsilon_j}(0) = x_0$ and $\bar{x}_{\varepsilon_j}(\bar{t}) = x_1$, and, in fact, $\bar{y}_{\varepsilon_j}(\bar{t}) = y_{\varepsilon_j}(\bar{t})$, and such that $\bar{u}_{\varepsilon_j}(t)$ takes values in the extreme points $extK$ of K. Furthermore, since the bang-bang principle for the perturbed system covers the fast trajectory as well, the argument can be applied also to any small subinterval, and if this is done successively on small enough intervals, the uniform integrability of the admissible trajectories is maintained. The uniform integrability implies that a subsequence converges in the sense of Young measures, and that the corresponding slow trajectories converge uniformly, say to $(x_1(\cdot), \mu_1(\cdot))$. Then, clearly, $x_1(\bar{t}) = x_1$. Since the family $\bar{u}_{\varepsilon_j}(t)$ takes values in the extreme points $extK$ of K it follows that the support of the u-marginal of $\mu_1(\cdot)$ is in the closure of $extK$. This completes the proof. ∎

REMARK 5.2 As was pointed out earlier, the arguments go through when, rather than having a time invariant constraint set, one allows a time varying one, $K(t)$. Care, however, should be taken when the bang-bang result is verified. The argument in the preceding proof implies a weaker result. Namely, let E be the set $\{(t, u) : t \in [t_0, t_1], u \in extK(t)\}$. Then the support of the u-marginal of $\mu_1(t)$ is in the t-section of the closure of E. A more careful argument would show, however, that the consequence of Proposition 5.3 is valid, namely, the support of $M_u(\mu_1(t))$ is in the closure of $extK(t)$. This follows by applying the convergence of Young measures on closed subsets of $[t_0, t_1]$ on which (according to Lusin's theorem) the closure of $extK(t)$ is a continuous set-valued map.

The preceding result implies in particular that the bang-bang controls which generate the terminal state for the perturbed system, converge (in distribution in the sense of Young measures) to the bang-bang control which generates the terminal state in the limit system. A similar result is not valid when one examines the extremal solutions which generate the terminal state according to Olech's theory. The following example illustrates the displayed bang-bang result and the lack of convergence of Olech's extremal solutions.

EXAMPLE 5.2 Consider again the example where x, y and u are scalars, $u \in [-1, 1]$, and the system is

$$\begin{aligned} &\frac{dx}{dt} = y \\ &\varepsilon \frac{dy}{dt} = u \\ &x(0) = 0,\ y(0) = 0. \end{aligned} \tag{22}$$

The point 0 is in the attainable set of the limit system for $\bar{t} = 1$. Indeed, it is reached by employing the control with the constant value 0. It is also reached by employing the bang-bang control which assigns equal weights to -1 and 1. It is easy to see that the resulting trajectory of the limit system is generated by the perturbed equation, as guaranteed by the preceding derivations. For each fixed ε the state $x = 0$ at the time 1 can also be reached by switching once between two extremal trajectories, namely, employing the controls $+1$ and -1 on, respectively, the two halves of the time interval. This is a particular case of Olech's fundamental analysis in Olech (1966, 1967). Following this strategy for a sequence $\varepsilon_j \to 0$ would not result in a sequence satisfying Assumption 2.1; rather, a blow up in the limit of the fast dynamics will occur.

We conclude with some comments on the generation of the fast state of the attainable set of the limit system via bang-bang controls.

REMARK 5.3 The fast state variables y attainable by the singularly perturbed system at a given time $\bar{t}$ extend beyond the support of the measure $M_y(\mu_0(\bar{t}))$, namely, the y-marginal part of the trajectory of the limit dynamics. Indeed, on the fast scale the controller may steer the fast dynamics to a prescribed point without affecting the slow dynamics. This has ramifications in, say, optimization problems of the Meyer type, as pointed out and analyzed by Dontchev and Veliov (1983, 1985a,b). We are interested in the bang-bang facet of the issue. To this end it is sufficient to note that any attainable point $y_\varepsilon(\bar{t})$ can be written as

$$\begin{aligned} y_\varepsilon(\bar{t}) \quad = \quad & \Phi_\varepsilon(\bar{t}, \bar{t} - r\varepsilon) y_\varepsilon(\bar{t} - r\varepsilon) \\ & + \int_{\bar{t} - r\varepsilon}^{\bar{t}} (\Phi_\varepsilon(\bar{t}, \bar{t} - \tau)(A_2(\tau) x_\varepsilon(\tau) + B_2(\tau) u_\varepsilon(\tau))) d\tau, \end{aligned} \tag{23}$$

with r any fixed number and where $\Phi_\varepsilon(t, \tau)$ is the transition matrix of the homogeneous part of the perturbed fast dynamics. If one can deduce that $y_{\varepsilon_j}(\bar{t} - r\varepsilon_j)$ converges, say to $\bar{y}$, then the standard change of time scale to the fast scale would yield

$$y_0(\bar{t}) = e^{F_2(\bar{t})r}\bar{y} + \int_0^r e^{F_2(\bar{t})(r-s)}(A_2(\bar{t})x_0(\bar{t}) + B_2(\bar{t})u(\sigma))d\sigma \tag{24}$$

for some control function $u(\sigma)$. In some particular cases Dontchev and Veliov have found that the convergence sought after indeed occurs. (In case $F_2(\bar{t})$ is stable the limit as $r \to \infty$ can also be taken, resulting in the closed form (18) that we copied from Dontchev and Veliov, 1983.) In any case, the form (24) assures that when the limit is tractable, (e.g., (18) holds), the classical bang-bang principle can be applied and we may include the fast state part of the dynamics in the statement concerning bang-bang controls.

References

Artstein, Z. (1999) Invariant measures of differential inclusions applied to singular perturbations. *J. Differential Equations* **152**, 289-307.

Artstein, Z. (2000) The chattering limit of singularly perturbed optimal control problems. *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, 564-569.

Artstein, Z. (2002) An occupational measure solution to a singularly perturbed optimal control problem. *Control and Cybernetics* **31**, 623-642.

Artstein, Z. (2004a) On impulses induced by singular perturbations. In: M. de Queiroz, M. Malisoff and P. Wolenski, eds., *Optimal Control, Stabilization, and Nonsmooth Analysis, Lecture Notes in Control and Information Sciences* **301**, Springer-Verlag, Heidelberg, 61-72.

Artstein, Z. (2004b) Invariant measures and their projections in nonautonomous dynamical systems. *Stochastics and Dynamics* **4**, 439-459.

Artstein, Z. (2004c) On the value function of singularly perturbed optimal control systems. *Proceedings of the 43rd IEEE Conference on Decision and Control*, Paradise Island, Bahamas, 432-437.

Artstein, Z. and Gaitsgory, V. (1997) Linear-quadratic tracking of coupled slow and fast targets. *Math. Cont. Sign. Syst.* **10**, 1-30.

Artstein, Z. and Vigodner, A. (1996) Singularly perturbed ordinary differential equations with dynamic limits. *Proceedings of the Royal Society of Edinburgh* **126A**, 541-569.

Balder, E.J. (2000) Lectures on Young measure theory and its applications to economics. *Rend. Istit. Mat. Univ. Trieste* **31**, supplemento 1, 1-69.

Dontchev, A.L. and Veliov, V.M. (1983) Singular perturbation in Mayer's problem for linear systems. *SIAM J. Control Optim.* **21**, 566-581.

Dontchev, A.L. and Veliov, V.M. (1985a) Singular perturbations in linear control systems with weakly coupled stable and unstable fast subsystems. *J. Math. Anal. Appl.* **110**, 1-30.

DONTCHEV, A.L. and VELIOV, V.M. (1985b) On the order reduction of linear optimal control systems in critical cases. In: A. Bagehi and H.Th. Jongen, eds., *Systems and optimization*. *Lecture Notes in Control and Inform. Sci.* **66**, Springer, Berlin, 61-73.

HERMES, H. and LASALLE, J.P. (1969) *Functional Analysis and Time Optimal Control*. Academic Press, New York.

KOKOTOVIC, P.V., KHALIL, H.K. and O'REILLY, J. (1999) *Singular Perturbation Methods in Control: Analysis and Design.* Academic Press, London, 1986. Reprinted as *Classics in Applied Mathematics* **25**, SIAM Publications, Philadelphia.

LASALLE, J.P. (1959) The time optimal control problem. *Theory of Nonlinear Oscillations*, **5**, Princeton University Press, Princeton, NJ, 1-24.

OLECH, C. (1966) Extremal solutions of a control system. *J. Differential Equations* **2**, 74-101.

OLECH, C. (1967) Lexicographical order, range of integrals and "bang-bang" principle. In: A.V. Balakrishnan and L.W. Neustadt, eds., *Mathematical Theory of Control*, *Proc. Conf. in Univ. of Southern California*, Jan. 30–Feb. 1, 1967, 35-45, Academic Press, New York.

VALADIER, M. (1994) A course on Young measures. *Rend. Istit. Mat. Univ. Trieste* **26** supp., 349-394.

VIGODNER, A. (1997) Limits of singularly perturbed control problems with statistical dynamics of fast motions. *SIAM J. Control Optim.* **35**, 1-28.