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Inversion of multifunctions and differential inclusions

by

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Abstract: We present a new inverse mapping theorem for correspondences. It uses a notion of differentiability for multifunctions which seems to be new. We compare it with previous versions. We provide an application to differential inclusions.

Keywords: derivative, differentiability, differential inclusion, multimapping, inverse mapping, implicit function, set-valued mapping.

1. Introduction

The extension of the classical inverse mapping theorem to correspondences (or relations, or multifunctions or multimappings) has been the subject of numerous works (see references from Aubin, Frankowska, 1987 to Borwein, 1986; Frankowska, 1987; Fabian, 1979; Gautier, 1990; Gautier, Isac, Penot, 1983; Grande, 1994; Khanh, 1986; Klatte, Kummer, 2001, 2002; Kummer, 1991a,b; Ledyaev, Zhu, 1999; Lemaréchal, Zowe, 1991; Levy, 2001b; Martelli, Vignoli, 1974; Penot, 1985, 1995; Robinson, 1976, 1991; Sach, 1998; Serovaiskij, 1995; Silin, 1997; Szilágyi, 1989; Yen, 1987; Zhang, Treiman, 1995, ...). The variety of concepts and methods involved corresponds to the multiplicity of purposes. Here our method relies on a fixed point theorem, as in the classical case of mappings, and not on the Ekeland variational principle, as in many works using nonsmooth analysis methods; in fact, it is known that there exists a link between these two tools. For that purpose, we introduce some notions of differentiability for correspondences which are rather stringent, but close to the familiar one for mappings. They slightly differ in order to capture different situations created by multivaluedness: the simplest notion is seldom satisfied in practice, as shown by the examples we present. A modification involving a neighborhood of the image point makes the concept more usable; it corresponds to the change from Lipschitz multimappings to pseudo-Lipschitz multimappings, so that we call it pseudo-differentiability. We consider this notion central. However, in our results, such a modification is not always satisfied and we have to further weaken the condition to what we call quasi-differentiability. For single-valued maps, these three concepts collapse to ordinary Fréchet differentiability. We also use strengthenings of these concepts which correspond to what is called strong or strict or differentiability; here we speak of peridifferentiability in order to avoid confusions when strict inequalities are present.

We briefly compare these concepts to some other existing notions, without claim of completeness, as such notions abound in the literature and have various purposes. Among these aims are existence results for equations Khanh (1986), Pták (1982), Sengupta (1997), optimality conditions for constrained optimization Halkin (1974, 1976), Penot (1982) sensitivity analysis Klatte, Kummer (2002), Levy (2001a), Penot (1984), Robinson (1976). Here, as an application, we extend the approach of Choquet (1960), Penot (1970), Robbin (1968) to existence results for ordinary differential equations to existence results for differential inclusions as in Filippov (1967), Hermes (1970), Himmelberg, Van Vleck (1973), Zhu (1991)... Such a study reinforces the links between generalized differential calculus, differential inclusions and optimization (see Brown, Bartholomew-Biggs, 1989a, b for some other links between these last two subjects). Other tracks deal with generalized equations and applications of inverse mapping theorems to stability and optimal control (see references from Dontchev, 1996 to Dontchev, Rockafellar, 1997; Halkin, 1974; Klatte, Kummer, 2002; Ledyaev, Zhu, 1999; Robinson, 1976, 1991, ...).

2. Preliminaries

In this section we recall from Azé, Penot (2005) a fixed point theorem for correspondences and a perturbation result. For such an aim, we need to fix some notation and conventions. The open ball with center x and radius r in a metric space (X, d) is denoted by $B(x, r)$; if X is a normed vector space (n.v.s.) the closed unit ball is denoted by $\overline{B}_X$. We endow the product $Z := X \times Y$ of two metric spaces (resp. two normed vector spaces (n.v.s.)) with a *product distance*(resp. product norm), i.e. a distance (resp. norm) on Z for which the canonical projections and the insertions $x \mapsto (x, b)$, $y \longmapsto (a, y)$ are nonexpansive for any $a \in X$, $b \in Y$. Given a metric space (X, d) and given subsets C,

$D \subset X$ we take

$$d(x, D) = \inf_{y \in D} d(x, y) \text{ with the convention } \inf_{\varnothing} = +\infty,$$

$$e(C, D) = \sup_{x \in C} d(x, D) \text{ if } D \neq \varnothing,\ e(C, \varnothing) = +\infty \text{ if } C \neq \varnothing,\ e(\varnothing, D) = 0$$

$$d(C, D) = \max(e(C, D), e(D, C)).$$

The following definition captures the notion of Lipschitzian behavior for set-valued maps, which encompasses the notion of pseudo-Lipschitzian behavior (or Aubin property) but is more versatile since one can take for V a member of a family of bounded subsets or even an arbitrary subset.

Definition 2.1 *A multimapping $F : X \rightrightarrows Y$ between two metric spaces is said to be quasi-Lipschitzian on a subset U of X with respect to (w.r.t.) a subset V of Y if there exist $\kappa, \delta > 0$ such that for any $x, x' \in U$ satisfying $d(x, x') < \delta$ one has $e(F(x) \cap V, F(x')) \leq \kappa d(x, x')$. If the restriction $d(x, x') < \delta$ is removed, F is said pseudo-Lipschitzian on U w.r.t. V. When $\kappa \in [0, 1)$, $X = Y$, $V = U$, F is said to be pseudo-κ-contractive w.r.t. U.*

Given $F : X \rightrightarrows Y$ and $(x_0, y_0) \in \mathrm{Gr}(F)$, we say that F is *quasi-Lipschitzian* (resp. *pseudo-Lipschitzian*) *around* (x_0, y_0) if there are neighborhoods U, V of x_0 and y_0 respectively such that F is quasi-Lipschitzian (resp. pseudo-Lipschitzian) on U w.r.t. V. Then, since U can be shrunk to a smaller neighborhood of x_0, both concepts coincide. This is not the case in general for quasi-Lipschitzness for an arbitrary set U when U may be large. Clearly, if F is quasi-Lipschitzian around (x_0, y_0), then it is *lower semicontinuous at* (x_0, y_0) in the sense that $d(y_0, F(x)) \to 0$ as $x \to x_0$. Observe that F is lower semicontinuous at (x_0, y_0) if, and only if, there exists a selection f of F on some neighborhood of x_0 such that $f(x_0) = y_0$ and f is continuous at x_0 (take $f(x) \in F(x) \cap B(y_0, r(x))$ with $r(x) > d(y_0, F(x))$ for $x \in X \backslash \{x_0\}$, for instance $r(x) := d(y_0, F(x)) + d(x_0, x)$). Recall that F is said lower semicontinuous at x_0 if F is lower semicontinuous at (x_0, y_0) for every $y_0 \in F(x_0)$.

The existence of fixed points for pseudo-contractive multifunctions is well known (see Dontchev, Hager, 1993, 1996; Ioffe, Tihomirov, 1979, Lemma 1, p. 31; Penot, 1982, Prop. 2.5). Let us recall that result for the sake of clarity.

Proposition 2.1 (Nadler, 1969; Ioffe, Tihomirov, 1979; Penot, 1982; Azé, Penot, 2005) *Let (X, d) be a complete metric space and let $G : X \rightrightarrows X$ be a multifunction with closed, nonempty values. Suppose that G is pseudo-κ-contractive with respect to some open ball $B(x_0, r_0)$ for some $\kappa \in [0, 1)$ and $r := (1 - \kappa)^{-1} d(x_0, G(x_0)) < r_0$. Then the fixed point set $\Phi_G := \{x \in X : x \in G(x)\}$ of G is nonempty and*

$$d(x_0, \Phi_G \cap B(x_0, r_0)) \leq r. \tag{1}$$

Thus, for any $r' > r$ the set $\Phi_G \cap B(x_0, r')$ is nonempty.

In Azé, Penot (2005) it is just asserted that the set $\Phi_G \cap B(x_0, r_0)$ is nonempty and the looser estimate $d(x_0, \Phi_G) \leq r_0$ is given. However, replacing r_0 by $r'_0 \in (r, r_0)$ the estimate $d(x_0, \Phi_G) \leq r'_0$ for any $r'_0 \in (r, r_0)$ shows that $d(x_0, \Phi_G) \leq r$; whence for any $r' \in (r, r_0)$ the set $\Phi_G \cap B(x_0, r')$ is nonempty and contained in $B(x_0, r_0)$, so that $d(x_0, \Phi_G \cap B(x_0, r_0)) = d(x_0, \Phi_G) \leq r$: (1) holds.

We also need a sensitivity result about the fixed point set Φ_G when G varies in the set of multifunctions from X to X. In order to do so, identifying a multifunction with its graph, one could endow the power set $2^{X \times X}$ (the hyperspace of subsets of $X \times X$) with some topology or convergence. We avoid doing that and focus our attention on metric estimates. The result we give here is a variant of Prop. 2.3 from Azé, Penot (2005); it has a stronger assumption, akin to the one in Lemma 1 in Lim (1985), a stronger conclusion and a simpler proof. It suffices for our needs.

Proposition 2.2 *Let (X, d) be a complete metric space. Let $G : X \rightrightarrows X$ be a multifunction with closed nonempty values which is assumed to be pseudo-κ-contractive with respect to $B(x_0, r_0)$ for some $r_0 > 0$, and some $\kappa \in [0, 1)$. Then for any $r \in (0, r_0)$ and for any multifunction $H : X \rightrightarrows X$ satisfying*

$$e(H(x) \cap B(x_0, r), G(x)) < (1 - \kappa)(r_0 - r) \qquad \forall x \in B(x_0, r)$$

one has

$$\begin{aligned} e(\Phi_H \cap B(x_0, r), \Phi_G \cap B(x_0, r_0)) &\leq (1-\kappa)^{-1} \sup_{x \in B(x_0, r)} e(H(x) \cap B(x_0, r), G(x)) \\ &\leq r_0 - r. \end{aligned}$$

Proof. Let $x \in \Phi_H \cap B(x_0, r)$ (if there is no such x, there is nothing to prove since we adopted the convention $d(\varnothing, D) = 0$ for any subset D) and let $t > e(H(x) \cap B(x_0, r), G(x))$ be such that $t < (1 - \kappa)(r_0 - r)$. Since $x \in H(x) \cap B(x_0, r)$, we have

$$d(x, G(x)) < t < (1 - \kappa)(r_0 - r).$$

Since G is pseudo-κ-contractive with respect to $B(x, r_0 - r) \subset B(x_0, r_0)$, it follows from (1), from the preceding estimate and from Proposition 2.1 in which x_0 and r_0 are replaced with x and $r_0 - r$, respectively, that

$$d(x, \Phi_G \cap B(x_0, r_0)) \leq d(x, \Phi_G \cap B(x, r_0 - r)) \leq (1-\kappa)^{-1} d(x, G(x)) \leq (1-\kappa)^{-1} t,$$

Since t is arbitrarily close to $e(H(x) \cap B(x_0, r), G(x))$, the result follows by taking the infimum over t and then the supremum over $x \in \Phi_H \cap B(x_0, r)$. ■

Corollary 2.1 *Let (X, d), (Y, d_Y) be metric spaces, X being complete, and let $x_0 \in X$, $r_0 > 0$. Let $(G_y)_{y \in Y}$ be a family of multifunctions from $B(x_0, r_0)$ to X*

which are pseudo-κ-contractive with respect to $B(x_0, r_0)$, with closed nonempty values. Suppose that for some $r \in (0, r_0)$, $\lambda > 0$ and any $y, y' \in Y$ one has

$$e(G_{y'}(x) \cap B(x_0, r), G_y(x)) \leq \lambda d_Y(y, y') \qquad \forall x \in B(x_0, r),$$

Then the multimapping $F : y \mapsto \Phi_{G_y}$ is quasi-Lipschitz on Y w.r.t. $V := B(x_0, r)$, with Lipschitz rate $\lambda(1-\kappa)^{-1}$.

Proof. Let $\delta := \lambda^{-1}(1-\kappa)(r_0 - r)$. Given $y, y' \in Y$ such that $d_Y(y, y') < \delta$, let us set $G := G_y$ and $H := G_{y'}$, so that

$$e(H(x) \cap V, G(x)) \leq \lambda d_Y(y, y') < (1-\kappa)(r_0 - r) \qquad \forall x \in B(x_0, r).$$

Then, by Proposition 2.2,

$$\begin{aligned} e(\Phi_{G_{y'}} \cap V, \Phi_{G_y}) &\leq (1-\kappa)^{-1} \sup_{x \in B(x_0, r)} e(H(x) \cap B(x_0, r), G(x)) \\ &\leq (1-\kappa)^{-1} \lambda d_Y(y, y'). \end{aligned}$$ ■

3. Differentiability of multifunctions

From now on, unless otherwise specified, X, Y and Z are n.v.s. and $L(X, Y)$ denotes the set of continuous linear maps from X to Y. As mentioned above, in the multivalued case, a number of notions of differentiability can be given. The one which seems to be the most closely related to the notion of Fréchet differentiability is as follows. We incorporate to it a continuity condition, since such a requirement is satisfied in the single-valued case and since dropping it would lead to cases in which the estimate would be trivial. Here we say that a function $o : X \to \mathbb{R}$ is a *remainder* if $\lim_{x \to 0, x \neq 0} \|x\|^{-1} o(x) = 0$. Equivalently, o is a remainder if there exists a *modulus* $\mu : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ (i.e. a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ continuous at 0 with $\mu(0) = 0$) such that $\|o(x)\| \leq \mu(\|x\|) \|x\|$; without loss of generality, one may assume that μ is nondecreasing. This concept has been adopted by a number of authors. On the other hand the following notions seem to be new (albeit related to the concept of conical differentiability of Mignot, 1976). Note that they are outer notions in the sense that they impose a certain control of the expansion of the multimapping F around x_0, but do not require such a precise control for its shrinking. Since the terminology of the paper is rather heavy, we do not mention the word "outer" in these definitions. Replacing the excess by the Hausdorff distance would give more stringent notions which will be considered later on, in a still more strengthened form. We consider that the first notion we introduce is a crude, restrictive concept. The second one is more realistic.

Definition 3.1 *Let X, Y be n.v.s. and let X_0 be an open subset of X. A multifunction $F : X_0 \rightrightarrows Y$ with domain X_0 is said to be differentiable at $x_0 \in X_0$*

if it is lower semicontinuous at x_0 and if there exists some $A \in L(X,Y)$ such that the function o given by

$$o(x) := e(F(x_0 + x), F(x_0) + A(x)). \tag{2}$$

is a remainder, or, equivalently, for some modulus μ,

$$F(x_0 + x) \subset F(x_0) + A(x) + \mu(\|x\|)\,\|x\|\,\overline{B}_Y.$$

Then A is called a derivative of F at x_0. The set of derivatives of F at x_0 is denoted by $\mathcal{D}F(x_0)$.

It is said to be pseudo-differentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ if it is lower semicontinuous at (x_0, y_0) and if there exist some neighborhood V of y_0 and a continuous linear map $A : X \to Y$ called a derivative of F at (x_0, y_0) such that the function o_V defined as follows is a remainder

$$o_V(x) := e(F(x_0 + x) \cap V, F(x_0) + A(x)). \tag{3}$$

Equivalently, F is pseudo-differentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ if it is lower semicontinuous at (x_0, y_0) and if there exist some neighborhood V of y_0, a continuous linear map $A : X \to Y$ and a modulus μ such that for $x \in X$ one has

$$F(x_0 + x) \cap V \subset F(x_0) + A(x) + \mu(\|x\|)\,\|x\|\,\overline{B}_Y.$$

Clearly, a multifunction F, which is differentiable at x_0, is pseudo-differentiable at (x_0, y_0) for any $y_0 \in F(x_0)$; moreover, it is upper Lipschitz at x_0, therefore upper semicontinuous at x_0 in the Hausdorff sense, hence in the usual sense when $F(x_0)$ is compact. Conversely, when $F(x_0)$ is compact, when F is upper semicontinuous at x_0, and when F is pseudo-differentiable at (x_0, y_0) for any $y_0 \in F(x_0)$ with derivative A, then F is differentiable at x_0. Let us also observe that when $F(x_0) = \{y_0\}$, x_0 being in the interior of the domain of F and when o defined in (2) is a remainder, then F is lower semicontinuous at x_0 and differentiable at x_0. Such an observation shows that when F is single-valued, the preceding notions reduce to the classical concept of (Fréchet) differentiability.

Of course, in spite of their simplicities, these notions are not adapted to all situations which can be considered as smooth enough, although the presence of the set V increases applicability. In particular, a weaker notion may prevail.

Definition 3.2 *Let X, Y be n.v.s. and let X_0 be an open subset of X. A multifunction $F : X_0 \rightrightarrows Y$ is said to be quasi-differentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$, with derivative $A \in L(X,Y)$, if it is lower semicontinuous at (x_0, y_0) and if for any $\varepsilon > 0$ there exist some $\beta(\varepsilon) > 0$, $\delta(\varepsilon) > 0$ such that for $x \in \delta(\varepsilon)\overline{B}_X$ one has*

$$F(x_0 + x) \cap B(y_0, \beta(\varepsilon)) \subset F(x_0) + A(x) + \varepsilon\,\|x\|\,\overline{B}_Y. \tag{4}$$

The preceding definition is coherent with the usual definition of differentiability when F is a single-valued map, as easily seen.

EXAMPLE 3.1 Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $F(x) = [a - x^2, b + x^2]$ for each $x \in \mathbb{R}$, with $a \leq b$ in $\mathbb{R}$. Then F is differentiable at 0 and pseudo-differentiable at (x_0, y_0) for every $(x_0, y_0) \in \mathrm{Gr}(F)$; for $x_0 \neq 0$, it is not differentiable at x_0.

EXAMPLE 3.2 More generally, let $g : X \to Y$, $r : X \to \mathbb{R}$ be differentiable at x_0 and such that r takes positive values. Then the multimapping $F : X \rightrightarrows Y$ given by $F(x) := g(x) + r(x)\overline{B}_Y$ is differentiable at x_0 if $Dr(x_0) = 0$ and is pseudo-differentiable at (x_0, y_0) when $\|y_0 - g(x_0)\| < r(x_0)$, but it is not pseudo-differentiable at (x_0, y_0) when $Dr(x_0) \neq 0$ and $\|y_0 - g(x_0)\| = r(x_0)$. On the other hand, for any $y_0 \in F(x_0)$, F is quasi-differentiable at (x_0, y_0) : given $\varepsilon > 0$, picking $b_0 \in \overline{B}_Y$ such that $y_0 = g(x_0) + r(x_0)b_0$, setting $c := \|Dr(x_0)\|$, taking $\beta(\varepsilon) \in (0, \varepsilon r(x_0)/9c)$ one can find $\delta(\varepsilon) > 0$ such that for any $x \in \delta(\varepsilon)\overline{B}_X$ one has $\|g(x_0 + x) - g(x_0)\| < \beta(\varepsilon)$, $|r(x_0 + x) - r(x_0)| \leq \beta(\varepsilon)$ and

$$\begin{aligned} &g(x_0 + x) - g(x_0) - Dg(x_0)x \in (\varepsilon/3) \|x\| \overline{B}_Y, \\ &|r(x_0 + x) - r(x_0) - Dr(x_0)x| \leq (\varepsilon/3) \|x\|, \end{aligned}$$

so that, for any $y \in F(x_0 + x) \cap B(y_0, \beta(\varepsilon))$ one can write $y := g(x_0 + x) + r(x_0 + x)b$ for some $b \in \overline{B}_Y$ with $\|b - b_0\| < \varepsilon/3c$ since

$$\begin{aligned} \|r(x_0)b - r(x_0)b_0\| &\leq \|r(x_0)b - r(x_0 + x)b\| + \|r(x_0 + x)b - r(x_0)b_0\| \\ &\leq |r(x_0 + x) - r(x_0)| + \|y - y_0\| + \|g(x_0 + x) - g(x_0)\| \\ &< 3\beta(\varepsilon), \end{aligned}$$

hence $\|b - b_0\| < 3\beta(\varepsilon)/r(x_0) < \varepsilon/3c$ and, since $\|Dr(x_0)xb - Dr(x_0)xb_0\| \leq c\|x\| \|b - b_0\| \leq (\varepsilon/3) \|x\|$,

$$y - y_0 \in Dg(x_0)x + Dr(x_0)xb_0 + \varepsilon \|x\| \overline{B}_Y$$

and F is quasi-differentiable at (x_0, y_0) with derivative A given by $A(x) = Dg(x_0)x + Dr(x_0)xb_0$.

EXAMPLE 3.3 Let $g : X \to Y$ be a mapping between two n.v.s., which is differentiable at x_0. Then for any subset C of Y the multimapping $F : X \rightrightarrows Y$ given by $F(x) := g(x) + C$ is differentiable at x_0. Note that the inverse of F is of interest, especially when C is a closed convex cone, as the feasible set of a mathematical programming problem depending on a parameter is of that type. Such a case motivates our study.

The following lemma sheds some light over the preceding notions when $F(x_0)$ is a singleton.

LEMMA 3.1 *Let $F : X_0 \rightrightarrows Y$ be a multifunction such that $F(x_0) = \{y_0\}$ and let $A : X \to Y$ be a continuous linear map. Then among the following assertions one has the equivalences (a)⇔(b), (a')⇔(b'):*

(a) F is differentiable at x_0 with derivative A;

(a') F is quasi-differentiable at (x_0, y_0) with derivative A;

(b) any selection f of F is differentiable at x_0 with derivative A;

(b') F is lower semicontinuous at $(x_0, y_0) \in \mathrm{Gr}(F)$ and any selection f of F continuous at x_0 is differentiable at x_0 with derivative A.

Proof. The implication (a')⇒(b') is obvious. Let us show by contraposition that (b')⇒(a'). Suppose F is not quasi-differentiable at (x_0, y_0) with derivative A. Then there exist $\alpha > 0$, a sequence (x_n) with limit 0 and a sequence (y_n) such that $y_n \in F(x_0 + x_n) \cap B(y_0, 2^{-n})$ and $y_n \notin \left(y_0 + A(x_n) + \alpha \|x_n\| \overline{B}_Y\right)$ for each $n \geq 1$. Since F is lower semicontinuous at (x_0, y_0), it has a selection g, which is continuous at x_0. Let us set $f(x) := y_n$ for $x = x_0 + x_n$, $f(x) := g(x)$ for $x \in X \backslash \{x_0 + x_n : n \geq 1\}$. Then f is a selection of F which is continuous at x_0 and f is not differentiable at x_0 with derivative A. The equivalence (a)⇔(b) is similar and simpler. ■

COROLLARY 3.1 *When $F(x_0)$ is a singleton $\{y_0\}$ and F is quasi-differentiable at (x_0, y_0), the mapping A appearing in (4) is unique.*

Proof. This is a consequence of the uniqueness of the derivative of a mapping and of the fact that F has a selection which is continuous at x_0, hence is differentiable at x_0 by the preceding lemma. ■

When $F(x_0)$ is not a singleton, uniqueness may fail.

EXAMPLE 3.4 Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $F(x) = [-1, 1]$ for each $x \in \mathbb{R}$. Then F is differentiable at x_0 for every $x_0 \in \mathbb{R}$; for $y_0 \in (-1, 1)$ any $A \in L(\mathbb{R}, \mathbb{R})$ can be taken in the definition of pseudo-differentiability at (x_0, y_0).

EXAMPLE 3.5 Let X, Y be n.v.s. and let Y_0 be a linear subspace of Y, not reduced to $\{0\}$. The constant multimapping $F : X \rightrightarrows Y$ with value Y_0 is differentiable at each point and any continuous linear map from X into Y_0 can be taken as a derivative. Since F can be considered as very smooth, one sees that multivaluedness has new effects, even for the strongest notion among those we introduced.

A pleasant feature of the preceding concepts lies in the fact that they have simple calculus rules. We state them without proofs, since the proofs are similar to the ones for mappings and to the ones presented below for a stronger notion of differentiability. We use the notion of uniformly differentiable map: a map $g : Y \to Z$ between two n.v.s. is said to be *uniformly differentiable* on $Y_0 \subset Y$ if it is differentiable at each $y_0 \in Y_0$ and if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g(y) - g(y_0) - Dg(y_0)(y - y_0)\| \leq \varepsilon \|y - y_0\| \qquad \forall y_0 \in Y_0, \ \forall y \in B(y_0, \delta).$$

When Y_0 is finite, obviously this condition is a consequence of the differentiability of g at each point of Y_0.

Proposition 3.1 *(a) Let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be differentiable at $x_0 \in X$, with derivatives A and B, respectively. Then $H := (F, G) : X \rightrightarrows Y \times Z$ is differentiable at x_0 with derivative (A, B). If F and G are pseudo-differentiable (resp. quasi-differentiable) at $(x_0, y_0) \in \mathrm{Gr}(F)$ and $(x_0, z_0) \in \mathrm{Gr}(G)$ respectively, then $H := (F, G)$ is pseudo-differentiable (resp. quasi-differentiable) at $(x_0, (y_0, z_0))$.*

(b) Let $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$ be (pseudo-, resp. quasi-) differentiable at $x_0 \in X$ and $y_0 \in Y$ respectively (resp. at $(x_0, y_0) \in \mathrm{Gr}(F)$ and $(y_0, z_0) \in \mathrm{Gr}(G)$ respectively). Suppose $F(x_0) = \{y_0\}$. Then $H := G \circ F$ is (pseudo-, resp. quasi-) differentiable at x_0 (resp. (x_0, z_0)).

(c) Let $F : X \rightrightarrows Y$ be differentiable at $x_0 \in X$ and let $g : Y \to Z$ be a uniformly differentiable map on $F(x_0)$ with constant derivative. Then $g \circ F$ is differentiable at x_0.

(d) Let $F : X \rightrightarrows Y$ be pseudo-differentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ and let $g : Y \to Z$ be differentiable at y_0. Suppose g is open at y_0 and is injective. Then $g \circ F$ is pseudo-differentiable at $(x_0, g(x_0))$.

(e) Let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Y$ be differentiable at x_0. Then $F + G$ is differentiable at x_0.

(f) Let $F : X \rightrightarrows Y$ be pseudo (resp. quasi)-differentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ and let $g : X \to Y$ be differentiable at x_0. Then $H := F + g$ is pseudo (resp. quasi)-differentiable at $(x_0, y_0 + g(x_0))$.

A mean value theorem is available with the differentiability concept we defined.

Proposition 3.2 *Let $F : X \rightrightarrows Y$ be nonempty-valued and differentiable on an open convex subset X_0 of X. If there exists $c \in \mathbb{R}_+$ such that $\inf\{\|A\| : A \in \mathcal{D}F(x)\} \leq c$ for each $x \in X_0$, then F is Lipschitzian with rate c on X_0 : for every $x_0, x_1 \in X_0$ one has*

$$d(F(x_1), F(x_0)) \leq c \|x_1 - x_0\| .$$

Proof. Given $x_0, x_1 \in X_0$ it suffices to prove that $e(F(x_1), F(x_0)) \leq c' \|x_1 - x_0\|$ for every $c' > c$. Let $G : \mathbb{R} \rightrightarrows Y$ be given by $G(r) = F((1-r)x_0 + rx_1)$; then G is differentiable on $[0, 1]$. For $r \in [0, 1)$ and $s \in (0, 1-r)$ we have

$$e(G(r+s), G(0)) \leq e(G(r+s), G(r)) + e(G(r), G(0)),$$

so that, setting $x_r := (1-r)x_0 + rx_1$, $g(r) := e(G(r), G(0))$ and picking some $A \in \mathcal{D}F(x_r)$ such that $\|A\| < c'$, we get

$$\begin{aligned} g(r+s) - g(r) &\leq e(G(r+s), G(r)) \\ &\leq e(G(r) + sA(x_1 - x_0), G(r)) + o(s) \\ &\leq s \|A(x_1 - x_0)\| + o(s). \end{aligned}$$

It follows that the right upper Dini derivative of g is bounded above by $c' \|x_1 - x_0\|$ on $[0,1)$. Therefore $g(1) - g(0) \le c' \|x_1 - x_0\|$ and as $g(0) = 0$, the required inequality is proved. ■

The following consequence can be derived in applying the statement to $F(\cdot) - A(\cdot)$.

Corollary 3.2 *Let $F : X \rightrightarrows Y$ be nonempty-valued and differentiable on an open convex subset X_0 of X. Let $\varepsilon \in \mathbb{R}_+$ and $A \in L(X,Y)$ be such that $d(A, \mathcal{D}F(x)) < \varepsilon$ for each $x \in X_0$. Then, for every $x_0, x_1 \in X_0$ one has*

$$F(x_1) \subset F(x_0) + A(x_1 - x_0) + \varepsilon \|x_1 - x_0\| \overline{B}_Y.$$

The following result bearing on inverse multimappings is similar to a classical result but some care about neighborhoods is needed. Recall that $F : X \rightrightarrows Y$ is *open at* $(x_0, y_0) \in \mathrm{Gr}(F)$ if, for any neighborhood U of x_0, the set $F(U)$ is a neighborhood of y_0.

Proposition 3.3 *Let $F : X_0 \rightrightarrows Y$ be such that $F(x_0) = \{y_0\}$. Suppose F is open at (x_0, y_0) and is quasi-differentiable (resp. pseudo-differentiable) at (x_0, y_0) with derivative A, an isomorphism from X onto Y. Then F^{-1} is quasi-differentiable (resp. pseudo-differentiable) at (y_0, x_0) with derivative A^{-1}.*

Proof. Without loss of generality, using translations and replacing F by $F \circ A^{-1}$ if necessary, we may assume that $x_0 = 0$, $y_0 = 0$ and that $X = Y$, $A = I$. Since F is open at (x_0, y_0), F^{-1} is lower semicontinuous at (y_0, x_0). When F is quasi-differentiable at (x_0, y_0), given $\varepsilon \in (0,1)$ we can find $\beta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ such that

$$F(x) \cap \beta(\varepsilon)\overline{B}_Y \subset x + \frac{\varepsilon}{2} \|x\| \overline{B}_Y \qquad \forall x \in \eta(\varepsilon)\overline{B}_X. \tag{5}$$

Let $\delta(\varepsilon) := \min(\beta(\varepsilon), \eta(\varepsilon)/2)$. For any $y \in \delta(\varepsilon)\overline{B}_Y$, $x \in F^{-1}(y) \cap B(0, \eta(\varepsilon))$ we have $y \in F(x) \cap \beta(\varepsilon)\overline{B}_Y \subset x + \frac{1}{2}\|x\| \overline{B}_Y$, hence $\|x - y\| \le \frac{1}{2}\|x\|$. Thus

$$\|x\| \le \|x - y\| + \|y\| \le \frac{1}{2}\|x\| + \|y\|$$

and $\|x\| \le 2\|y\|$. Since $x \in \eta(\varepsilon)\overline{B}_X$ and $y \in F(x) \cap \beta(\varepsilon)\overline{B}_Y$ it follows from (5) that

$$\|x - y\| \le \frac{\varepsilon}{2}\|x\| \le \varepsilon \|y\|. \tag{6}$$

We have proved that

$$F^{-1}(y) \cap B(0, \eta(\varepsilon)) \subset y + \varepsilon \|y\| \overline{B}_X \qquad \forall y \in \delta(\varepsilon)\overline{B}_Y,$$

so that F^{-1} is quasi-differentiable at (y_0, x_0) with derivative A^{-1}.

When F is pseudo-differentiable at (x_0, y_0) we may suppose that β is constant. Let us set $\gamma := \eta(1)$, $\delta(\varepsilon) := \min(\beta, \eta(\varepsilon)/2)$ and $U := B(0,\gamma)$. For any $y \in \delta(\varepsilon)\overline{B}_Y$, $x \in F^{-1}(y) \cap B(0,\gamma)$ we have again $\|x\| \leq 2\|y\|$, hence $\|x\| \leq 2\delta(\varepsilon) \leq \eta(\varepsilon)$. Then (5) holds and (6) follows. We have proved that

$$F^{-1}(y) \cap U \subset y + \varepsilon \|y\| \overline{B}_X \qquad \forall y \in \delta(\varepsilon)\overline{B}_Y,$$

so that F^{-1} is pseudo-differentiable at (y_0, x_0) with derivative A^{-1}. ∎

It is known that mere differentiability is not enough to get an inverse mapping theorem (but for the finite dimensional case, as in Halkin, 1974, and Thm 4.1 in Penot, 1982). The following notion is a multivalued version of the concept of strict differentiability (henceforth called peridifferentiability) which, in the single-valued case, is effective for such an aim. As in the case of the definition of pseudo-differentiability and in the case of the pseudo-Lipschitz property, it is localized in the range space in order to increase its applicability, since it is known that the use of the Pompeiu-Hausdorff distance or excess is very restrictive.

Definition 3.3 *Let X, Y be n.v.s., let X_0 be an open subset of X and let V be a subset of Y. A multifunction $F : X_0 \rightrightarrows Y$ with nonempty values is said to be pseudo-peridifferentiable at $x_0 \in X_0$ w.r.t. V if there exists some continuous linear map $A : X \to Y$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ for which*

$$F(x) \cap V - A(x) \subset F(x') - A(x') + \varepsilon \|x - x'\| \overline{B}_Y \qquad \forall x, x' \in B(x_0, \delta). \quad (7)$$

The multifunction F is said to be pseudo-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ if there exists some neighborhood V of y_0 such that F is pseudo-peridifferentiable at $x_0 \in X$ w.r.t. V.

The multifunction F is said to be peridifferentiable at $x_0 \in X_0$ if F is pseudo-peridifferentiable at x_0 w.r.t. Y.

Thus, F is pseudo-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ iff there exist some $\beta > 0$, some continuous linear map A from X to Y and some modulus μ such that for $r > 0$ small enough to have $B(x_0, r) \subset X_0$ and for any $x, x' \in B(x_0, r)$ one has

$$e(F(x) \cap B(y_0, \beta), F(x') + A(x - x')) \leq \mu(r) \|x - x'\|.$$

The following consequence of Corollary 3.2 is reminiscent of the well known fact that a continuously differentiable (single-valued) map is peridifferentiable.

Proposition 3.4 *Let $F : X \rightrightarrows Y$ be nonempty-valued and differentiable on an open neighborhood U of some $\overline{x} \in X$. Suppose that $\liminf_{u \to \overline{x}} \mathcal{D}F(u)$ is nonempty. Then F is peridifferentiable at $\overline{x}$.*

Proof. Let us show that any $A \in \liminf_{u \to \overline{x}} \mathcal{D}F(u)$ is a periderivative of F at $\overline{x}$ in the sense that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that relation (7) is satisfied with $V = Y$. By definition of a limit inferior, given $\varepsilon > 0$ we can find $\delta > 0$ such that $B(\overline{x}, \delta) \subset U$ and for every $u \in B(\overline{x}, \delta)$ there exists some $A_u \in \mathcal{D}F(u)$ such that $\|A_u - A\| \leq \varepsilon$. Then, by Corollary 3.2, for every $x_0, x_1 \in B(\overline{x}, \delta)$ one has

$$F(x_1) \subset F(x_0) + A(x_1 - x_0) + \varepsilon \|x_1 - x_0\| \overline{B}_Y. \qquad \blacksquare$$

We will need a weakening of Definition 3.3 in which the neighborhood V depends on the accuracy of the required approximation.

Definition 3.4 *Let X, Y be n.v.s., let X_0 be an open subset of X. A multifunction $F : X_0 \rightrightarrows Y$ is said to be quasi-peridifferentiable at $(x_0, y_0) \in Gr(F)$ if there exists some continuous linear map $A : X \to Y$ such that for any $\varepsilon > 0$ there exist $\beta, \delta > 0$ for which*

$$e(F(x) \cap B(y_0, \beta) - A(x), F(x') - A(x')) \leq \varepsilon \|x - x'\| \tag{8}$$

whenever $x, x' \in B(x_0, \delta)$.

When F is single-valued, Definitions 3.3 and 3.4 coincide with the usual notion of peridifferentiability, often called strict differentiability or strong differentiability. When F is multivalued, it is no more the case that both definitions coincide.

Example 3.6 Let F be given as in Example 3.3. If g and r are peridifferentiable at x_0, following the line of Example 3.3, one can show that F is quasi-peridifferentiable at (x_0, y_0) for any $y_0 \in F(x_0)$. However, if $Dr(x_0) \neq 0$, and $\|y_0 - g(x_0)\| = r(x_0)$, F is not pseudo-differentiable at (x_0, y_0), hence is not peridifferentiable at (x_0, y_0).

The following lemma discloses a simple but useful observation; a similar result holds in the quasi-peridifferentiable case (with W depending on ε).

Lemma 3.2 *The multifunction $F : X_0 \rightrightarrows Y$ is pseudo-peridifferentiable at $(x_0, y_0) \in Gr(F)$ iff there exist some continuous linear map $A : X \to Y$ and some neighborhood W of $z_0 := y_0 - A(x_0)$ such that, setting $R(\cdot) := F(\cdot) - A(\cdot)$, for any $\eta > 0$ there exists $\rho > 0$ for which*

$$e(R(x) \cap W, R(x')) \leq \eta \|x - x'\| \tag{9}$$

whenever $x, x' \in B(x_0, \rho)$.

Proof. Suppose F is pseudo-peridifferentiable at (x_0, y_0) and let $\eta > 0$ be given. Let us take $\varepsilon = \eta$ and pick $\beta > 0$ and $\delta > 0$ such that relation (7) holds for

$x, x' \in B(x_0, \delta)$, with $V = B(y_0, \beta)$. Taking $W := B(z_0, \gamma)$ with $\gamma \in (0, \beta)$ and $\rho \in (0, \delta)$ such that $\rho \|A\| + \gamma \leq \beta$, for any $x, x' \in B(x_0, \rho)$, we have

$$\begin{aligned} R(x) \cap B(z_0, \gamma) &\subset F(x) \cap B(y_0, \beta) - A(x) \\ &\subset F(x') - A(x') + \eta \|x - x'\| \overline{B}_Y \end{aligned}$$

and (9) holds when $x, x' \in B(x_0, \rho)$.

Conversely, suppose the condition of the lemma holds and let $\varepsilon > 0$ be given. We may suppose $W := B(z_0, \gamma)$ for some $\gamma > 0$. For any $\eta \in (0, \varepsilon)$ let $\rho > 0$ be such that (9) holds when $x, x' \in B(x_0, \rho)$. Then, taking $V := B(y_0, \beta)$ with $\beta \in (0, \gamma)$, $\delta \in (0, \rho)$ such that $\delta \|A\| + \beta \leq \gamma$, relation (7) holds for $x, x' \in B(x_0, \delta)$. ■

Let us point out some elementary properties. The first one is an analogue of the continuity of differentiable mappings. It implies that F is quasi (resp. pseudo)-differentiable at (x_0, y_0) whenever it is quasi (resp. pseudo)-peridifferentiable at (x_0, y_0).

LEMMA 3.3 *Let X, Y be n.v.s. and $F : X \rightrightarrows Y$ be quasi-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$. Then F is lower semicontinuous at (x_0, y_0).*

Proof. Taking A as in Definition 3.4, and $\varepsilon = 1$, we can find $\beta, \delta > 0$ such that (8) holds for any $x, x' \in B(x_0, \delta)$. Thus, since $y_0 \in F(x_0) \cap B(y_0, \beta)$, for $x \in B(x_0, \delta)$, we have

$$\begin{aligned} d(y_0, F(x)) &\leq e(F(x_0) \cap B(y_0, \beta), F(x) + A(x_0) - A(x)) + \|A(x_0) - A(x)\| \\ &\leq (\|A\| + 1) \|x - x_0\| . \end{aligned}$$

Therefore, $d(y_0, F(x)) \to 0$ as $x \to x_0$ and F is lower semicontinuous at (x_0, y_0). ■

PROPOSITION 3.5 *(a) Let X, Y, Z be n.v.s. and let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be pseudo (resp. quasi)-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ and $(x_0, z_0) \in \mathrm{Gr}(G)$ respectively. Then $H := (F, G) : X \rightrightarrows Y \times Z$ is pseudo (resp. quasi)-peridifferentiable at $(x_0, (y_0, z_0))$.*

(b) Let X, Y be n.v.s., let $F : X \rightrightarrows Y$ be pseudo (resp. quasi)-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ and let $g : X \to Y$ be peridifferentiable at x_0. Then $H := F + g$ is pseudo (resp. quasi)-peridifferentiable at $(x_0, (y_0 + g(x_0)))$.

Proof. Assertion (a) is obvious. Let us prove (b) in the case F is pseudo-peridifferentiable at (x_0, y_0). Let $\alpha > 0$ and let $A : X \to Y$ be a linear continuous map such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \cap B(y_0, \alpha) \subset F(x') + A(x - x') + \varepsilon \|x - x'\| \overline{B}_Y \qquad \forall x, x' \in B(x_0, \delta).$$

Let $\eta > 0$ be given. We pick $\varepsilon \in (0, \eta)$ and we choose $\beta, \gamma > 0$ such that $\beta + \gamma \leq \alpha$. We may suppose that δ is small enough to guarantee that $g(x) - g(x') \in \gamma \overline{B}_Y$ for $x, x' \in B(x_0, \delta)$ and

$$g(x) - g(x') - B(x - x') \in (\eta - \varepsilon) \|x - x'\| \overline{B}_Y$$

for $B := Dg(x_0)$. Then, for any $x, x' \in B(x_0, \delta)$ and for any $z \in H(x) \cap B(z_0, \beta)$ we have $y := z - g(x) \in F(x) \cap B(y_0, \alpha)$, so that there exists $y' \in F(x')$ such that $y - y' - A(x - x') \in \varepsilon \|x - x'\| \overline{B}_Y$. Then we have

$$z = y + g(x) \in y' + g(x') + (A + B)(x - x') + \eta \|x - x'\| \overline{B}_Y$$

hence $H(x) \cap B(y_0, \alpha) \subset H(x') + (A + B)(x - x') + \eta \|x - x'\| \overline{B}_Y$ for any $x, x' \in B(x_0, \delta)$. ■

EXAMPLE 3.7 Let X, Y be n.v.s., $r > 0$ and let $g : X \to Y$ be peridifferentiable at $x_0 \in X$. Let $F : X \rightrightarrows Y$ be given by $F(x) := g(x) + r\overline{B}_Y$. Then, for any $y_0 \in F(x_0)$, F is pseudo-peridifferentiable at (x_0, y_0).

EXAMPLE 3.8 As in Example 3.6, let $g : X \to Y$ be peridifferentiable at x_0, let $r : X \to \mathbb{R}$ be peridifferentiable at x_0 with $r(x_0) > 0$. Then the multimapping $F : X \rightrightarrows Y$ given by $F(x) := g(x) + r(x)\overline{B}_Y$ is quasi-peridifferentiable at (x_0, y_0) for any $y_0 \in F(x_0)$. In fact, one can check that both summands are quasi-peridifferentiable at x_0, what simplifies the analysis of Example 3.6.

Now let us give chain rules. The first one is as follows.

PROPOSITION 3.6 *Let X, Y, Z be n.v.s. and let $f : X \to Y$, $G : Y \rightrightarrows Z$, $H = G \circ f$. Suppose f is peridifferentiable at $x_0 \in X$, G is pseudo-peridifferentiable at $y_0 = f(x_0)$ w.r.t. $V \subset Z$. Then H is pseudo-peridifferentiable at $x_0 \in X$ w.r.t. V.*

A similar result holds for quasi-peridifferentiability.

Proof. By the assumptions, there exist a neighborhood V of y_0 and two continuous linear mappings $A : X \to Y$, $B : Y \to Z$ and two moduli $\alpha(\cdot), \beta(\cdot)$ such that, for $r, s \in \mathbb{R}_+$,

$$\begin{aligned}
&\|f(x) - f(x') - A(x - x')\| \leq \alpha(r) \|x - x'\| \qquad \forall x, x' \in B(x_0, r), \\
&G(y) \cap V \subset G(y') + B(y - y') + \beta(s) \|y - y'\| B_Z \qquad \forall y, y' \in B(y_0, s).
\end{aligned}$$

Let $\rho > 0$ be such that $\alpha(r) \leq 1$ for $r \leq \rho$ and let $c := \|A\| + 1$. Then, for $r \in [0, \rho]$, $x, x' \in B(x_0, r)$ one has $\|f(x) - f(x')\| \leq \|A(x - x')\| + \alpha(r) \|x - x'\| \leq c \|x - x'\|$, in particular $f(x), f(x') \in B(y_0, cr)$, hence

$$\begin{aligned}
G(f(x)) \cap V \subset\ & G(f(x')) + B(A(x - x')) \\
& \qquad + \|B\| \alpha(r) \|x - x'\| B_Z + \beta(cr) \|f(x) - f(x')\| B_Z \\
\subset\ & G(f(x')) + (B \circ A)(x - x') \\
& \qquad + (\|B\| \alpha(r) + c\beta(cr)) \|x - x'\| B_Z.
\end{aligned}$$

Therefore H is pseudo-peridifferentiable at x_0 w.r.t. V with derivative $B \circ A$. ■

The assumptions of the following two statements are rather restrictive, but they are satisfied when g is a linear isomorphism.

PROPOSITION 3.7 *Let X, Y, Z be n.v.s. and let $F : X \rightrightarrows Y$, $g : Y \to Z$. Suppose F is pseudo-peridifferentiable at $x_0 \in X$ w.r.t. $V := g^{-1}(W)$ for some subset W of Z and g is a uniformly differentiable map on V with constant derivative B. Then $H := g \circ F$ is pseudo-peridifferentiable at x_0 w.r.t. W, with derivative $B \circ A$, A being the derivative of F at x_0.*

Proof. Let $A : X \to Y$, $B : Y \to Z$ be continuous linear maps and let $\alpha(\cdot)$, $\beta(\cdot)$ be moduli such that, for any $r, s \in \mathbb{R}_+$,

$$\begin{aligned} F(x) \cap V \subset F(x') + A(x - x') + \alpha(r) \|x - x'\| \overline{B}_Y \qquad & \forall x, x' \in B(x_0, r), \\ g(y) \in g(y') + B(y - y') + \beta(s) \|y - y'\| \overline{B}_Z \qquad & \forall y \in V,\ y' \in B(y, s). \end{aligned}$$

Let $\rho > 0$ be such that $\alpha(r) \leq 1$ for $r \leq \rho$ and let $c := \|A\| + 1$. Then, for $r \in [0, \rho]$ and for any $x, x' \in B(x_0, r)$ and any $y \in F(x) \cap V$ we can find $y' \in F(x')$ such that

$$y'' := y - y' - A(x - x') \in \alpha(r) \|x - x'\| \overline{B}_Y$$

and thus $\|y - y'\| \leq c \|x - x'\| \leq 2cr$. Then, we have

$$\begin{aligned} g(y) &\in g(y') + B(A(x - x')) + B(y'') + \beta(2cr)c \|x - x'\| B_Z \\ &\subset g(F(x')) + B(A(x - x')) + [\|B\| \alpha(r) + c\beta(2cr)] \|x - x'\| B_Z. \end{aligned}$$

Since any $z \in H(x) \cap W$ is obtained as $z = g(y)$ for some $y \in F(x) \cap V$, we obtain that $H := g \circ F$ is pseudo-peridifferentiable at $x_0 \in X$ w.r.t. W. ■

PROPOSITION 3.8 *Let X, Y, Z be n.v.s. and let $F : X \rightrightarrows Y$, $g : Y \to Z$. Suppose F is pseudo-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ and g is injective, open at y_0 and peridifferentiable at y_0. Then $H := g \circ F$ is pseudo-peridifferentiable at $(x_0, g(y_0))$.*

Proof. The proof of this statement is similar to the proof of the preceding one. This time, we can take for V some ball $B(y_0, \sigma)$ and set $W := g(V)$, a neighborhood of $g(y_0)$. We assume that

$$g(y) \in g(y') + B(y - y') + \beta(s) \|y - y'\| B_Z \qquad \forall y,\ y' \in B(y_0, s).$$

Since g is injective, for any $x, x' \in B(x_0, r)$ and $z \in H(x) \cap W$ we have $z = g(y)$ for some $y \in F(x) \cap V$ and the same estimates hold. ■

4. An inversion theorem

The following statement is our central result:

THEOREM 4.1 *Let X and Y be Banach spaces and let $F : X_0 \rightrightarrows Y$ be a multifunction defined on some open subset X_0 of X with closed nonempty values. Suppose F is quasi-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ and such that some derivative A of F at (x_0, y_0) is invertible. Then F is open at (x_0, y_0) in the sense that for any neighborhood U of x_0 there exists a neighborhood V of y_0 such that $V \subset F(U)$. Moreover, upon taking U and V small enough, F^{-1} is quasi-Lipschitz on V w.r.t. U and is quasi-peridifferentiable at (y_0, x_0).*

Proof. Without loss of generality, using translations and taking the composition with $DF(x_0, y_0)^{-1}$ if necessary, thanks to Propositions 3.6, 3.8, we may assume that $x_0 = 0$, $y_0 = 0$ and that $X = Y$, $DF(x_0, y_0) = I$, the identity mapping. Let us set $R(x) := F(x) - x$ for $x \in X_0$. Given $\kappa \in (0, 1)$, in view of Lemma 3.2, we can find some $\sigma > 0$, $\rho \in (0, \sigma)$ such that $B(0, \rho) \subset X_0$ and

$$R(x) \cap \sigma\overline{B}_Y \subset R(x') + \kappa \|x - x'\| \overline{B}_Y \qquad \forall x, x' \in \rho\overline{B}_X. \tag{10}$$

Without loss of generality, we may assume that U is a ball $B(0, r)$ with $r \in (0, \rho)$. Let $s := \min(\sigma - r, (1-\kappa)r)$, $V := B(0, s)$. Let us observe that for $(x, y) \in U \times V$, we have

$$y \in F(x) \Leftrightarrow x \in G_y(x) := y - R(x).$$

Now, for any $y \in V$, the multifunction G_y is pseudo-κ-contractive w.r.t. U: for $x, x' \in U$ we have

$$G_y(x) \cap U \subset G_y(x') + \kappa \|x - x'\| \overline{B}_Y \tag{11}$$

since for any $u \in G_y(x) \cap U$ we have $y - u \in R(x)$ and $\|y - u\| \le \|u\| + \|y\| < r + s \le \sigma$, so that there exists some $z' \in R(x')$ with $\|(y - u) - z'\| \le \kappa \|x - x'\|$, hence $u' := y - z' \in G_y(x')$ and $\|u' - u\| \le \kappa \|x - x'\|$. Moreover, since $0 \in R(0)$,

$$d(0, G_y(0)) \le \|y\| < s \le (1 - \kappa)r.$$

The assumptions of Proposition 2.1 being satisfied, G_y has a fixed point $x \in B(0, r)$ and $y \in F(x) \subset F(U)$.

To prove that F^{-1} is quasi-Lipschitz on V, we will apply Proposition 2.2 or Corollary 2.1, observing that, for $y, y' \in V$, satisfying $\|y - y'\| < (1 - \kappa)(\rho - r)$, we have

$$e(G_{y'}(x) \cap B(0, r), G_y(x)) \le \|y - y'\| < (1 - \kappa)(\rho - r) \qquad \forall x \in B(0, r),$$

so that, by Proposition 2.2, with $G := G_y$, $H := G_{y'}$, we get

$$\begin{aligned} e(\Phi_H \cap B(0, r), \Phi_G) &\le (1 - \kappa)^{-1} \sup_{x \in B(0,r)} e(H(x) \cap B(0, r), G(x)) \\ &\le (1 - \kappa)^{-1} \|y - y'\|. \end{aligned}$$

or, since $\Phi_G = F^{-1}(y)$ and $\Phi_H = F^{-1}(y')$,

$$e(F^{-1}(y') \cap B(0,r), F^{-1}(y)) \leq (1-\kappa)^{-1} \|y - y'\| . \tag{12}$$

Finally, let us show that F^{-1} is quasi-peridifferentiable at (y_0, x_0). Given $\varepsilon \in (0, \kappa)$, there exist some $s_\varepsilon \in (0, s)$, $r_\varepsilon \in (0, r) \cap (0, s_\varepsilon)$ such that

$$R(x) \cap B(0, s_\varepsilon) \subset R(x') + \varepsilon(1-\varepsilon) \|x - x'\| \overline{B}_Y \quad \forall x, x' \in B(0, r_\varepsilon). \tag{13}$$

Setting $V_\varepsilon := B(0, r_\varepsilon/4)$, $S(y) := F^{-1}(y) - y$, let us show that

$$e(S(y) \cap V_\varepsilon, S(y')) \leq \varepsilon \|y - y'\| \qquad \forall y, y' \in V_\varepsilon. \tag{14}$$

Let $y, y' \in V_\varepsilon$ and let $u \in S(y) \cap V_\varepsilon$. By definition, there exists $x \in F^{-1}(y)$ such that $u = x - y$. Thus $y \in F(x) = x + R(x)$ and $-u \in R(x)$. Let us set $w := x + y' - y = y' + u \in B(0, r_\varepsilon/2)$, so that $\|w - x\| = \|y' - y\|$. Since $-u \in R(x) \cap V_\varepsilon \subset R(x) \cap B(0, s_\varepsilon)$ and $x, w \in B(0, r_\varepsilon/2)$, relation (13) yields some $z \in R(w)$ such that $\|z - (-u)\| \leq \varepsilon(1-\varepsilon) \|w - x\|$. Thus

$$\begin{aligned} d(w, G_{y'}(w)) &\leq \|w - (y' - z)\| = \|(y' + u) - (y' - z)\| = \|z + u\| \\ &\leq \varepsilon(1-\varepsilon) \|w - x\| , \end{aligned}$$

and $d(w, G_{y'}(w)) \leq t_\varepsilon := \varepsilon(1-\varepsilon) \|y' - y\| \leq \varepsilon(1-\varepsilon) r_\varepsilon/2$. Since, by (13), $G_{y'}$ is an ε-contraction w.r.t. $B(w, r_\varepsilon/2) \subset B(0, r_\varepsilon)$, using Proposition 2.1, we can find some $x' \in B(w, (1-\varepsilon)^{-1} t_\varepsilon)$ such that $x' \in G_{y'}(x')$. Then, setting $u' := x' - y'$, we have $u' \in S(y')$ since $x' \in y' - R(x')$ or $x' \in F^{-1}(y') = y' + S(y')$, and

$$\|u - u'\| = \|(x - y) - (x' - y')\| = \|w - x'\| \leq (1-\varepsilon)^{-1} t_\varepsilon = \varepsilon \|y' - y\| .$$

so that relation (14) is satisfied. ■

Corollary 4.1 *Suppose $F : X_0 \rightrightarrows Y$ is as in Theorem 4.1 and is pseudo-differentiable at (x_0, y_0) with $F(x_0) = \{y_0\}$. Then F^{-1} is pseudo-differentiable at (y_0, x_0) with derivative A^{-1}, where A is the derivative of F at x_0.*

Proof. This is a consequence of Proposition 3.3, since by Theorem 4.1 F is open at (x_0, y_0). ■

A refinement of the preceding result can be given.

Proposition 4.1 *Suppose $F : X_0 \rightrightarrows Y$ is as in Theorem 4.1 and is pseudo-peridifferentiable at (x_0, y_0) with $F^{-1}(y_0) \cap B(x_0, r_0) = \{x_0\}$ for some $r_0 > 0$. Then F^{-1} is pseudo-peridifferentiable at (y_0, x_0) with derivative A^{-1}, where A is the derivative of F at x_0.*

Proof. We may suppose that relation (10) is satisfied with $\kappa = 1/2$ and $\rho \in (0, r_0]$. Let $r := \rho/2$. By assumption, there exists some $s > 0$ such that, given $\varepsilon \in (0, 1/2)$ we can find some $r_\varepsilon \in (0, r) \cap (0, s)$ for which

$$R(x) \cap U \subset R(x') + \varepsilon(1-\varepsilon) \|x - x'\| \overline{B}_Y \qquad \forall x, x' \in B(0, r_\varepsilon) \tag{15}$$

for $U := B(0,s)$. Since F^{-1} is quasi-Lipschitz, relation (12) shows that for $y \in B(0,\rho/4)$ we have

$$e(F^{-1}(y) \cap B(0,r), F^{-1}(0)) \leq 2\,\|y\|\,.$$

Thus, for any $y \in B(0,\rho/4)$, $x \in F^{-1}(y) \cap B(0,r)$ we can find $x' \in F^{-1}(0)$ such that $\|x - x'\| < \rho/2$, hence $\|x'\| < r + \rho/2 \leq r_0$ and $x' = 0$, $\|x\| \leq 2\,\|y\|$.

Let us show that for $y, y' \in V_\varepsilon := B(0, r_\varepsilon/4)$ satisfying $\|y - y'\| \leq r_\varepsilon/4$ we have, with $S(y) := F^{-1}(y) - y$,

$$e(S(y) \cap U, S(y')) \leq \varepsilon\,\|y - y'\|\,. \tag{16}$$

We proceed as in the proof of Theorem 4.1. Let $y, y' \in V_\varepsilon$ satisfy $\|y - y'\| \leq r_\varepsilon/4$ and let $u \in S(y) \cap U$. By definition, there exists $x \in F^{-1}(y)$ such that $u = x - y$. Thus $y \in F(x) = x + R(x)$ and $-u \in R(x)$. We have just proved that $\|x\| \leq 2\,\|y\| < r_\varepsilon/2$. Let us set $w := y' + u = x + y' - y \in B(0, 3r_\varepsilon/4)$. Since $-u \in R(x) \cap U$ and $x, w \in B(0, r_\varepsilon)$, relation (15) yields some $z \in R(w)$ such that $\|z - (-u)\| \leq t := \varepsilon(1-\varepsilon)\,\|w - x\|$. Thus

$$\begin{aligned} d(w, G_{y'}(w)) &\leq \|w - (y' - z)\| = \|(y' + u) - (y' - z)\| = \|u + z\| \\ &\leq \varepsilon(1-\varepsilon)\,\|w - x\| \leq \varepsilon(1-\varepsilon) r_\varepsilon/4 \end{aligned}$$

since $\|w - x\| = \|y' - y\| \leq r_\varepsilon/4$. Since, by (15), $G_{y'}$ is an ε-contraction w.r.t. $B(w, r_\varepsilon/4) \subset B(0, r_\varepsilon)$, using Proposition 2.1, we can find some $x' \in B(w, (1-\varepsilon)^{-1}t)$ such that $x' \in G_{y'}(x')$. Then, setting $u' := x' - y'$, we have $u' \in S(y')$ since $x' \in y' - R(x')$ or $x' \in F^{-1}(y') = y' + S(y')$, and

$$\|u - u'\| = \|(x - y) - (x' - y')\| = \|w - x'\| \leq (1-\varepsilon)^{-1} t = \varepsilon\,\|y' - y\|\,,$$

so that relation (16) is satisfied. ■

It would be interesting to find other conditions ensuring that F^{-1} is pseudo-differentiable at (y_0, x_0) without assuming that $F(x_0)$ or $F^{-1}(y_0)$ is a singleton. In the following example we make use of the order of $\mathbb{R}$ to reach that conclusion.

EXAMPLE 4.1 Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $F(x) = [a(x), b(x)]$ for $x \in \mathbb{R}$ where $a, b : \mathbb{R} \to \mathbb{R}$ are peridifferentiable at $x_0 \in \mathbb{R}$ with $a(x_0) < b(x_0)$ and we can find an open interval U containing x_0 such that $a(x) < b(x)$ for each $x \in U$. Then F is pseudo-peridifferentiable at (x_0, y_0) for any $y_0 \in F(x_0)$, with derivative $A = a'(x_0)$ if $y_0 < b(x_0)$ and $A = b'(x_0)$ if $y_0 > a(x_0)$ (and any A if y_0 satisfies both conditions). If moreover, a and b have non null derivatives at x_0, we may suppose that a and b are invertible on U. Then, for any $y \in Y$ one has $F^{-1}(y) \cap U = [b^{-1}(y), a^{-1}(y)]$, where a^{-1} stands for $(a \mid U)^{-1}$ and b^{-1} stands for $(b \mid U)^{-1}$. Then F and F^{-1} are pseudo-differentiable at (x_0, y_0) and (y_0, x_0) respectively. When $a'(x_0) = b'(x_0)$, F is peridifferentiable at x_0. If, moreover, a and b are increasing, F^{-1} is also peridifferentiable at any $y_0 \in F(x_0)$; it is not so if a and b are not globally increasing.

5. A parametric inversion theorem

In order to deal with a parametric version of Theorem 4.1 and with an implicit mapping theorem, let us introduce a notion of partial peridifferentiability.

DEFINITION 5.1 *Let X, Y be n.v.s., Z be a topological space and let Ω be an open subset of $X \times Z$. A multifunction $F : \Omega \rightrightarrows Y$ is said to be partially quasi-peridifferentiable at $((x_0, z_0), y_0) \in \mathrm{Gr}(F) \subset \Omega \times Y$ w.r.t. X, if:*

(a) the multifunction $z \mapsto F(x_0, z)$ is lower semicontinuous at (z_0, y_0);

(b) there exist a continuous linear map $A : X \to Y$ such that for any $\varepsilon > 0$ there are $\delta_\varepsilon > 0$ and neighborhoods V_ε, W_ε of y_0 and z_0, respectively, for which

$$e(F(x, z) \cap V_\varepsilon - A(x), F(x', z) - A(x')) \le \varepsilon \|x - x'\| \tag{17}$$

whenever $x, x' \in B(x_0, \delta_\varepsilon)$ and $z \in W_\varepsilon$. The mapping A will be denoted by $D_1F(x_0, z_0)$.

If in condition (b) V_ε can be chosen independently of ε (resp. $V_\varepsilon = Y$), we say that F is partially pseudo-peridifferentiable at $((x_0, z_0), y_0)$ (resp. partially peridifferentiable at $((x_0, z_0), y_0)$).

When F is single-valued, these notions coincide with the concept of circa (or strict) partial differentiability. Moreover, we have the following observation:

LEMMA 5.1 *If $F : \Omega \rightrightarrows Y$ is partially quasi-peridifferentiable at $((x_0, z_0), y_0)$, then F is jointly lower semicontinuous at $((x_0, z_0), y_0)$.*

Proof. Given $\alpha > 0$, let us find some $\beta > 0$ and some neighborhood U of z_0 such that $d(y_0, F(x, z)) \le \alpha$ for any $(x, z) \in B(x_0, \beta) \times U$. Let V_1 and W_1 be the neighborhoods of y_0 and z_0, respectively, associated with $\varepsilon = 1$ in the preceding definition. Since $F(x_0, \cdot)$ is lower semicontinuous at (z_0, y_0), we can find some neighborhood U of z_0 contained in W_1 such that for each $z \in U$ there exists some $y \in F(x_0, z)$ satisfying $\|y - y_0\| < \min(\alpha/2, \rho)$, where $\rho > 0$ is such that $B(y_0, \rho) \subset V_1$. Let us set $\beta := \min(\delta_1, \alpha/2(\|A\|+1))$. Given $x \in B(x_0, \beta)$, $z \in U$, taking $y \in F(x_0, z)$ satisfying $\|y - y_0\| < \min(\alpha/2, \rho)$, we have

$$\begin{aligned} d(y, F(x, z)) &\le e(F(x_0, z) \cap V_1, F(x, z) + A(x_0 - x)) + \|A(x - x_0)\| \\ &\le \|x - x_0\| + \|A(x_0 - x)\| < \alpha/2, \end{aligned}$$

hence $d(y_0, F(x, z)) < \alpha$. ∎

THEOREM 5.1 *(Parametric inversion theorem) Let X, Y be Banach spaces, let Z be a topological space and let $F : \Omega \rightrightarrows Y$ be a multifunction with closed nonempty values defined on some open subset Ω of $X \times Z$. Suppose F is partially quasi-peridifferentiable at $((x_0, z_0), y_0) \in (\Omega \times Y) \cap \mathrm{Gr}(F)$ w.r.t. X, with an invertible partial derivative A. Then for any neighborhood U of x_0 one can find a neighborhood V of y_0 and a neighborhood W of z_0 such that $V \times W \subset$*

$\Psi(U\times W)$, where $\Psi(x,z) := F(x,z)\times\{z\}$. Moreover, one can ensure that for each $z \in W$ the multimapping $F(\cdot,z)^{-1}$ is quasi-Lipschitzian. Furthermore, the first component H of the inverse multimapping $(y,z) \mapsto \Psi^{-1}(y,z) := H(y,z) \times \{z\}$ of Ψ is partially quasi-peridifferentiable at $((y_0,z_0),x_0)$ w.r.t. Y.

Proof. Let us adapt the proof of Theorem 4.1. As in that proof, we may suppose $x_0 = 0, y_0 = 0$, $X = Y$ and $A = I$. Let us set $R(x,z) = x - F(x,z)$, $G_{y,z}(x) = y - R(x,z)$. Given $\kappa \in (0,1)$, we can find some $\sigma > 0, \rho \in (0,\sigma)$ and some neighborhood W of z_0 in Z such that

$$R(x,z) \cap \sigma\overline{B}_Y \subset R(x',z) + \kappa\,\|x-x'\|\,\overline{B}_Y \quad \forall x,x' \in \rho\overline{B}_X,\ z \in W. \tag{18}$$

Let U be a neighborhood of x_0. Shrinking it, if necessary, we may suppose $U := B(0,r)$ for some $r \in (0,\rho)$. Let $s < s' := \min(\sigma - r, (1-\kappa)r)$, $V := B(0,s)$. Since $z \mapsto F(0,z)$ is lower semicontinuous at $(z_0,0)$, shrinking W, if necessary, we may assume that $d(0,F(0,z)) < s' - s$ for $z \in W$. Let us observe that for $(x,y,z) \in U \times V \times W$, we have

$$y \in F(x,z) \Leftrightarrow x \in G_{y,z}(x) := y - R(x,z).$$

Again, for any $(y,z) \in V \times W$, the multifunction $G_{y,z}$ is pseudo-κ-contractive w.r.t. U: for $x,x' \in U$ we have

$$G_{y,z}(x) \cap U \subset G_{y,z}(x') + \kappa\,\|x-x'\|\,\overline{B}_Y\,. \tag{19}$$

Moreover, since $0 \in F(0,z_0) = -R(0,z_0)$,

$$d(0,G_{y,z}(0)) = d(0,y-F(0,z)) \leq \|y\| + d(0,F(0,z)) < s + (s'-s) \leq (1-\kappa)r.$$

The assumptions of Proposition 2.1 being satisfied, $G_{y,z}$ has a fixed point $x \in B(0,r) = U$ and $y \in F(x,z) \subset F(U \times \{z\})$. Thus $V \times W \subset \Psi(U \times W)$ where $\Psi(x,z) := F(x,z) \times \{z\}$.

The second assertion is obtained as in the proof of Theorem 4.1: replacing G_y by $G_{y,z}$ for $(y,z) \in V \times W$, we obtain that for each $z \in W$ the multimapping $F_z^{-1} := F(\cdot,z)^{-1}$ is quasi-Lipschitzian with rate $(1-\kappa)^{-1}$ on V. More precisely, for any $y,y' \in V$ satisfying $\|y-y'\| < (1-\kappa)(\rho - r)$ and any $z \in W$, we have

$$e(F_z^{-1}(y') \cap B(0,r), F_z^{-1}(y)) \leq (1-\kappa)^{-1}\,\|y-y'\|\,.$$

We deduce from this fact that $z \mapsto H(y_0,z)$ is l.s.c. at (z_0,x_0), i.e. that $z \mapsto F_z^{-1}(y_0)$ is l.s.c. at (z_0,x_0) : using the lower semicontinuity of $F(0,\cdot)$, for z near z_0 we pick some $y_z \in F(0,z)$ such that $y_z \to 0$ as $z \to z_0$ and we note that $0 \in F_z^{-1}(y_z)$, hence

$$d(0,F_z^{-1}(0)) \leq e(F_z^{-1}(y_z) \cap B(0,r), F_z^{-1}(0)) \leq (1-\kappa)^{-1}\,\|y_z\| \to 0 \text{ as } z \to z_0.$$

Finally, let us show that H is partially quasi-peridifferentiable at $((y_0, z_0), x_0)$ w.r.t. Y. Given $\varepsilon \in (0, \kappa)$, there exist some $s_\varepsilon \in (0, s)$, $r_\varepsilon \in (0, r) \cap (0, s_\varepsilon)$ and some neighborhood W_ε of z_0 such that

$$R(x, z) \cap B(0, s_\varepsilon) \subset R(x', z) + \varepsilon(1-\varepsilon) \|x - x'\| \overline{B}_Y \qquad \forall x, x' \in B(0, r_\varepsilon),\ z \in W_\varepsilon. \tag{20}$$

Setting $U_\varepsilon := B(0, r_\varepsilon/4)$, $V_\varepsilon := B(0, r_\varepsilon/4)$, $S(y, z) := F_z^{-1}(y) - y$, we can show as in the proof of Theorem 4.1 that

$$e(S(y, z) \cap U_\varepsilon, S(y', z)) \leq \varepsilon \|y - y'\| \qquad \forall y, y' \in V_\varepsilon,\ z \in W_\varepsilon.$$

Thus H, is partially quasi-peridifferentiable at $((y_0, z_0), x_0)$ w.r.t. Y. ■

One can deduce from the preceding result an implicit multimapping theorem.

THEOREM 5.2 *Let X, Y be Banach spaces, let Z be a topological space and let $F : \Omega \rightrightarrows Y$ be a multifunction with closed nonempty values defined on some open subset Ω of $X \times Z$. Suppose F is partially quasi-peridifferentiable at $((x_0, z_0), y_0) \in \mathrm{Gr}(F)$ w.r.t. X, with an invertible partial derivative. Then there exist a neighborhood W of z_0 such that the implicit multimapping $M : W \rightrightarrows X$ given by*

$$M(z) := \{x \in X : (x, z) \in \Omega,\ y_0 \in F(x, z)\}$$

has nonempty values for $z \in W$. Moreover M is lower semicontinuous at (z_0, x_0).

Proof. Clearly, $M(z) = H(y_0, z)$. The lower semicontinuity of M has been shown during the proof of the preceding theorem; it is also a consequence of Lemma 5.1, H being partially quasi-peridifferentiable at $((y_0, z_0), x_0)$ w.r.t. Y. ■

COROLLARY 5.1 *Let X, Y, Z be n.v.s., X, Y being complete, and let $F : \Omega \rightrightarrows Y$ be a multifunction with closed nonempty values defined on some open subset Ω of $X \times Z$. Let $(x_0, z_0, y_0) \in \Omega \times Y$ be such that $F(x_0, z_0) = \{y_0\}$. Suppose F is pseudo-(resp. quasi-)differentiable at $((x_0, z_0), y_0)$ and is partially quasi-peridifferentiable at $((x_0, z_0), y_0)$ w.r.t. X, with an invertible partial derivative A. Then, the implicit multifunction M of the preceding statement is pseudo (resp. quasi)-differentiable at (z_0, x_0).*

Proof. Let $\Psi : X \times Z \rightrightarrows Y \times Z$ be given by $\Psi(x, z) := F(x, z) \times \{z\}$. Its inverse Ψ^{-1} is given by $\Psi^{-1}(y, z) := H(y, z) \times \{z\}$, hence is lower semicontinuous at $((y_0, z_0), (x_0, z_0))$. Therefore Ψ is open at $((x_0, z_0), (y_0, z_0))$. Moreover, the derivative of Ψ at $((x_0, z_0), (y_0, z_0))$ is obviously given by

$$D\Psi((x_0, z_0), (y_0, z_0))(u, v) = (Au + Bv, v).$$

for some continuous linear mapping $B : Z \to Y$. Thus, it is invertible and Proposition 3.3 applies. Hence H is pseudo (resp. quasi)-differentiable at $((y_0, z_0), x_0)$ and M has the same differentiability property at (z_0, x_0). ■

6. Comparisons with other notions

Many notions of differentiability of multimappings exist (see references from Aubin, 1981 to Azé, Corvellec, 2004; Banks, Jacobs, 1970; Dien, Yen, 1991; Gautier, 1989, 1990; Hukuhara, 1967; Klatte, Kummer, 2001 to Lemaréchal, Zowe, 1991; Martelli, Vignoli, 1974; Penot, 1984; Polovinkin, Smirnov, 1986; Sach, 1998 to Silin, 1997, ...). It is the purpose of this section to present a short comparison. In this section $F : X \rightrightarrows Y$ is a multimapping between two normed vector spaces with domain an open subset X_0 of X and (x_0, y_0) is a point in $\mathrm{Gr}(F)$. The notion of differentiability we have adopted ensures a strong approximation property of the graph.

PROPOSITION 6.1 *Suppose F is quasi-differentiable at $z_0 := (x_0, y_0)$ with derivative A and $F(x_0) = \{y_0\}$. Let $Z_0 := \{x_0\} \times F(x_0)$. Then $\mathrm{Gr}(A) + Z_0$ is an approximation of $\mathrm{Gr}(F)$ at z_0 in the sense of Maurer, Zowe (1979): there exists a mapping $h : \mathrm{Gr}(F) \to \mathrm{Gr}(A) + Z_0$ such that $h(z) - z = o(\|z - z_0\|)$.*

If, moreover, F is quasi-peridifferentiable at (x_0, y_0) then $\mathrm{Gr}(F)$ is an approximation of $\mathrm{Gr}(A) + Z_0$ at z_0.

Proof. The first assertion follows from the equivalence proved in Agadi, Penot (2005) taking z_0 as a base point in $X \times Y$, and setting $e_r(C, D) := e(C \cap B(z_0, r), D)$ for $r > 0$, $C, D \subset X \times Y$, the existence of h is equivalent to the relation

$$\lim_{r \to 0_+} \frac{1}{r} e_r(\mathrm{Gr}(F), \mathrm{Gr}(A) + Z_0) = 0.$$

Such a relation is satisfied since

$$e_r(\mathrm{Gr}(F), \mathrm{Gr}(A) + Z_0) \leq \sup_{x \in B(x_0, r)} e(F(x) \cap B(y_0, r), F(x_0) + A(x - x_0)) \leq o(r).$$

If F is quasi-peridifferentiable at (x_0, y_0) we also have, for $c := \|A\| + 1$

$$e_r(\mathrm{Gr}(A) + Z_0, \mathrm{Gr}(F)) \leq \sup_{x \in B(x_0, r)} e((F(x_0) + A(x - x_0)) \cap B(y_0, cr), F(x)) \leq o(r).$$

■

A comparison with the notion of contingent derivative used in Aubin (1981) follows easily.

COROLLARY 6.1 *If F is quasi-differentiable at $z_0 := (x_0, y_0)$ with derivative A and if $F(x_0) = \{y_0\}$, then the graph of the contingent derivative of F at (x_0, y_0) is contained in $\mathrm{Gr}(A)$. If, moreover, F is pseudo-peridifferentiable at (x_0, y_0) then the graph of the contingent derivative of F at (x_0, y_0) coincides with $\mathrm{Gr}(A)$.*

Proof. The graph of the contingent derivative of F at (x_0, y_0) being the tangent cone to $\mathrm{Gr}(F)$ at z_0, the result follows from the fact (obtained by using the mapping h of the preceding proposition) that when a subset S' of Z is an approximation of a subset S of Z at some $z_0 \in S \cap S'$, the tangent cone to S at z_0 is contained in the tangent cone to S' at z_0 . A direct proof is also easy. ∎

Now, let us turn to a comparison with a notion which gave rise to an inverse multimapping theorem.

Definition 6.1 (Azé, Chou, 1995) *A multifunction $F : X \rightrightarrows Y$ between two n.v.s. is said to be strictly lower pseudo-differentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ if there exists a multifunction $L : X \rightrightarrows Y$ whose graph $\mathrm{Gr}(L)$ is a closed cone such that*

$$\lim_{((x,y),(u,v)) \overset{\mathrm{Gr}(F)\times\mathrm{Gr}(L)}{\to} ((x_0,y_0),(0,0))} \|(u,v)\|^{-1} d((x+u, y+v), \mathrm{Gr}(F)) = 0. \quad (21)$$

Proposition 6.2 *Let X, Y be n.v.s. If a multifunction $F : X \rightrightarrows Y$ is quasi-peridifferentiable at $(x_0, y_0) \in \mathrm{Gr}(F)$ then F is strictly lower pseudo-differentiable at (x_0, y_0).*

Proof. Let A be as in Definition 3.4. Define $L : X \rightrightarrows Y$ by $L(x) = \{A(x)\}$ for $x \in X$, so that $\mathrm{Gr}(L) = \mathrm{Gr}(A)$ which is a closed subset of $X \times Y$. Let us show that condition (21) is satisfied. We know that for any $\varepsilon > 0$ there exists $\delta > 0$ and a neighborhood V of y_0 for which

$$e(F(x) \cap V - A(x), F(x') - A(x')) \leq \varepsilon \|x - x'\|$$

whenever $x, x' \in B(x_0, \delta)$. Thus for $x \in B(x_0, \delta/2)$ and $u \in B(0, \delta/2)$ we have

$$e(F(x) \cap V - A(x), F(x+u) - A(x+u)) \leq \varepsilon \|u\|$$

so that, for any $y \in F(x) \cap V$ there exists $y' \in F(x+u)$ such that

$$d(y - A(x), y' - A(x+u)) \leq \varepsilon \|u\|.$$

Then, $z := (x+u, y') \in \mathrm{Gr}(F)$ satisfies

$$d((x+u, y+A(u)), z) = \|y + A(u) - y'\| \leq \varepsilon \|u\|$$

which proves that for any $(x, y) \in (B(x_0, \delta/2) \times V) \cap \mathrm{Gr}(F)$, $u \in B(0, \delta/2)$ and $v := A(u)$ one has

$$d((x+u, y+v), \mathrm{Gr}(F)) \leq \varepsilon \|(u, v)\|,$$

so that F is strictly lower pseudo-differentiable at (x_0, y_0). ∎

Similarly, it can be shown that peridifferentiability of F at x_0 implies strict differentiability of F at x_0 in the sense of Azé (1988) and that any periderivative is a (continuous linear) selection of the periderivative of Aubin, Frankowska (1987) defined as the multimapping whose graph is the Clarke's tangent cone to the graph of F.

It follows from that comparison that the openness result of Theorem 3.3 is a consequence of the main result of Azé, Chou (1995). We have been informed by D. Azé (in personal communication) that a part of the inversion theorem of the present paper can also be deduced from the notion of (strong) slope; see Azé, Corvellec (2004) and Azé, Corvellec, Lucchetti (2002) for recent accounts about this notion and its links with metric regularity, a topic treated in many works, for instance Borwein (1986), Borwein, Dontchev (2003), Jourani, Thibault (1995), Klatte, Kummer (2002), Penot (1982, 1989). However, Theorem 3.3 brings some information about the inverse mapping and is closer to the classical inverse mapping theorem.

When $F(x_0)$ is convex, the multimapping $A(\cdot) + F(x_0)$ is affine in the sense of Gautier (1989), hence is eclipsing in the sense of Lemaréchal, Zowe (1991) and Gautier (1990). Note yet that we do not make here a convexity assumption on $F(x_0)$.

We will not make a comparison with other notions. Let us note, however, that the notions we introduce do not suffer from the deficiency of the concepts in Banks, Jacobs (1970), Hukuhara (1967), Martelli, Vignoli (1974) which impose that the values of F at points x near x_0 be larger than $F(x_0)$.

7. Application to differential inclusions

Let us illustrate our results by an application to the well-known Filippov existence theorem for differential inclusions. It is well known that fixed point theorems yield existence results for ordinary differential equations or inclusions (see Arino, Gautier, Penot, 1984, Azé, Penot, 2004, for instance). Here we give a perturbation result which may be useful for the study of reachable sets (see Frankowska, Olech, 1982, Olech, 1975, 1983 ...).

Let E be a separable Banach space, $\xi \in E$ and $T := [0, \theta]$, with $\theta > 0$. Let us consider the question of the existence of solutions to the differential inclusion:

$$(D.I.) \qquad \begin{aligned} \dot{w}(t) &\in \Phi(t, w(t)) \\ w(0) &= \xi \end{aligned}$$

where $\Phi : T \times E \rightrightarrows E$ is a multifunction. A solution $w(\cdot)$ of the differential inclusion (D.I.) is an element of the space $X := W^{1,1}(T, E)$ of continuous functions $w : T \to E$ such that there exists $u \in L_1(T, E)$ (the space of Bochner integrable functions from T into E) satisfying $u(t) \in \Phi(t, w(t))$ a.e. $t \in T$ and

$$w(t) = w(0) + \int_0^t u(s)ds \qquad \forall t \in T.$$

Then w is differentiable a.e. on T and $\dot{w} = u$. Let us assume that the multifunction Φ satisfies the following assumptions in which $w_0 \in X$, $r > 0$, $k \in L_1(T, \mathbb{R}_+)$:

(H_1) for each $(t, e) \in T \times B(w_0(t), r)$, the set $\Phi(t, e)$ is closed, nonempty and $\Phi(\cdot, e)$ is mesurable;

(H_2) for a.e. $t \in T$, the multifunction $\Phi(t, \cdot)$ is $k(t)-$Lipschitzian on $B(w_0(t), r)$;

(H_3) $\rho(\cdot) := d(\dot{w_0}(\cdot), \Phi(\cdot, w_0(\cdot)))$ is in $L_1(T, \mathbb{R})$.

Such assumptions are classical since the work of Filippov (1967) in which the following result has been given; see also Aubin, Frankowska (1987), Azé, Penot (2005), Deimling (1992), Papageorgiou (1988), Polovinkin, Smirnov (1986), Tolstonogov (2000), Zhu (1991).

THEOREM 7.1 *Under assumptions* $(H_1), (H_2)$ *and* (H_3), *for each* $\xi \in E$ *close enough to* $w_0(0)$, *the differential inclusion* $(D.I.)$ *has a solution on some interval* $[0, \tau]$ *with* $\tau \in (0, \theta)$.

Proof. Changing Φ into $\widehat{\Phi}$ given by $\widehat{\Phi}(t, e) := \Phi(t, e + w_0(t)) - \dot{w}_0(t)$, we may suppose $w_0(t) = 0$ for $t \in T$. Setting $X = Y = W^{1,1}(T, E)$ endowed with the norm given by $\|x\|_X = \|x(0)\| + \int_0^\theta \left\|\dot{x}(s)\right\| ds$, $Z = [0, 1] \times E$ and $X_0 = \{x \in X : \forall t \in T,\ x(t) \in B(w_0(t), r)\}$, let us define the multifunction $F : X_0 \times Z \rightrightarrows X$ as follows:

$$
\begin{aligned}
&y \in F(x, z) \Longleftrightarrow \exists u \in L_1(T, E) \text{ such that } u(t) \in \Phi(\zeta t, x(t)) \\
&\text{a.e. } t \in T,\ y(t) = x(t) - \xi - \int_0^t \zeta u(s) ds.
\end{aligned}
$$

As in Lemma 3.3 of Azé, Penot (2005) one can check that F has closed nonempty values. Moreover for $x_0 := 0$, $z_0 := (0, 0)$, one has $F(x_0, z_0) = \{y_0\}$ with $y_0 := 0$. Let us show that F is partially-peridifferentiable at (x_0, z_0) w.r.t. x.

Let us first check that the multifunction $z \mapsto F(x_0, z)$ is lower semicontinuous at (z_0, y_0). For every $z := (\zeta, \xi) \in Z$, by Thm 14.60 in Rockafellar, Wets (2000) we can interchange integration and minimization over the set of selections of $s \mapsto \Phi(\zeta s, x_0(s))$ and get, using (H_3),

$$
\begin{aligned}
&d(0, F(x_0, z)) := \inf\{\|y\| : y \in F(x_0, z)\} \\
&= \inf\{\|\xi\| + \int_0^\theta \zeta \left\|u(s)\right\| ds : u(\cdot) \in \Phi(\zeta \cdot, x_0(\cdot))\} \\
&= \|\xi\| + \int_0^\theta \zeta d(0, \Phi(\zeta s, x_0(s))) ds \leq \|\xi\| + \int_0^\theta \zeta \rho(\zeta s) ds \leq \|\xi\| + \int_0^{\zeta\theta} \rho(t) dt
\end{aligned}
$$

so that $d(0, F(x_0, z)) \to 0$ as $z \to 0$.

Let us show that F is partially peridifferentiable at $((x_0, z_0), y_0)$ with respect to x and $D_1F(x_0, z_0) = I$. Given $\varepsilon > 0$, let us take $\xi \in E$ and $\zeta \in [0, 1]$ so small that $m(\zeta\theta) \leq \varepsilon$, where $m(t) := \int_0^t k(s)ds$, let us prove that for $z = (\zeta, \xi)$ and $x_1, x_2 \in B(x_0, r)$, we have

$$e(F(x_1, z) - x_1, F(x_2, z) - x_2) \leq \varepsilon \left\| x_1 - x_2 \right\|_X . \tag{22}$$

Let $y_i \in F(x_i, z)$ for $i = 1, 2$, so that $y_i(t) = x_i(t) - \xi - \int_0^t \zeta u_i(s)ds$ with $u_i(t) \in \Phi(\zeta t, x_i(t))$ for a.e. $t \in T$; then $(y_1(0) - x_1(0)) - (y_2(0) - x_2(0)) = 0$ and

$$d(y_1 - x_1, y_2 - x_2) \leq \int_0^\theta |\zeta| \left\| u_1(s) - u_2(s) \right\| ds.$$

Thus, using again Theorem 14.60 of Rockafellar, Wets (2000) and assumption (H_2), we get

$$\begin{aligned} d(y_1 - x_1, F(x_2, z) - x_2) &\leq \int_0^\theta \zeta e(\Phi(\zeta s, x_1(s)), \Phi(\zeta s, x_2(s)))ds \\ &\leq \int_0^\theta \zeta k(\zeta s) \left\| x_1(s) - x_2(s) \right\| ds \leq m(\zeta\theta) \left\| x_1 - x_2 \right\|_\infty \leq \varepsilon \left\| x_1 - x_2 \right\|_X . \end{aligned}$$

Thus the multifunction F satisfies the assumptions of Theorem 5.1 and there exist neighborhoods U, V, W of $x_0, 0$ and z_0 respectively with $U \times W \subset X_0 \times Z$ and some mapping $h : V \times W \to U$ such that $y \in F(h(y, z), z)$ for any $(y, z) \in V \times W$. In particular, for all $z \in W$ there exists $h(0, z)$ in U for which $0 \in F(h(0, z), z)$; thus there exists $u \in L_1(T, E)$ such that $u(t) \in \Phi(\zeta t, h(0, z)(t))$ for a.e. $t \in T$ and $h(0, z)(t) = \xi + \int_0^t \zeta u(s)ds$. We may suppose that W is some product $[0, \zeta] \times W_E$, where $\zeta > 0$ is small enough, and W_E is a neighborhood of 0 in E. Then, for $\xi \in W_E$, setting $z := (\zeta, \xi)$, $w(t) := h(0, z)(t/\zeta)$, $v(t) = u(t/\zeta)$ for $t \in [0, \zeta\theta]$, we have $v(\cdot) \in L_1([0, \zeta\theta], E)$, $v(t) \in \Phi(t, x(t))$ a.e. $t \in [0, \zeta\theta]$ and

$$w(t) = \xi + \int_0^t v(s)ds.$$

Thus, for $\tau := \zeta\theta$, and for every $\xi \in W_E$, the differential inclusion $(D.I.)$ has a solution on $[0, \tau]$. ■

Since the set $S(\xi)$ of solutions of $(D.I.)$ is obtained by a change of parametrization from the value $M(\zeta, \xi)$ of the implicit multimapping M defined by the inclusion $0 \in F(x, z)$, under additional assumptions, we can deduce from Corollary 5.1 a differentiability result about $S(\cdot)$. Similarly, a differentiability result can be obtained when Φ depends on parameters.

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References

Agadi, A. and Penot, J.-P. (2005) A comparative study of various notions of approximation of sets. *J. Approx. Th.* **134**, 80-101.

Arino, O., Gautier, S. and Penot, J.-P. (1984) A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations. *Funkc. Ekvacioj, Ser. Int.* **27**, 273-279.

Aubin, J.-P. (1981) Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions. *Adv. Math., Suppl. Stud.* **7A**, 159-229.

Aubin, J.-P. and Frankowska, H. (1987) On inverse function theorems for set-valued maps. *J. Maths. Pures Appl.* **66**, 71-89.

Auslender A. and Cominetti, R. (1991) A comparative study of multifunction differentiability with applications in mathematical programming. *Math. Oper. Res.* **16** (2), 240-258.

Azé, D. (1988) An inversion theorem for set-valued maps. *Bull. Aust. Math. Soc.* **37**, 411-414.

Azé, D. and Chou, C.C. (1995) On a Newton type iterative method for solving inclusions. *Math. Oper. Res.* **20** (4), 790-800.

Azé, D. and Corvellec, J.-N. (2004) Characterizations of error bounds for lower semicontinuous functions on metric spaces. *ESAIM Control Optim. Calc. Var.* **10**, to appear.

Azé, D., Corvellec, J.-N. and Lucchetti, R.E. (2002) Variational pairs and applications to stability in nonsmooth analysis. *Nonlinear Anal., Theory Methods Appl.* **49A** (5), 643-670.

Azé, D. and Penot J.-P. (2005) On the dependence of fixed point sets of pseudo-contractive multifunctions. Applications to differential inclusions. To appear in *Nonlinear Dynamics and Systems Theory.*

Banks, H.T. and Jacobs, Q.M. (1970) A differential calculus for multifunctions. *J. Math. Anal. Appl.* **29**, 246-272.

Borwein, J. (1986) Stability and regular points of inequality systems. *J. Optim. Th. Appl.* **48**, 9-52.

Borwein, J.M. and Dontchev, A.L. (2003) On the Bartle-Graves theorem. *Proc. Am. Math. Soc.* **131**, (8), 2553-2560.

Brown, A.A. and Bartholomew-Biggs, M.C. (1989a) Some effective methods for unconstrained optimization based on the solution of systems of ordinary differential equations. *J. Optim. Theory Appl.* **62** (2), 211-224.

Brown, A.A. and Bartholomew-Biggs, M.C. (1989b) ODE versus SQP methods for constrained optimization. *J. Optim. Theory Appl.* **62** (3), 371-386.

Choquet, G. (1960) *Cours d'analyse*, mimeographied notes. Centre de Documentation Universitaire, Paris.

Deimling, K. (1992) *Multivalued Differential Equations.* De Gruyter, Berlin, New York.

Dien, P. H. and Yen, N.D. (1991) On implicit function theorems for set valued maps and their application to mathematical programming under inclusion constraints. *Applied Math. Optim.* **24**, 35-54.

Dontchev, A.L. (1995) Implicit function theorems for generalized equations. *Math. Program.* **70A** (1), 91-106.

Dontchev, A.L. (1996) The Graves theorem revisited. *J. Convex Anal.* **3** (1), 45-53.

Dontchev, A.L. and Hager, W.W. (1993) Lipschitzian stability in nonlinear control and optimization. *SIAM J. Control Opt.* **31** (3), 569-603.

Dontchev, A.L. and Hager, W.W. (1994a) Implicit functions, Lipschitz maps, and stability in optimization. *Math. Oper. Res.* **19** (3), 753-768.

Dontchev, A.L. and Hager, W.W. (1994b) An inverse mapping theorem for set-valued maps. *Proc. Amer. Math. Soc.* **121**, 481-489.

Dontchev, A.L. and Hager, W.W. (1996) Implicit functions defined by generalized equations. In: V. Lakshmikantham, ed., *World Congress of Nonlinear Analysis '92. Proceedings of the first world congress, Tampa, FL, USA, August 19-26, 1992.* de Gruyter, Berlin, 2289-2298.

Dontchev, A.L. and Hager, W.W. (1998) Lipschitzian stability for state constrained nonlinear optimal control. *SIAM J. Control Opt.* **36** (2), 698-718.

Dontchev, A.L. and Rockafellar, R.T. (1997) Characterizations of Lipschitzian stability in nonlinear programming. In: A. Fiacco, ed., *Mathematical programming with data perturbations. 17th symposium, George Washington University, Washington, DC, USA, May 1995.* New York, NY: Marcel Dekker. *Lect. Notes Pure Appl. Math.* **195**, 65-82.

Fabian, M. (1979) Concerning an interior mapping theorem. *Comm. Math. Univ. Carol.* **20**, 345-356.

Filippov, A.F. (1967) Classical solutions of differential equations with multivalued right-hand side. *SIAM J. Control* **5**, 609-621.

Frankowska, H. (1987) An open mapping theorem for set-valued maps. *J. Math. Anal. Appl.* **127**, 172-180.

Frankowska, H. and Olech, C. (1982) Boundary solutions of differential inclusion. *J. Differ. Equations* **44**, 156-165.

Gautier, S. (1989) *Contributions à l'analyse non-linéaire: inclusions différentielles, différentiabilité des multiapplications.* Thèse d'Etat, Univ. of Pau.

Gautier, S. (1990) Affine and eclipsing multifunctions. *Numer. Funct. Anal. Optimization* **11** (7/8), 679-699.

Gautier, S., Isac, G. and Penot, J.-P. (1983) Surjectivity of multifunctions under generalized differentiability assumptions. *Bull. Aust. Math. Soc.* **28**, 13-21.

Grande, Z. (1994) On the inverse multifunction. *Real Anal. Exch.* **19** (2), 598-603.

HERMES, H. (1970) The generalized differential equation $\dot{x} \in R(t,x)$. *Adv. Math.* **4**, 149-169.

HALKIN, H. (1974) Implicit functions and optimization problems without continuous differentiability of the data. *SIAM J. Control* **12**, 229-236.

HALKIN, H. (1976) Interior mapping theorem with set-valued derivatives. *J. Anal. Math.* **30**, 200-207.

HIMMELBERG, C.J. and VAN VLECK, F.S. (1973) Lipschitzian generalized differential equations. *Rend. Sem. Mat. Univ. Padova* **48**, 159-169.

HUKUHARA, M. (1967) Integration of measurable maps with compact, convex set values. *Funkcial. Ekvac.* **10**, 205-223.

IOFFE, A.D. and TIHOMIROV, V.M. (1979) *Theory of Extremal Problems.* Studies in Mathematics and its applications **6**, North Holland, Amsterdam.

JOURANI, A. and THIBAULT, L. (1995) Metric regularity and subdifferential calculus in Banach spaces. *Set-Valued Anal.* **3** (1), 87-100.

KHANH, P.H. (1986) An induction theorem and general open mapping theorems. *J. Math. Anal. Appl.* **118**, 519-534.

KLATTE, D. and KUMMER, B. (2001) Contingent derivatives of implicit (multi-) functions and stationary points. *Ann. Oper. Res.* **101**, 313-331.

KLATTE, D. and KUMMER, B. (2002) *Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications.* Nonconvex Optimization and Its Applications **60**. Kluwer, Dordrecht.

KUMMER, B. (1991a) Lipschitzian inverse functions, directional derivatives and applications in $C^{1,1}$ optimization. *J. Optim. Th. Appl.* **70** (3), 561-581.

KUMMER, B. (1991b) An implicit function theorem for $C^{0,1}$-equations and parametric $C^{1,1}$-optimization. *J. Math. Anal. Appl.* **158**, 35-46.

LEDYAEV, Y.S. and ZHU, Q.J. (1999) Implicit multifunction theorems. *Set-Valued Anal.* **7**, 209-238.

LEMARÉCHAL, C. and ZOWE, J. (1991) The eclipsing concept to approximate a multi-valued mapping. *Optimization* **22** (1), 3-37.

LEVY, B. (2001a) Solution sensitivity from general principles. *SIAM J. Control Optimization* **40** (1), 1-38.

LEVY, B. (2001b) Lipschitzian multifunctions and a Lipschitzian inverse mapping theorem. *Math. Oper. Res.* **26** (1), 105-118.

LIM, T.-C. (1985) On fixed point stability for set-valued contractive mappings with applications to generalized differential equations. *J. Math. Anal. Appl.* **110**, 436-441.

MARTELLI, M. and VIGNOLI, A. (1974) On differentiability of multi-valued maps. *Boll. Unione Mat. Ital.* **IV**, 10, 701-712.

MAURER, H. and ZOWE, J. (1979) First and second-order necessary and sufficient conditions in mathematical programming and optimal control. *Math. Program.* **16**, 98-110.

MIGNOT, F. (1976) Contrôle dans les inéquations variationnelles elliptiques. *J. Funct. Anal.* **22**, 130-185.

NADLER, S.B. (1969) Multivalued contraction mappings. *Pacific J. Maths.* **30**, 475-488.

OLECH, C. (1975) Existence of solutions of non-convex orientor fields. *Boll. Unione Mat. Ital.* **IV**, 11, Suppl. Fasc. 3, 189-197.

OLECH, C. (1983) Boundary solutions of differential inclusion. In: *Mathematical Theories of Optimization, Proc. Conf., Genova 1981*, Lect. Notes Math. **979**, 236-239.

PAPAGEORGIOU, N.S. (1988) Convergence theorems for fixed points of multifunctions and solutions of differential inclusions in Banach spaces. *Glas. Mat.* **23**, 247-257.

PENOT, J.-P. (1970) Sur le théorème de Frobenius. *Bull. Soc. Math. France* **98**, 47-80.

PENOT, J.-P. (1982) On regularity conditions in mathematical programming. *Math. Prog. Study* **19**, 167-199.

PENOT, J.-P. (1984) Differentiability of relations and differential stability of perturbed optimization problems. *SIAM J. Control and Optim.* **22** (4), 529-551. Erratum, idem, 260 (4) (1986).

PENOT, J.-P. (1985) Open mapping theorems and linearization stability. *Numerical Funct. Analysis and Optimization* **8** (1-2), 21-36.

PENOT, J.-P. (1989) Metric regularity, openness and Lipschitzian behavior of multifunctions. *Nonlinear Anal., Theory Methods Appl.* **13** (6), 629-643.

PENOT, J.-P. (1995) Inverse function theorems for mappings and multimappings. *South East Asia Bull. Math.* **19** (2), 1-16.

POLOVINKIN, E.S. and SMIRNOV, G.V. (1986) An approach to the differentiation of many-valued mappings, and necessary conditions for optimization of solutions of differential inclusions. *Diff. Uravneniya* **22** (6), 944-954.

PTÁK, V. (1982) A nonlinear subtraction theorem. *Proc. Royal Irish Acad. Sect.* **A 62**, 47-53.

ROBBIN, J.W. (1968) On the existence theorem for differential equations. *Proc. Am. Math. Soc.* **19**, 1005-1006.

ROBINSON, S.M. (1976) Stability theory for systems of inequalities, part II: pseudo-differentiable nonlinear systems. *SIAM J. Numer. Anal.* **13**, 497-513.

ROBINSON, S.M. (1991) An implicit-function theorem for a class of nonsmooth functions. *Math. Oper. Res.* **16** (2), 292-309.

ROCKAFELLAR, R.T. and WETS, R.JB. (2000) *Variational Analysis.* Springer, New York.

SACH, P.H. (1998) Differentiability of set-valued maps in Banach spaces. *Math. Nachr.* **139** (188), 215-235.

SENGUPTA, A. (1997) Multifunction and generalized inverse. *J. Inverse Ill-Posed Probl.* **5** (3), 265-285.

SEROVAISKIJ, S.YA. (1995) Inverse function theorem and extended differentiability in Banach spaces. *Russ. Math.* **39** (8), 37-46 (1995); translation from *Izv. Vyssh. Uchebn. Zaved., Mat.* 1995 (8), 39-49 (1995).

SILIN, D.B. (1997) On set-valued differentiation and integration. *Set-Valued Anal.* **5** (2), 107-146.

SZILÁGYI, T. (1989) Existence and Lipschitzian dependence theorems in complete spaces. *Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Math.* **32**, 191-202.

TOLSTONOGOV, A. (2000) *Differential Inclusions in a Banach Space.* Kluwer, Dordrecht.

YEN, N.D. (1987) Implicit function theorems for set-valued maps. *Acta Math. Vietnamica* **12** (2), 17-28.

ZHANG, R. and TREIMAN, J. (1995) Upper-Lipschitz multifunctions and inverse subdifferentials. *Nonlinear Anal., Theory Methods Appl.* **24** (2), 273-286.

ZHU, Q.J. (1991) On the solution set of differential inclusions in Banach spaces. *J. Differ. Equations* **93**, 213-237.