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*Professor Czesław Olech*

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# Phase portraits of planar control-affine systems*

by

**Bronisław Jakubczyk**[1 †] **and Witold Respondek**[2]

[1]Institute of Applied Mathematics and Mechanics
Warsaw University
ul. Banacha 2, 02-097 Warsaw, Poland
e-mail: B.Jakubczyk@impan.gov.pl

[2]Laboratoire de Mathématiques
INSA de Rouen
Pl. Emile Blondel, 76131 Mont Saint Aignan, France
e-mail: wresp@insa-rouen.fr

**Abstract:** We study nonlinear control systems in the plane, affine with respect to control. We introduce two sets of feedback equivariants forming a phase portrait $\mathcal{PP}$ and a parameterized phase portrait $\mathcal{PPP}$ of the system. The phase portrait $\mathcal{PP}$ consists of an equilibrium set $E$, a critical set $C$ (parameterized, for $\mathcal{PPP}$), an optimality index, a canonical foliation and a drift direction. We show that under weak generic assumptions the phase portraits determine, locally, the feedback and orbital feedback equivalence class of a system. The basic role is played by the critical set $C$ and the critical vector field on $C$. We also study local classification problems for systems and their families.

**Keywords:** control system, family of control systems, invariants, phase portrait, critical trajectories, feedback equivalence, bifurcation.

## 1. Introduction

The phase portrait of a dynamical system allows to understand the most important features of the system. It gives a practical method of analysis of dynamical systems in the plane. Can a similar notion be defined and used for control systems?

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For optimal control problems in the plane this was already done in the book of Pontriagin, Boltianskii, Gamkrelidze and Mishchenko and is still a subject of intensive research (see Baitmann, 1978a, 1978b; Boscain and Piccoli, 2004; Bressan and Piccoli, 1998; Sussmann, 1987a, 1987b). In all those works one draws optimal synthesis or optimal trajectories, subject to specified constraints on the control and specified objective function. Even if the objective function is canonical (time), still the optimal portrait (synthesis) depends heavily on the constraints and can be very complicated (compare Sussmann, 1987a, 1987b; Bressan and Piccoli, 1998; Boscain and Piccoli, 2004). A qualitative analysis of planar systems with constraints which was partially independent of the optimality point of view was proposed in Davydov (1998, 1994).

In this paper we consider smooth planar control systems

$$\Sigma : \qquad \dot{z} = f(z) + ug(z), \qquad z \in X \subset \mathbb{R}^2,\ u \in \mathbb{R},$$

where $z = (z_1, z_2)$ is the state, $u$ is the control, $X \subset \mathbb{R}^2$ is an open subset, and $f$ and $g$ are $C^\infty$-smooth vector fields on $X$. There are no specified control constraints. The aim is to understand the structure of systems $\Sigma$ without introducing a particular optimality problem. Instead, we use the natural feedback equivalence of systems (which preserves the set of trajectories). Our aim is to define a phase portrait of a system so that the following holds:

*If two systems have the same phase portraits then they are feedback equivalent.*

When the above statement holds, knowing the phase portrait allows one to analyze all feedback invariant properties of the system (like local or global controllability, stabilizability, time-maximal and time-minimal trajectories etc). The phase portrait that we propose will consists of the equilibrium set $E$, the critical set $C$ (which is formed, roughly, by time-critical curves), the phase portrait of $g$, called the canonical foliation (or the foliation of fast trajectories), the discriminant set $D$, the optimality index $\tau$ (indicating if the curve is time-maximal or time-minimal) and a drift direction.

Our analysis is performed in the domain where $g$ does not vanish and we use smooth feedback and orbital feedback equivalence. A topological classification of generic degenerations around points where $g$ vanishes has recently been obtained by Rupniewski (2005).

The set of equilibria, the set of (time) critical trajectories, and the optimality index appear in several problems concerning control-affine planar systems. In constructing the time-optimal synthesis on $\mathbb{R}^2$ for a system $\dot{x} = f(x)+ug(x)$ with constraints $|u| \leq 1$, both the equilibria set and the critical set play an important role, see Baitmann (1978a, 1978b), Boscain and Piccoli, (2004), Bressan and Piccoli (1998), Sussmann (1987a, 1987b). In this case the set of fast trajectories is not significant. In studying generic controllability problems and singularities of the boundary of the reachable set for such systems (Davydov, 1994, 1998) the equilibrium set is important, while the other two invariants do not appear. Instead, the main role is played by two vector fields $X_+$ and $X_-$, $X_\pm = f \pm g$ and by two foliations of oriented orbits of these vector fields, called limiting directions (Davydov, 1994). In that case the problem is reduced to a local classification of two generic, oriented foliations.

Our aim is somewhat different: using information about specific trajectories (stationary, time-optimal, fast) encoded in the phase portrait we want to determine the system and, as a consequence, all its trajectories. It turns out that, indeed, the phase portrait determines uniquely generic systems (up to orbital feedback equivalence) and the parameterized phase portrait (we add the canonical parametrization on time-optimal trajectories) determines (up to feedback or orbital feedback) all systems from even a much bigger class.

The paper is organized as follows. We define feedback and orbital feedback transformations in Section 2. Then we introduce in Section 3 basic notions of the paper: phase portrait and parameterized phase portrait. Those notions lead to main results of the paper: for the phase portrait in Section 5 and for parameterized phase portrait in Section 6 (for systems) and in Section 7 (for families). We also recall a classification of generic systems in Section 4 and of generic families in Section 8, and their bifurcations in Section 9. In Appendix, we give a result on equivalence of deformations of functions on which our proofs are based.

## 2. Feedback and orbital feedback equivalence

Together with $\Sigma$, consider another smooth system

$$\tilde{\Sigma}: \qquad \dot{\tilde{z}} = \tilde{f}(\tilde{z}) + \tilde{u}\tilde{g}(\tilde{z})$$

on $\tilde{X} \subset \mathbb{R}^2$. We call $\Sigma$ and $\tilde{\Sigma}$ *feedback equivalent* if there is a $C^\infty$-smooth diffeomorphism $(\phi, \psi) : X \times \mathbb{R} \to \tilde{X} \times \mathbb{R}$, called *feedback transformation*, which is affine with respect to $u$, i.e., of the form

$$\tilde{z} = \phi(z), \quad u = \psi^{-1}(z, \tilde{u}) = \alpha(z) + \beta(z)\tilde{u},$$

and which brings $\Sigma$ into $\tilde{\Sigma}$. The resulted transformation of the dynamics is

$$\Gamma \ : \quad \tilde{f} = \phi_*(f + \alpha g), \qquad \tilde{g} = \phi_*(\beta g),$$

where $\alpha$ and $\beta$ are $C^\infty$-smooth functions of $z$, with $\beta(z) \neq 0$, and $\psi^{-1}$ stands for the inverse of $\psi$ with respect to $u$. Here for any vector field $f$ and a diffeomorphism $\tilde{z} = \phi(z)$ we denote $(\phi_* f)(\tilde{z}) = d\phi(z) \cdot f(z)$, with $z = \phi^{-1}(\tilde{z})$. The transformation $\Gamma$ will be shortly denoted $\Gamma = (\phi, \alpha, \beta)$.

If $\phi$ is a local diffeomorphism, $\phi(z_0) = \tilde{z}_0$ and the above identity holds locally around $\tilde{z}_0$, then $\Sigma$ and $\tilde{\Sigma}$ are called *locally feedback equivalent* at $z_0$ and $\tilde{z}_0$.

An *orbital feedback transformation* $\Gamma_{orb} = (\phi, \alpha, \beta, h)$ contains additionally a positive valued $C^\infty$-smooth function $h$ on $X$ which changes the time scale of the system according to

$$\frac{dt}{d\tau} = h(z).$$

Thus, by definition, $\Gamma_{orb}$ brings $\Sigma$ into $\tilde{\Sigma}$ with $\tilde{f} = \phi_*(hf + h\alpha g)$ and $\tilde{g} = \phi_*(h\beta g)$. We can incorporate the action of $h$ on $g$ by choosing $\tilde{\alpha} = h\alpha$ and $\tilde{\beta} = h\beta$ and the transformation formula becomes

$$\Gamma_{orb} \;:\quad \tilde{f} = \phi_*(hf + \tilde{\alpha} g), \quad \tilde{g} = \phi_*(\tilde{\beta} g).$$

Throughout the paper we assume that $h$ is positive valued and constant on the trajectories of $g$, i.e.,

$$L_g h = 0,$$

where $L_g$ denotes the directional (Lie) derivative along $g$

Definition 2.1 *Systems* $\Sigma$ *and* $\tilde{\Sigma}$ *are called* feedback equivalent *(resp.* locally feedback equivalent*) if one can be transformed into the other via a global (resp. local) feedback transformation* $\Gamma$*. They are called* orbitally feedback equivalent *(resp.* locally orbitally feedback equivalent*) if one can be transformed into the other via a global (resp. local) orbital feedback transformation* $\Gamma_{orb}$*, where* $h$ *satisfies* $L_g h = 0$*,* $h > 0$*.*

The feedback equivalence preserves the set of all trajectories of the system (understood as time-parameterized curves) reparameterizing that set with respect to controls. The orbital feedback equivalence also preserves the set of all trajectories (also reparameterizing that set with respect to controls) but changes the time-parameterization of trajectories. Due to the condition on $h$, which satisfies $L_g h = 0$, the orbital feedback equivalence does not change the basic properties of the system, as we shall see later.

## 3. Fundamental equivariants and phase portrait

Consider

$$\Sigma : \qquad \dot{z} = f(z) + u g(z), \qquad z \in X \subset \mathbb{R}^2,\ u \in \mathbb{R},$$

where $z = (z_1, z_2)$. In these coordinates we identify the vector fields $f = f_1 \partial/\partial z_1 + f_2 \partial/\partial z_2$ and $g = g_1 \partial/\partial z_1 + g_2 \partial/\partial z_2$ with the column vectors $f = (f_1, f_2)^T$ and $g = (g_1, g_2)^T$. We introduce the functions

$$\begin{aligned} e &= \det(f, g), \\ c &= \det([g, f], g), \\ d &= L_g c. \end{aligned}$$

Here $L_g c = \frac{\partial c}{\partial z_1} g_1 + \frac{\partial c}{\partial z_2} g_2$ denotes the derivative of $c$ along $g$ and $[g, f]$ is the Lie bracket of $g$ and $f$, with the i-th component $[g, f]_i = \sum_j (\frac{\partial f_i}{\partial z_j} g_j - \frac{\partial g_i}{\partial z_j} f_j)$.

The four following objects play a crucial role in analyzing planar systems $\Sigma$. We define the sets

$$\begin{aligned} E &= \{z \in X : e(z) = 0\}, \\ C &= \{z \in X : c(z) = 0\}, \\ D &= \{z \in X : c(z) = d(z) = 0\}, \end{aligned}$$

called, respectively, the *equilibrium set*, the *critical set* and the *discriminant set*. Finally, we define the *foliation of fast trajectories* or the *canonical foliation* as the set $\mathcal{G}$ of phase curves (unparameterized trajectories) of the vector field $g$. Away from stationary points of $g$ (nonstationary points of $g$ will also be called control-regular points), $\mathcal{G}$ consists of regular curves in $X$.

On the critical set $C$ we define the *optimality index*

$$\tau(z) = \operatorname{sgn}(e\, d)(z), \qquad z \in C.$$

Note that $\tau$ encodes the subset $(E \cup D) \cap C$ in $C$, namely $(E \cup D) \cap C = \{z \in C : \tau(z) = 0\}$.

Given $\Sigma$ and a point $z \notin E$, the set of trajectories of $g$ near $z$ (that is, the set of leaves of the canonical foliation $\mathcal{G}$) can be parameterized by a 1-dimensional parameter with values in an interval. We define the transversal *drift direction* of $\Sigma$ at $z$, denoted $\mathcal{DD}(z)$, as one of the two possible orientations of this interval, the one given by the vector $f(z)$. The drift direction $\mathcal{DD}(z)$ defines an order on the set of local trajectories of $g$, in a neighborhood of $z$. This order shows that passage between different trajectories of $g$ is possible (in a neighborhood of $z \notin E$) "in one direction" only, the direction defined by the vector $f(z)$.

Definition 3.1 *The* phase portrait *of* $\Sigma$ *is the 6-tuple* $\mathcal{PP} = (E, C, \mathcal{G}, D, \tau, \mathcal{DD})$.

The first three members of the phase portrait are basic, the remaining three play auxiliary role. What is perhaps surprising is that the critical set $C$ is the most powerful invariant (equivariant) in $\mathcal{PP}$.

We interpret the components of $\mathcal{PP}$ below. The set $E$ is the set of points $p$ which can be made equilibrium points, with a suitable feedback control $u$ so that $\tilde{f}(z) = f(z) + u(z)g(z) = 0$. The critical set $C$ consists of points at which the motion transversal to the trajectories of $g$ admits its critical velocity, in particular, locally minimal or locally maximal velocity (we shall explain this below). Both, $E$ and $C$ are, generically, curves. The set $D$ consists of those points where the critical set $C$ either degenerates or it is tangent to the canonical foliation $\mathcal{G}$. Finally, the leaves $S_\alpha$ of the canonical foliation $\mathcal{G}$ are exactly those 1-dimensional submanifolds of the state space $X$, which can be arbitrarily closely approximated by trajectories of $\Sigma$ (with large controls) and, moreover, the system can follow them approximately in both directions in $S_\alpha$, with arbitrarily large speed.

The optimality index $\tau = \operatorname{sgn}(e\,d)$ defines three categories of points on $C$

$$C_0 = \{z \in C : \tau(z) = 0\}, \quad C_+ = \{z \in C : \tau(z) > 0\},$$
$$C_- = \{z \in C : \tau(z) < 0\}$$

and the partition $C = C_0 \cup C_+ \cup C_-$ determines $\tau$. Note that $C_0 = C \cap (E \cup D)$. We shall see later that $C_+$ and $C_-$ consist of time-maximal and time-minimal curves, respectively.

The just defined objects are particularly simple for the following prenormal form $\Sigma_{pre}$. If $g(z) \neq 0$, then there exist local coordinates $(x, y)$ around $z$ such that $g = \partial/\partial y$. The system equations become $\dot{x} = f_1(x, y)$, $\dot{y} = f_2(x, y) + u$. Applying the feedback transformation $u \mapsto u - f_2(x, y)$ we obtain the following proposition, stated here for further reference.

Proposition 3.1 *If $g(z) \neq 0$, then $\Sigma$ is locally feedback equivalent at $p$ to the prenormal form*

$$S_{pre}: \quad \dot{x} = f_1(x, y), \quad \dot{y} = u.$$

For the system $\Sigma_{pre}$ the condition $L_g h = 0$ means that $h$ is a function of the variable $x$, only. We also have $f = (f_1, 0)^T$, $g = (0, 1)^T$, $[g, f] = (\partial f_1/\partial y, 0)^T$, and

$$e = f_1, \qquad c = \frac{\partial f_1}{\partial y}, \qquad d = \frac{\partial^2 f_1}{\partial y^2}.$$

Thus

$$E = \{f_1 = 0\}, \quad C = \{\ \frac{\partial f_1}{\partial y} = 0\ \}, \quad D = \{\ \frac{\partial^2 f_1}{\partial^2 y} = 0\ \}.$$

and the canonical foliation is given by

$$\mathcal{G} = \{S_\alpha\}_{\alpha \in \mathbb{R}}, \quad \text{where} \quad S_\alpha = \{x = \alpha = \text{const}\}.$$

Finally,

$$\tau(x, y) = \operatorname{sgn}\left(f_1 \frac{\partial^2 f_1}{\partial y^2}\right)(x, y), \qquad (x, y) \in C.$$

The velocity $\dot{x} = f_1(x, y)$ can be identified with the velocity transversal to the leaves of $\mathcal{G}$. This means that $C$ consists of the points where the motion transversal to the trajectories of $g$ admits its critical velocity. This interpretation is the starting point for an approach to the feedback classification problem based on its relations with the time-optimal control problem (see Bonnard, 1991, Jakubczyk, 1998). If $\partial^2 f_1/\partial y^2(x, y) \neq 0$ then the curve $C = \{\partial f_1/\partial y(x, y) = 0\}$ has, locally, a parametrization $y = \varphi(x)$. The velocity $\dot{x} = f_1(x, y)$ of $x$ is locally minimal or locally maximal, as a function of $y$, when $(x, y)$ lies on the critical curve $C = \{y = \varphi(x)\}$. Maximality or minimality depends on the sign of $\partial^2 f_1/\partial y^2$ and, actually, the curve $y = \varphi(x)$ is time-minimal if $\tau(x, y) < 0$ and time-maximal if $\tau(x, y) > 0$. This shows that:

*$C_+$ consists of locally time-maximal and $C_-$ of locally time-minimal trajectories of $\Sigma$.*

We have shown this for a system $\Sigma_{pre}$. The same holds for a general system $\Sigma$ since $C$ and $\tau$ are invariant, by the proposition below.

Proposition 3.2 *If $\Sigma$ and $\tilde{\Sigma}$ are feedback equivalent, under the feedback transformation $\Gamma = (\phi, \alpha, \beta)$, then*

$$\tilde{E} = \phi(E), \quad \tilde{C} = \phi(C), \quad \tilde{D} = \phi(D), \quad \tilde{\mathcal{G}} = \phi(\mathcal{G}), \quad \text{and} \quad \tau = \tilde{\tau} \circ \phi \quad \text{on} \quad C.$$

*The same holds if $\Sigma$ and $\tilde{\Sigma}$ are orbitally feedback equivalent. In particular, the phase portrait does not change under the transformation $\Gamma_{orb} = (id, \alpha, \beta, h)$ and it is transformed by $\phi$, when $\Gamma_{orb} = (\phi, \alpha, \beta, h)$.*

The above property of $E$, $C$, $\mathcal{G}$ and of the ideals $(e)$, $(c)$ is called *equivariance* or, by abuse of language, *invariance*. Thus $E$, $C$, $\mathcal{G}$ are said to be *equivariant* or, by abuse of language, *invariant* with respect to feedback equivalence.

Note that from the invariance of $C$ and $\tau$ we also get $\tilde{C}_+ = \phi(C_+)$ and $\tilde{C}_- = \phi(C_-)$.

Recall that the Lie bracket has two basic properties:

$$[\phi_* f, \phi_* g] = \phi_*[f, g],$$
$$[a\, f, b\, g] = ab\, [f, g] + aL_f b\, g - bL_g a\, f,$$

where $\phi$ is a diffeomorphism, $f$, $g$ are vector fields, and $a$, $b$ are smooth functions. Using the second property we see that if $\tilde{f} = hf + \alpha g$ and $\tilde{g} = \beta g$, then the condition $L_g h = 0$ implies

$$[\tilde{f}, \tilde{g}] = h\beta[f, g] + \varphi\, g,$$

where $\varphi = hL_f(\beta) + \alpha L_g(\beta) - \beta L_g(\alpha)$.

*Proof of Proposition 3.2.* Recall that the sets $E$, $C$, $D$ are defined as zeros of the functions $e = \det(f, g)$, $c = \det([g, f], g)$, $d = L_g c$. Replacing $f$ and $g$ by the equivalent pair $\tilde{f} = hf + g\alpha$, $\tilde{g} = \beta\, g$ gives $[\tilde{f}, \tilde{g}] = h\beta[f, g]\ mod\, g$ and thus changes $e$, $c$, and $d$ for

$$\tilde{e} = h\beta\, e, \qquad \tilde{c} = h\beta^2 c, \qquad \tilde{d} = h\beta^3 d + c\beta L_g(h\beta^2),$$

respectively. Thus, the ideals $I(e)$, $I(c)$, and $I(c, d)$ generated, respectively, by $e$, $c$, and by $c$ and $d$, do not change under the orbital feedback transformation $\Gamma_{orb} = (id, \alpha, \beta, h)$. It follows from the property of Lie bracket

$[\phi_*^{-1}g, \phi_*^{-1}f] = \phi_*^{-1}[g,f]$ that under the transformation $\Gamma_{orb} = (\phi, \alpha, \beta, h)$, the ideals are transformed by the coordinate change $\phi$. Thus $E$, $C$ and $D$ (being the zero level sets of $e$, $c$, and of $c$ and $d$) are also transformed by the coordinate change $\phi$ and so they satisfy the relations stated in the proposition. Finally, under $\Gamma_{orb} = (id, \alpha, \beta, h)$ we have on $C = \tilde{C}$:

$$\tilde{\tau} = \operatorname{sgn}(\tilde{e}\,\tilde{d}) = \operatorname{sgn}(h^2\beta^4 ed) = \operatorname{sgn}(ed) = \operatorname{sgn}\tau$$

and the transformation $\Gamma_{orb} = (\phi, \alpha, \beta, h)$ gives $\tau = \tilde{\tau}\circ\phi$ on $C$. ■

The phase portrait determines basic qualitative properties of the system, like controllability and stabilizability, and will be used for defining bifurcations in Section 9. We will show in Section 5 that the phase portrait determines locally the system up to orbital equivalence. Analogous results for feedback equivalence and for families of systems will be given in the consecutive sections.

## 4. Feedback classification of generic systems

In order to set the stage for further considerations we recall generic local classification results (Jakubczyk and Respondek, 1990).

For functions $h_1, h_2$ we denote their Jacobian $j(h_1,h_2) = \det(\partial h_i/\partial z_j)$. We introduce the following conditions at a point $p \in X$

$$(GS1) \qquad (e,c,j(e,c))(p) \neq (0,0,0)$$
$$(GS2) \qquad (c,d,j(c,d))(p) \neq (0,0,0)$$

Below, by $*$ we denote arbitrary nonzero numbers.

THEOREM 4.1 *A smooth system $\Sigma$, at any point $p$ at which $g(p) \neq 0$ and (GS1), (GS2) hold, is locally feedback equivalent to one of the following systems at $0 \in \mathbb{R}^2$:*

| | | | | |
|---|---|---|---|---|
| (O) | $\dot{x} = y+1,$ | $\dot{y} = v,$ | iff | $(e,c) = (*,*)$ at $p$; |
| (E) | $\dot{x} = y,$ | $\dot{y} = v,$ | iff | $(e,c) = (0,*)$ at $p$; |
| $(\mathrm{C})^{\pm}$ | $\dot{x} = y^2 \pm 1,$ | $\dot{y} = v,$ | iff | $(e,c,d) = (*,0,*)$ at $p$; |
| $(\mathrm{EC})_\lambda$ | $\dot{x} = y^2 + \lambda x,$ | $\dot{y} = v,$ | iff | $(e,c,j(e,c)) = (0,0,*)$ at $p$; |
| $(\mathrm{C}\mathcal{G})_{\mathrm{a}}$ | $\dot{x} = y^3 + xy + a(x),$ | $\dot{y} = v,$ | iff | $(e,c,d,j(c,d)) = (*,0,0,*)$ at $p$, |

*where $\lambda \neq 0$ and $a(0) \neq 0$. The same holds for orbital feedback equivalence, with the last two normal forms replaced, respectively, by*

| | | |
|---|---|---|
| $(\mathrm{EC})^{\pm}$ | $\dot{x} = y^2 \pm x,$ | $\dot{y} = v;$ |
| $(\mathrm{C}\mathcal{G})^{\pm}$ | $\dot{x} = y^3 + xy \pm 1,$ | $\dot{y} = v.$ |

The phase portraits of the canonical systems are illustrated in Figs. 1,2. Vertical lines represent the canonical foliation and arrows indicate the drift direction.

The proof of the first part is given in Jakubczyk and Respondek (1990). The second part (orbital equivalence) can be deduced from the first one. The above Theorem gives a local classification of generic systems (i.e., satisfying (GS1) and (GS2) at any $p \in X$). Arbitrary analytic, control-affine systems were also classified (Jakubczyk and Respondek, 1991, see also Respondek, 1998).

Above, the constant $\lambda \in \mathbb{R}$ is invariant, namely, it is the eigenvalue of the uncontrollable mode of the linear approximation of $\Sigma$ at $p$. The smooth function $a(x)$ has the following interesting invariance property: two systems of the form $(\mathrm{C}\mathcal{G}_a)$, given by $y^3 + xy + a(x)$ and $y^3 + xy + \tilde{a}(x)$, respectively, are equivalent if and only if

$$a(x) = \tilde{a}(x) \qquad \text{for} \quad x \leq 0.$$

We interpret the numerical invariant $\lambda$ and the functional invariant $a(x)$ in terms of critical trajectories in Section 6 (compare Jakubczyk and Respondek, 1990, and Zhitomirskii, 1985).

Remark 4.1 *If we drop the condition (GS1), then a smooth system $\Sigma$, under the condition (GS2) only (which remains generic), is locally feedback equivalent, around any control-regular point, to one of the following systems at $0 \in \mathbb{R}^2$: (O), (E), $(\mathrm{C})^{\pm}$, $(\mathrm{C}\mathcal{G})_{\mathrm{a}}$ (with an arbitrary $a(x)$), or*

$$(\mathrm{EC})_{\mathrm{a}} \qquad \dot{x} = y^2 + a(x), \quad \dot{y} = v, \qquad \text{iff} \quad (e, c, d) = (0, 0, \star) \ \text{ at } p$$

*(with $a(0) = 0$). If there exists a positive integer $k$ such that $a^{(k)}(0) \neq 0$, then by applying an additional feedback transformation we can normalize $a(x)$ as $a(x) = \pm x^k + \lambda x^{2k-1}$, where $\lambda \in \mathbb{R}$ (or as $a(x) = \pm x^k$ in the case of orbital feedback).*

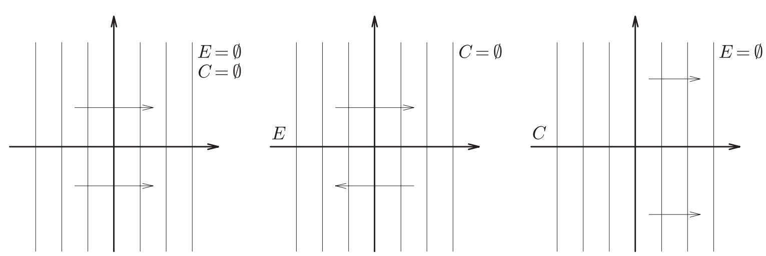


Figure 1. Normal form (O), normal form (E), normal form $(\mathrm{C})^{\pm}$

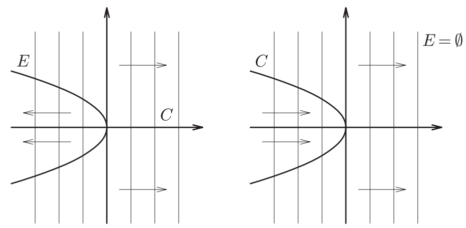


Figure 2. Normal form $(\mathrm{EC})_\lambda$, $\lambda > 0$, normal form $(\mathrm{C}\mathcal{G})^{\pm}$, $a(0) > 0$

## 5. Phase portrait and orbital feedback equivalence

In this section we will show that locally the phase portrait $\mathcal{PP}$ determines generic systems up to orbital feedback equivalence. We will use the genericity conditions (GS1), (GS2) (Section 4). Theorem 4.1 and Proposition 3.2 imply the following result:

THEOREM 5.1 *The following conditions are equivalent for two smooth systems $\Sigma$ and $\tilde{\Sigma}$ satisfying (GS1) and (GS2) around points $p$, $\tilde{p}$ such that $g(p) \neq 0$, $\tilde{g}(\tilde{p}) \neq 0$.*

- (i) *The systems $\Sigma$ and $\tilde{\Sigma}$ are locally orbitally feedback equivalent at $p$ and $\tilde{p}$, respectively.*
- (ii) *There exists a local diffeomorphism $\phi$, such that $\phi(p) = \tilde{p}$, transforming the phase portrait $\mathcal{PP}$ of $\Sigma$ into the phase portrait $\widetilde{\mathcal{PP}}$ of $\tilde{\Sigma}$.*
- (iii) *There exists a local homeomorphism $\phi$, such that $\phi(p) = \tilde{p}$, transforming the phase portrait $\mathcal{PP}$ of $\Sigma$ into the phase portrait $\widetilde{\mathcal{PP}}$ of $\tilde{\Sigma}$.*

*Proof.* (i)⇒(ii) Assume $\Sigma$ and $\tilde{\Sigma}$ be orbitally feedback equivalent via $\Gamma_{orb} = (\phi, \alpha, \beta, h)$. Then, by Proposition 3.2, the diffeomorphism $\phi$ maps $E$ into $\tilde{E}$, $C$ into $\tilde{C}$, $\tau$ into $\tilde{\tau}$, $\mathcal{G}$ into $\tilde{\mathcal{G}}$. It is clear that $\phi$ maps $\mathcal{DD}$ into $\widetilde{\mathcal{DD}}$.

Obviously, (ii)⇒(iii) and in the remaining part of the proof we will show that (iii)⇒(i). Consider two systems $\Sigma$ and $\tilde{\Sigma}$ and suppose that their respective phase portraits $\mathcal{PP}$ and $\widetilde{\mathcal{PP}}$ are equivalent via a homeomorphism. Without loss of generality, we can assume that $\Sigma$ and $\tilde{\Sigma}$, satisfying (GS1) and (GS2), are represented by two of the normal forms (O), (E), $(\mathrm{C})^{\pm}$, $(\mathrm{EC})^{\pm}$, $(\mathrm{C}\mathcal{G})^{\pm}$ listed in Theorem 4.1. Thus it is enough to compare, case by case, all pairs of the list in order to exclude the possibility that they are represented by two different normal forms. This exercise is done below, for completeness.

*1st case.* $\Sigma$ is given by (O) so $E = \emptyset$ and $C = \emptyset$ while $\tilde{\Sigma}$ is given by one of the remaining normal forms but for all of them either $E$ or $C$ is nonempty. Hence a homeomorphism $\phi$ conjugating the phase portraits $\mathcal{PP}$ of $\Sigma$ and $\widetilde{\mathcal{PP}}$ of $\tilde{\Sigma}$ cannot exist.

*2nd case.* $\Sigma$ is given by (E) so $E$ is nonempty and the only other normal forms with a nonempty $E$ are $(\mathrm{EC})^{\pm}$. We have, however, $C = \emptyset$ for (E) but $C \neq \emptyset$ for $(\mathrm{EC})^{\pm}$ thus contradicting the existence of a conjugating homeomorphism.

*3rd case.* $\Sigma$ is given by $(\mathrm{C})^+$ or by $(\mathrm{C})^-$, so $C \neq \emptyset$ and $E = \emptyset$. The only other normal form with an empty $E$ and a nonempty $C$ is $(\mathrm{C}\mathcal{G})^{\pm}$. For $(\mathrm{C})^+$, we have $C = C_+$ (and $C_- = \emptyset$) so there does not exit a homeomorphism $\phi$ which would conjugate it with $(\mathrm{C})^-$ (for which, $C = C_-$ and $C_+ = \emptyset$). To prove that $\tilde{\Sigma}$ cannot be of the form $(\mathrm{C}\mathcal{G})^{\pm}$, either, notice that for the latter each leaf of $\mathcal{G}$ intersects the curve $C$ either twice or not at all (and only one leaf intersects $C$ just one time) while for $(\mathrm{C}^+)$ and $(\mathrm{C}^-)$ each leaf of $\mathcal{G}$ intersects $C$ one time. Thus a homeomorphism conjugating phase portraits cannot exist.

*4th case.* $\Sigma$ is given by $(\mathrm{EC})^+$ or by $(\mathrm{EC})^-$, which are the only forms such that both $C$ and $E$ are nonempty. In both cases $E = \{y^2 \pm x = 0\}$ is a parabola and $C = \{y = 0\}$ is a line. They intersect at $0 \in \mathbb{R}^2$. The critical line $C$ has a distinguished part $C^*$ the points of which lie "between the arms of $E$" (the points in $C$ that are intersected by those trajectories of $g$ which also intersect $E$). On the $C^*$ the drift directions $\mathcal{DD}(z)$, represented by $f(z)$, point toward 0, in the case $(\mathrm{EC})^-$, and away of 0 in the case $(\mathrm{EC})^+$. Thus $(\mathrm{EC})^+$ and $(\mathrm{EC})^-$ cannot be equivalent.

*5th case.* $\Sigma$ is given by $(\mathrm{C}\mathcal{G})^+$ or by $(\mathrm{C}\mathcal{G})^-$ and hence $E = \emptyset$ while $C \neq \emptyset$. The only other normal forms with an empty $E$ and nonempty $C$ are $(\mathrm{C})^{\pm}$, which were excluded in case 3, so we can assume that $\Sigma$ is given by $(\mathrm{C}\mathcal{G})^+$ and $\tilde{\Sigma}$ by $(\mathrm{C}\mathcal{G})^-$. In these two cases the critical curve is the parabola $C = \{x + 3y^2 = 0\}$. However, the two phase portraits can not be equivalent since the drift direction at $0 \in \mathbb{R}^2$ points "inward the parabola $C$", in the case $(\mathrm{C}\mathcal{G})^-$, and outward the parabola $C$, in the case $(\mathrm{C}\mathcal{G})^+$. Here, "inside of $C$" is defined as the set of those points $z$ which lie inside segments of trajectories of $g$ meeting $C$. ∎

## 6. Parameterized phase portraits

If we use feedback equivalence, instead of orbital feedback equivalence, the phase portraits do not distinguish all locally nonequivalent generic systems. This follows from Theorem 4.1 and the remarks which follow it. Namely, all systems in the normal form $(\mathrm{C}\mathcal{G})_a$, with $a(0) > 0$, are locally orbitally feedback equivalent (and have equivalent phase portraits), while they are not locally feedback equivalent if $a(x) \not\equiv \tilde{a}(x)$, $x \leq 0$. We shall add an additional ingredient to the phase portrait (a critical vector field on $C \setminus D$) so that, for generic systems, the new portrait distinguishes nonequivalent systems under local feedback equivalence.

Consider the subset $C^{reg} = C \setminus D$ of $C$. Since $L_g c \neq 0$ at all points of $C^{reg}$, this set is a submanifold (curve) transversal to the trajectories of $g$ (the leaves of $\mathcal{G}$). The connected components $C_j$ of $C^{reg}$ are regular curves transversal to $\mathcal{G}$. Thus, there exists a unique control $u_j = u_j(z)$, defined on $C_j$ and $C^\infty$-smooth on $C_j$, such that the vector field $f_j^{crit}(z) = f(z) + u_j(z)g(z)$ is tangent to $C_j$.

In this way we obtain a vector field $f^{crit}$ on $C^{reg}$ which will be called *critical vector field* of $\Sigma$.

Recall that the optimality index $\tau = \mathrm{sgn}\,(e\,d)$ defines three categories of points on $C$: $C_0 = \{\tau = 0\}$, $C_+ = \{\tau > 0\}$, $C_- = \{\tau < 0\}$. Here $C_0 = C \cap (E \cup D)$ while $C_+$ and $C_-$ consist of locally time-maximal and time-minimal curves, respectively. The curves $C_+$ and $C_-$ are canonically parameterized by $f^{crit}$, as time-maximal and time-minimal trajectories of $\Sigma$, since $f^{crit}$ is nonzero on $C_+ \cup C_-$. On the other hand, $f^{crit}$ vanishes on $C^{reg} \cap E$.

DEFINITION 6.1 *We define the* parameterized phase portrait *of* $\Sigma$ *as the collection* $\mathcal{PPP} = (E, C, \mathcal{G}, D, \tau, f^{crit}, \mathcal{DD})$, *where* $f^{crit}$ *is the critical vector field on* $C \setminus D$.

EXAMPLE 6.1 *To illustrate the notion of critical vector field, we consider the normal form* $(\mathrm{EC})_\lambda$ *in Theorem 4.1, where* $\Sigma : \dot{x} = y^2 + \lambda x,\ \dot{y} = v$. *We have* $f_1(x, y) = y^2 + \lambda x$ *and hence* $E = \{y^2 + \lambda x = 0\}$, $C = \{y = 0\}$ *and the foliation* $\mathcal{G}$, *given by* $\{x = const\}$, *is transversal to* $C$ *at its every point, that is* $D = \emptyset$. *We have the critical vector field on* $C$*:*

$$f^{crit} = \lambda x \frac{\partial}{\partial x}.$$

*The function* $d = 2$ *and thus* $\tau(x, y) = \mathrm{sgn}\ (2(y^2 + \lambda x)) = \mathrm{sgn}\ \lambda x$ *for* $(x, y) \in C = \{y = 0\}$. *Assume* $\lambda > 0$. *Then the critical vector field parameterizes* $C_+ = \{y = 0, x > 0\}$ *as the time-maximal trajectory, and* $C_- = \{y = 0,\ x < 0\}$ *as the time-minimizing trajectory. For* $\lambda < 0$ *the situation is opposite.*

DEFINITION 6.2 *We say that the parameterized phase portraits* $\mathcal{PPP}$ *of* $\Sigma$ *and* $\widetilde{\mathcal{PPP}}$ *of* $\tilde{\Sigma}$ *are locally equivalent at* $p$ *and* $\tilde{p}$ *if there exits a local diffeomorphism* $\phi : X \to \tilde{X}$, *such that* $\phi(p) = \tilde{p}$, *which transforms* $\mathcal{PPP}$ *into* $\widetilde{\mathcal{PPP}}$. *Similarly,* $\mathcal{PPP}$ *and* $\widetilde{\mathcal{PPP}}$ *are called locally orbitally equivalent at* $p$ *and* $\tilde{p}$ *if there exist a local diffeomorphism* $\phi : X \to \tilde{X}$, $\phi(p) = \tilde{p}$, *and a positive valued function* $h$ *on* $C \setminus D$ *(having a smooth extension to a neighborhood of* $p$ *such that* $L_g h = 0$*) which transform* $\mathcal{PPP}$ *into* $\widetilde{\mathcal{PPP}}$.

Above the elements $E$, $C$, $D$, $\mathcal{G}$ and $\tau$ are transformed by $(\phi, h)$ according to the formulas in Proposition 3.2. The critical vector field is transformed via the formula $\phi_*(h\, f^{crit}) = \tilde{f}^{crit}$.

Let $\mathrm{grad}\, c := (\frac{\partial c}{\partial z_1}, \frac{\partial c}{\partial z_2})$. We impose the following conditions on the system $\Sigma$.

(AS1) $\mathrm{grad}\, c(p) \neq 0$ at each point $p \in C$.

(AS2) $D$ is nowhere dense in $C$.

Note that the condition (GS2) from Section 4 implies (AS1) and (AS2).

THEOREM 6.1 *Let $\Sigma$ and $\tilde{\Sigma}$ satisfy the conditions (AS1) and (AS2) around points $p$ and $\tilde{p}$ such that $g(p) \neq 0$, $\tilde{g}(\tilde{p}) \neq 0$, respectively. Then $\Sigma$ and $\tilde{\Sigma}$ are locally feedback equivalent (resp., locally orbitally feedback equivalent) at $p$ and $\tilde{p}$ if and only if their parameterized phase portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ are locally equivalent (resp., locally orbitally equivalent) at $p$ and $\tilde{p}$.*

*Proof.* Assume that $\Sigma$ and $\tilde{\Sigma}$ are locally feedback equivalent via $\Gamma = (\phi, \alpha, \beta)$ (resp. locally orbitally feedback equivalent via $\Gamma = (\phi, \alpha, \beta, h)$). From Proposition 3.2 it follows that feedback equivalent or orbital feedback equivalent systems have equivalent $E$, $C$, $\mathcal{G}$, $D$, $\tau$, and $\mathcal{DD}$. Recall that the critical vector field $f^{crit}$ on $C \setminus D$ is defined in a unique invariant way and thus it is mapped into $\tilde{f}^{crit}$ on $\tilde{C} \setminus \tilde{D}$ via $\phi_*$ (resp. via $\phi_*$ and $h$).

Now we will show that local equivalence of parameterized phase portraits implies local feedback equivalence of systems. We will consider the case of orbital feedback equivalence at the end of the proof. Consider two systems $\Sigma$ and $\tilde{\Sigma}$ and assume that they both satisfy (AS1), (AS2) and that their parameterized phase portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ are locally equivalent via a diffeomorphism $\phi$.

It is easy to see that if $p \notin C$ and $\tilde{p} \notin \tilde{C}$, then $\Sigma$ and $\tilde{\Sigma}$ are locally feedback linearizable and thus locally feedback equivalent to one of the first two canonical forms in Theorem 4.1 (depending on whether or not $p$ and $\tilde{p}$ are in the equilibrium set $E$ and $\tilde{E}$, respectively). We have $\phi(E) = \tilde{E}$ and thus $\Sigma$ and $\tilde{\Sigma}$ are locally feedback equivalent. We can restrict further considerations to the case where $p \in C$ and $\tilde{p} \in \tilde{C}$.

Local equivalence of the portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ means that there exists a local diffeomorphism $\phi$ which identifies the points $p$ with $\tilde{p}$ and makes the local phase portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ coincide. Let us transform $\Sigma$ by the diffeomorphism $\phi$. After applying $\phi$ to $\Sigma$ we have $\tilde{p} = p$ and $g(p) \neq 0 \neq \tilde{g}(p)$ and the vector fields $g$ and $\tilde{g}$ define the same canonical foliations $\mathcal{G} = \tilde{\mathcal{G}}$. Thus, we can apply another diffeomorphism, the same diffeomorphism to both systems, which rectifies $g$ and $\tilde{g}$ so that $g = g_2 \partial/\partial y$, $\tilde{g} = \tilde{g}_2 \partial/\partial y$. Now the feedback transformations $u \mapsto (g_2)^{-1}(u - f_2)$ applied to $\Sigma$ and $\tilde{u} \mapsto (\tilde{g}_2)^{-1}(u - \tilde{f}_2)$ applied to $\tilde{\Sigma}$ (not changing the phase portraits) bring the systems into the pre-normal forms

$$\Sigma_{pre}: \qquad \dot{x} = f_1(x, y), \quad \dot{y} = u,$$
$$\tilde{\Sigma}_{pre}: \qquad \dot{x} = \tilde{f}_1(x, y), \quad \dot{y} = u,$$

respectively, whose phase portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ coincide.

We will prove that equality of the portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ implies that the functions

$$F_0(y, w) := f_1(x, y), \quad F_1(y, w) := \tilde{f}_1(x, y),$$

with the identification $x = w$, satisfy the assumptions of Theorem 10.1 in Appendix. The local feedback equivalence of $\Sigma$ and $\tilde{\Sigma}$ will follow from this theorem.

We use notations from Appendix. Denote $E = E(\Sigma_{pre})$, $C = C(\Sigma_{pre})$, and $D = D(\Sigma_{pre})$ and, similarly, $\tilde{E} = \tilde{E}(\tilde{\Sigma}_{pre})$, $\tilde{C} = \tilde{C}(\tilde{\Sigma}_{pre})$, and $\tilde{D} = \tilde{D}(\tilde{D}_{pre})$. For $\Sigma_{pre}$ we have:

$$E = \{(x,y) : f_1(x,y) = 0\}, \qquad C = \{(x,y) : \frac{\partial f_1}{\partial y}(x,y) = 0\}.$$
$$D = \{(x,y) : \frac{\partial f_1}{\partial y}(x,y) = 0, \quad \frac{\partial^2 f_1}{\partial y^2}(x,y) = 0\}$$

and analogously for $\tilde{\Sigma}_{pre}$. Thus $E = \mathcal{Z}(f_1)$ is the set of zeros of $f_1$, $C = \mathcal{C}(f_1)$ is the critical set of $f_1$, and $D = \mathcal{D}(f_1)$ is the discriminant. The conditions (AS1),(AS2) yield the conditions (A1) and (A2) in Appendix, for $F(y,w) = f_1(w,y)$ and $F(y,w) = \tilde{f}_1(w,y)$.

Since the parameterized phase portraits are equal, we have $C = \tilde{C}$ and the assumption (i) in Theorem 10.1 is satisfied. The assumption (ii) is satisfied, too. Namely, we also have $D = \tilde{D}$, $C \setminus D = \tilde{C} \setminus \tilde{D}$ and $f^{crit} = \tilde{f}^{crit}$. Each connected component $C_j$ of $C \setminus D$ is of the form $C_j = \tilde{C}_j = \{y - y_j(x) = 0\}$, where $y_j$ is a smooth function. The equality of critical vector fields means that $f_1(x, y_j(x)) = \tilde{f}_1(x, y_j(x))$ on $C_j$. It follows that the critical values for $f_1$ and $\tilde{f}_1$ coincide on $C \setminus D$. Since $D$ is nowhere dense in $C$, $f_1(x, y(x))$ and $\tilde{f}_1(x, y(x))$ coincide everywhere on $C$. This means that the assumption (ii) in Theorem 10.1 is satisfied.

The assumption (iii) is also satisfied. Let us first assume that $E \cap C$ is nowhere dense in $C$, for $\Sigma$ and $\tilde{\Sigma}$. In this case the set $C_+ \cup C_-$ is dense in $C$. On this set the optimality index $\tau$ is nonzero. The fact that on $C_+ \cup C_-$ we have $\tau = \tilde{\tau}$ means that $sgn\,(f_1 \partial^2 f_1/\partial^2 y) = sgn\,(\tilde{f}_1 \partial^2 \tilde{f}_1/\partial^2 y)$. Since $f^{crit} = \tilde{f}^{crit} \neq 0$, we have that $f_1$ and $\tilde{f}_1$ coincide on the critical curves. Thus $sgn\, \partial^2 f_1/\partial^2 y$ and $sgn\, \partial^2 \tilde{f}_1/\partial^2 y$ are the same on $C \setminus (E \cup D) = \tilde{C} \setminus (\tilde{E} \cup \tilde{D})$. This means that $s_1 = s_2$ on $C_+ \cup C_- = C \setminus (D \cup E)$. By continuity and nowhere density of $E \cap C$ in $C$ we get $s_1 = s_2$ on $C_+ \cup C_- = C \setminus D = \mathcal{C} \setminus \mathcal{D}$. Thus (iii) is satisfied.

Assume now that $E \cap C$ has a nonempty interior in $C$. Consider a point $z \in E \cap C \setminus D$. We claim that the drift direction $\mathcal{DD}$ determines $sgn\, \partial^2 f_1/\partial^2 y$ at such point. Namely, since $z \in E \cap C$ and $z \notin D$, we have $f_1(z) = 0$ and $(L_g(c))(z) \neq 0$. Thus the transversal drift direction is the same on both sides of the curve $C$, when we traverse it along a trajectory of $g$ passing through $z$. If this drift direction points in the direction of growing $x$ then $\partial^2 f_1/\partial^2 y(z)$ is positive and $s_1(z) = 1$. If the drift direction on both sides of $C$ points in the direction of decreasing $x$, we have $s_1(z) = -1$. The same happens for the second system. We have established that $s_1 = s_2$ at all points in $C \setminus D = \mathcal{C} \setminus \mathcal{D}$. Thus (iii) is satisfied in this case, too.

Now we can apply Theorem 10.1. We can find a local diffeomorphism $(\tilde{x}, \tilde{y}) = (x, \psi(x,y))$ such that $f_1(x,y) = \tilde{f}_1(x, \psi(x,y))$. This diffeomorphism, completed with a suitable feedback, transforms $\Sigma$ into $\tilde{\Sigma}$ and shows their local feedback equivalence.

The proof of the case of orbital equivalence is the same except for the beginning: we rescale the system $\Sigma$ by $dt/d\tau = h(z)$, where $L_g h = 0$, and then the parameterized phase portraits of the rescaled system and of $\tilde{\Sigma}$ are equivalent by a diffeomorphism so we can apply the already proved part of the theorem. ■

Note that the drift direction $\mathcal{DD}$ is used in the above proof only in the case when $E \cap C$ has a nonempty interior in $C$. This proves the following

COROLLARY 6.1 *Theorem 6.1 holds with the drift direction $\mathcal{DD}$ removed from $\mathcal{PPP}$, if we additionally assume that $E \cap C$ is nowhere dense in $C$.*

Clearly, $E \cap C$ is nowhere dense in $C$ for all systems satisfying (GS1)-(GS2) but actually even for all systems satisfying (GS2) only (except for those equivalent to $\dot{x} = y^2 + a(x)$, $\dot{y} = v$, with $a$ vanishing on a set with nonempty interior).

In the above proof, equality of the equilibrium sets $E$ and $\tilde{E}$ was used only in the case $p \notin C$. The equivalence was established in the cases $p \notin C$ and $\tilde{p} \notin C$ without using the equilibrium set (the set $E \cap C$ was used but it can be recovered as the set of points where $f^{crit}$ vanishes). This implies the following "surprising"

COROLLARY 6.2 *If $p \in C$ and $\tilde{p} \in \tilde{C}$, then Theorem 6.1 remains true after withdrawing the equilibrium set from the parameterized phase portrait.*

The above theorem applies, for instance, to the class of systems satisfying at $p \in X$ the conditions: $e = 0$, $c = 0$, $d \neq 0$, $j(e,c) = 0$ but there exists an integer $k$ such that $(L_V^k e)(p) \neq 0$, where $V = [g,[g,f]]$. It follows that any such system is orbitally feedback equivalent to the form $\dot{x} = y^2 \pm x^k$, $\dot{y} = v$ and, moreover, any two such forms are orbitally equivalent if and only if their parameterized phase portraits $\mathcal{PPP}$ coincide. In particular, the systems $\dot{x} = y^2 + x^2$, $\dot{y} = v$ and $\dot{x} = y^2 + x^4$, $\dot{y} = v$ are not orbitally feedback equivalent because their critical vector fields $x^2 \frac{\partial}{\partial x}$ and $x^4 \frac{\partial}{\partial x}$ on $C = \{y = 0\}$ are not equivalent via a diffeomorphism and smooth time-rescaling. The phase portraits $\mathcal{PP}$ and $\widetilde{\mathcal{PP}}$ of the systems coincide because they define the same drift directions $\mathcal{DD}$. So, indeed, the phase portrait $\mathcal{PP}$ contains less information than the parameterized phase portrait $\mathcal{PPP}$.

EXAMPLE 6.2 *Consider the normal form $(\mathrm{C}\mathcal{G})_{\mathrm{a}}$ on $\mathbb{R}^2$ of the classification Theorem 4.1, given by $\dot{x} = y^3 + xy + a(x)$, $\dot{y} = v$ and assume $a(x) > 0$. We have $C = \{3y^2 + x = 0\}$ and the foliation $\mathcal{G}$, given by $\{x = const\}$, is transversal to $C$ at its every point, except for $(0,0) \in \mathbb{R}^2$, thus $D = \{(0,0)\}$. Since $d = 6y$, the two components of $C$ (transversal to $\mathcal{G}$) are thus $C_+ = \{3y^2 + x = 0,\ y > 0\}$ and $C_- = \{3y^2 + x = 0,\ y < 0\}$. On $C_+$ and $C_-$ we have the critical vector fields given by the same formula:*

$$f_{\pm}^{crit} = \left(y^3 + xy + a(x)\right)\left(\frac{\partial}{\partial x} - \frac{1}{6y}\frac{\partial}{\partial y}\right).$$

*Factoring $\mathbb{R}^2$ through the foliation $\mathcal{G}$, given by $\{x = const\}$, yields a one-dimensional manifold $\hat{X}$ (which can be identified with $\mathbb{R}$ equipped with the coordinate $x$). We denote by $\hat{X}^{crit}$ the subset of $\hat{X}$ consisting of those leaves of $\mathcal{G}$ whose intersection with $C$ is nonempty and transversal. On $\hat{X}^{crit} = \{x < 0\}$ we have two critical vector fields, obtained by plugging $y = \pm(-x/3)^{1/2}$ in $f_{\pm}^{crit}$ and projecting on $x$ (we denote $k = 2 \cdot 3^{-3/2}$):*

$$\hat{f}_{+}^{crit} = \big(a(x) + k(-x)^{3/2}\big)\frac{\partial}{\partial x}, \quad \text{and} \quad \hat{f}_{-}^{crit} = \big(a(x) - k(-x)^{3/2}\big)\frac{\partial}{\partial x}.$$

## 7. Phase portraits and families of systems

Consider a 1-parameter family of systems on $X \subset \mathbb{R}^2$

$$\Sigma^f \quad : \quad \dot{z} = f(z,\epsilon) + g(z,\epsilon)u, \tag{1}$$

where $u \in \mathbb{R}$ and $\epsilon \in I$, an open interval. Here $f(z,\epsilon) = f_\epsilon(z)$ and $g(z,\epsilon) = g_\epsilon(z)$ are families of vector fields on $X$, parameterized by $\epsilon$ and $C^\infty$-smooth with respect to $(z,\epsilon)$.

Consider a $C^\infty$ local invertible transformations $X \times \mathbb{R} \times I \to \tilde{X} \times \mathbb{R} \times \tilde{I}$ of the form

$$\Gamma^f \quad : \quad \begin{array}{rcl} \tilde{z} &=& \phi(z,\epsilon) = \phi_\epsilon(z) \\ \tilde{u} &=& \psi(z,u,\epsilon) = \psi_\epsilon(z,u) \\ \tilde{\epsilon} &=& \eta(\epsilon), \end{array}$$

where $\psi(z,\cdot,\epsilon)$ is affine with respect to $u$, i.e.,

$$u = \psi^{-1}(z,\tilde{u},\epsilon) = \alpha(z,\epsilon) + \beta(z,\epsilon)\tilde{u},$$

with $\alpha(z,\epsilon) = \alpha_\epsilon(z)$ and $\beta(z,\epsilon) = \beta_\epsilon(z)$ smooth with respect to $(z,\epsilon)$ and $\psi^{-1}$ standing for the inverse of $\psi$ with respect to $u$. Invertibility of $\Gamma^f$ at $(z_0,\epsilon_0)$ means that $d\phi_{\epsilon_0}(z_0)$ is of rank 2 and $\beta(z_0,\epsilon_0) \neq 0$, $\eta'(\epsilon_0) \neq 0$ (obviously, this invertibility is global with respect to $u$).

Definition 7.1 *We call two 1-parameter families of systems $\Sigma^f$ and $\widetilde{\Sigma^f}$* locally feedback equivalent *(or, simply,* equivalent*) at $(z_0,\epsilon_0)$ and $(\tilde{z}_0,\tilde{e}_0)$ if there exists a local, invertible at $(z_0,\epsilon_0)$, $C^\infty$-transformation $\Gamma^f = (\phi,\psi,\eta) : X \times \mathbb{R} \times I \to \tilde{X} \times \mathbb{R} \times \tilde{I}$ transforming $\Sigma^f$ into $\widetilde{\Sigma^f}$, that is,*

$$\tilde{f}_{\tilde{\epsilon}} = \phi_{\epsilon *}(f_\epsilon + \alpha_\epsilon g_\epsilon), \qquad \tilde{g}_{\tilde{\epsilon}} = \phi_{\epsilon *}(\beta_\epsilon g_\epsilon),$$

*and such that $(\phi,\eta)(z_0,\epsilon_0) = (\tilde{z}_0,\tilde{\epsilon}_0)$. Similarly, the families $\Sigma^f$ and $\tilde{\Sigma}^f$ are called* locally orbitally feedback equivalent *(or* orbitally equivalent*) at $(z_0,\epsilon_0)$ and $(\tilde{z}_0,\tilde{e}_0)$ if there exists a local, invertible at $(z_0,\epsilon_0)$, $C^\infty$-transformation $\Gamma^f$*

*and a positive valued function $h = h(z,\epsilon) = h_\epsilon(z)$ satisfying $L_{g_\epsilon} h_\epsilon = 0$, which gives*

$$\tilde{f}_{\tilde{\epsilon}} = \phi_{\epsilon *}(h_\epsilon f_\epsilon + \alpha_\epsilon g_\epsilon), \qquad \tilde{g}_{\tilde{\epsilon}} = \phi_{\epsilon *}(\beta_\epsilon g_\epsilon),$$

*where $\tilde{\epsilon} = \eta(\epsilon)$. Above, $\phi_{\epsilon *} = (\phi_\epsilon)_*$ stands, for any $\epsilon \in I$, for the tangent map of $\phi_\epsilon$.*

The parameterized phase portrait $\mathcal{PPP}$ for the family $\Sigma^f$ can be defined as the family of parameterized phase portraits $\mathcal{PPP}_\epsilon$ of the systems

$$\Sigma_\epsilon \ : \quad \dot{z} = f_\epsilon(z) + g_\epsilon(z)u,$$

with $\epsilon \in I$.

We make this definition more precise. Define the functions $e$, $c$, and $d$ for the family $\Sigma^f$ by the same formulas as for a single system $\Sigma$ in Section 3. Now they are functions of $(z,\epsilon) \in X \times I$. With the help of these functions we define $E = \{e = 0\}$, $C = \{c = 0\}$, and $D = \{d = 0\}$. These are subsets of $X \times I$. The corresponding subsets of $X$ given by a fixed value of $\epsilon$ are denoted $E_\epsilon$, $C_\epsilon$, and $D_\epsilon$. These are the corresponding equilibrium, critical and discriminant sets defined by the system $\Sigma_\epsilon$. The optimality index $\tau = \operatorname{sgn}(e\,d)$ is defined on $C$ and $\tau_\epsilon(z) = \tau(z,\epsilon)$ is well defined for $z \in C_\epsilon$.

The canonical foliation $\mathcal{G}_\epsilon$ is defined as the canonical foliation of the system $\Sigma_\epsilon$. The collection of the foliations $\mathcal{G}_\epsilon$, $\epsilon \in I$, is, by definition, the canonical foliation of the family $\Sigma^f$ and is denoted by $\mathcal{G}$. It is well defined away of points $(z,\epsilon)$ such that $g(z,\epsilon) = 0$. Alternatively, we can understand $\mathcal{G}$ as the foliation in $X \times I$, with each leaf contained in a surface $\epsilon = \mathit{const}$. In the same way we define the drift direction $\mathcal{DD}$ for $\Sigma^f$ as the collection of the drift directions $\mathcal{DD}_\epsilon$ of the systems $\Sigma_\epsilon$. It is immediate to observe that Proposition 3.2 holds for families $\Sigma^f$ (with $\tilde{\epsilon} = \eta(\epsilon)$).

Finally, we recall that each system $\Sigma_\epsilon$ has a well defined critical vector field $f_\epsilon^{crit}$ on the curve (the set of curves) $C_\epsilon \setminus D_\epsilon$ and the collection of critical vector fields $f_\epsilon^{crit}$, $\epsilon \in I$, defines a critical vector field $f^{crit}$ on $C \setminus D$.

Now we see that the *parameterized phase portrait* $\mathcal{PPP}$=$(E, C, \mathcal{G}, \tau, \mathcal{DD}, f^{crit})$ of the family $\Sigma^f$ is well defined and consists of analogous objects as in the case of a single system.

Definition 7.2 *We say that two phase portraits $\mathcal{PPP}$ of $\Sigma^f$ and $\widetilde{\mathcal{PPP}}$ of $\widetilde{\Sigma^f}$ are* locally equivalent *(respectively,* locally orbitally equivalent*) at $p_0 = (z_0, \epsilon_0)$ and $\tilde{p}_0 = (\tilde{z}_0, \tilde{\epsilon}_0)$ if there is a local diffeomorphism $(\phi, \eta) : X \times I \to \tilde{X} \times \tilde{I}$ (resp. a diffeomorphism $(\phi,\eta)$ and a time rescaling $h$ on $C \setminus D$), such that $\phi(z_0,\epsilon_0) = \tilde{z}_0$, $\eta(\epsilon_0) = \tilde{\epsilon}_0$, which transforms $\mathcal{PPP}$ into $\widetilde{\mathcal{PPP}}$. Here $\eta$ is assumed to be a function of $\epsilon \in \tilde{I}$, only, and $h$ is positive valued on $C \setminus D$, having smooth, positive extension to a neighborhood of $(z_0, \epsilon_0)$ such that $L_g h = 0$.*

In this definition, the transformation $(\phi, \eta)$, as well as $(\phi, \eta, h)$, transform the elements of $\mathcal{PPP}$ according to the formulas

$$\tilde{E} = \phi(E), \quad \tilde{C} = \phi(C), \quad \tilde{D} = \phi(D), \quad \tilde{\mathcal{G}} = \phi(\mathcal{G}), \quad \tau = \tilde{\tau} \circ (\phi, \eta).$$

The critical vector field is transformed via

$$\phi_{\epsilon *} f_{\epsilon}^{crit} = \tilde{f}_{\tilde{\epsilon}}^{crit} \;\; \text{(resp.} \;\; \phi_{\epsilon *}(h f_{\epsilon}^{crit}) = \tilde{f}_{\tilde{\epsilon}}^{crit}),$$

with $\tilde{\epsilon} = \eta(\epsilon)$.

Note that the feedback transformation $\Gamma^f = (\phi, \alpha, \beta)$ (resp. $\Gamma^f = (\phi, \alpha, \beta, h)$) transforms the elements of $\mathcal{PPP}$ exactly via the above formulas (this is Proposition 3.2 generalized to the case of families), that is, $\mathcal{PPP}$ does not change if we reparameterize the control via $u = \alpha + \beta v$.

Denote $\operatorname{Grad} c = (\frac{\partial c}{\partial z_1}, \frac{\partial c}{\partial z_2}, \frac{\partial c}{\partial \epsilon})$. We impose the following conditions on $\Sigma^f$:

(AF1) $\operatorname{Grad} c(p) \neq 0$ at each point $p = (z, \epsilon) \in C$.

(AF2) $D$ is nowhere dense in $C$.

Notice that the condition (GF2), stated in Section 8, implies (AF1) and (AF2).

THEOREM 7.1 *Suppose $\Sigma^f$ and $\tilde{\Sigma}^f$ fulfil the conditions (AF1), (AF2) at points $p_0 = (z_0, \epsilon_0)$ and $\tilde{p}_0 = (\tilde{z}_0, \tilde{\epsilon}_0)$, respectively, and $g(p_0) \neq 0$, $\tilde{g}(\tilde{p}_0) \neq 0$. Then they are locally feedback equivalent (resp., locally orbitally feedback equivalent) at $p_0$ and $\tilde{p}_0$ if and only if their parameterized phase portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ are locally equivalent (resp., locally orbitally equivalent) at $p_0$ and $\tilde{p}_0$.*

*Proof.* The proof follows the same line as that of Theorem 6.1. As we have mentioned, Proposition 3.2 holds for families, which proves necessity.

In order to prove sufficiency, consider two families $\Sigma^f$ and $\widetilde{\Sigma^f}$ with equivalent phase portraits. Analogously as in the proof of Theorem 6.1, we can bring them to the prenormal forms

$$\Sigma^f_{pre}: \qquad \dot{x} = f_1(x, y, \epsilon), \quad \dot{y} = u,$$
$$\widetilde{\Sigma^f}_{pre}: \qquad \dot{x} = \tilde{f}_1(x, y, \epsilon), \quad \dot{y} = u,$$

respectively, whose phase portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ coincide. The case when $C$ and $\tilde{C}$ are empty is easy since in this case $\partial f_1/\partial y \neq 0$ and the systems can be linearized to the normal form $\dot{x} = y+1$ or $\dot{x} = y$, depending if the equilibrium set $E$ is empty or not. Therefore, we can assume that $(z_0, \epsilon_0) \in C$ and $(\tilde{z}_0, \tilde{\epsilon}_0) \in \tilde{C}$.

We have to prove that equality of the portraits $\mathcal{PPP}$ and $\widetilde{\mathcal{PPP}}$ implies that the functions

$$F_0(y, w) := f_1(x, y, \epsilon), \quad F_1(y, w) := \tilde{f}_1(x, y, \epsilon),$$

with the identification $(x, \epsilon) = w$, satisfy the assumptions of Theorem 10.1, Appendix. Clearly, $C = \mathcal{C}(f_1)$ is the critical set of $f_1$ and $D = \mathcal{D}(f_1)$ is the discriminant. We see that the conditions (AF1), (AF2) imply (A1), (A2), for the functions $F_0 = f_1$ and $F_1 = \tilde{f}_1$.

The remaining part of the proof follows exactly the same line as that of Theorem 6.1. Namely, it can be verified that equality of the parameterized phase portraits of $\Sigma^f_{pre}$ and $\tilde{\Sigma}^f{}_{pre}$ implies that the assumptions (i), (ii) and (iii) of Theorem 10.1 are satisfied. We leave this verification to the reader. Applying Theorem 10.1 we see that there exists a function $\tilde{y} = \psi(x, y, \epsilon)$ such that $f_1(x, y, \epsilon) = \tilde{f}_1(x, \psi(x, y, \epsilon), \epsilon)$. This gives a transformation of $\Sigma^f_{pre}$ into $\widetilde{\Sigma^f}_{pre}$ and establishes local feedback equivalence of $\Sigma^f_{pre}$ and $\widetilde{\Sigma^f}_{pre}$ (after applying a suitable feedback). The case of orbital equivalence is analogous. ■

## 8. Orbital classification of families

In this section we will present a simplified version of a local classification theorem for generic families of systems $\Sigma^f$, obtained by the authors in Jakubczyk and Respondek (2005).

For three functions $h_i = h_i(z_1, z_2, z_3)$, $i = 1, 2, 3$, we denote

$$j(h_1, h_2) = \det \left\{ \frac{\partial h_i}{\partial z_j} \right\}_{i,j=1,2}, \qquad J(h_1, h_2, h_3) = \det \left\{ \frac{\partial h_i}{\partial z_j} \right\}_{i,j=1,2,3}.$$

We will identify $z_3 = \epsilon$. For $h = h(z_1, z_2, \epsilon)$ we denote $\operatorname{grad} h = (\partial h/\partial z_1, \partial h/\partial z_2)$ and $\operatorname{Grad} h = (\partial h/\partial z_1, \partial h/\partial z_2, \partial h/\partial \epsilon)$.

Put $c^0 = e$, $c^1 = c$ and, inductively, $c^{i+1} = L_g c^i$. Define $d_{mod} = d - (c^5/7c^4)c^1$. We will use the following conditions at $p = (z, \epsilon)$:

$$\begin{aligned} &(GF1) \qquad \mathcal{J}(e, c) := (e, c, j(e, c), J(e, c, j(e, c))) \neq (0,0,0,0) \\ &(GF2) \qquad \mathcal{J}(c, d) := (c, d, j(c, d), J(c, d, j(c, d))) \neq (0,0,0,0) \\ &(GF3) \qquad (e, c, d, \operatorname{grad} c) \neq (0,0,0,0) \\ &(GF4) \qquad (c, d, L_g d, \operatorname{grad} d_{mod}) \neq (0,0,0,0). \end{aligned}$$

Note that (GF1) and (GF2) can be viewed as a generalization of (GS1)-(GS2) to the case of families. Below, by $*$ we will denote arbitrary nonzero numbers.

Theorem 8.1 *Consider a family $\Sigma^f$ satisfying (GF1)-(GF4) at $p = (z_0, \epsilon_0)$ such that $g(z_0, \epsilon_0) \neq 0$. Then $\Sigma^f$ is orbitally feedback equivalent to one of the following canonical forms at $0 \in \mathbb{R}^2$ and $\epsilon = 0$. (Below the second equation is always $\dot{y} = v$. In the rightmost column we list the conditions, satisfied at $p$, which characterize the equivalence class.)*

| | | |
|---|---|---|
| $(O)$ | $\dot{x} = y + 1,$ | $(e, c) = (*, *)\,;$ |
| $(E)$ | $\dot{x} = y,$ | $(e, c) = (0, *)\,;$ |
| $(C)$ | $\dot{x} = \tau y^2 + 1,$ | $(e, c, d) = (*, 0, *)\,;$ |
| $(EC)$ | $\dot{x} = y^2 + \gamma x,$ | $(e, c, j(e, c)) = (0, 0, *)\,;$ |
| $(C\mathcal{G})$ | $\dot{x} = \delta y^3 + xy + 1,$ | $(e, c, d, j(c, d)) = (*, 0, 0, *)\,;$ |
| $(E_{bif})$ | $\dot{x} = \sigma_e y^2 + x^2 - \epsilon,$ | $\mathcal{J}(e, c) = (0, 0, 0, *), \quad d = *\,;$ |
| $(C_{bif})$ | $\dot{x} = \sigma_c y^3 + (x^2 - \epsilon) y + 1,$ | $\mathcal{J}(c, d) = (0, 0, 0, *),$ <br> $(e, L_g d) = (*, *)\,;$ |
| $(E\mathcal{G}_{bif})$ | $\dot{x} = y^3 + (x - \epsilon) y + \gamma x,$ | $\mathcal{J}(e, c) = (0, 0, 0, *),$ <br> $(d, j(c, d)) = (0, *)\,;$ |
| $(C\mathcal{G}_{bif})$ | $\dot{x} = y^4 + (\theta x - \epsilon) y^2 + xy + a(x, \epsilon),$ | $\mathcal{J}(c, d) = (0, 0, 0, *),$ <br> $(e, L_g d, L_g^2 d) = (*, 0, *),$ <br> $\operatorname{grad} d_{mod} = *\,.$ |

*Above $a$ is a smooth function of $(x, \epsilon)$ satisfying $a(0,0) \neq 0$ and $\operatorname{sgn} a(0,0) = \kappa$. The integers $\tau$, $\gamma$, $\delta$, $\sigma_e$, $\sigma_c$, $\theta$, and $\kappa$ take values $\pm 1$ and are orbitally feedback invariant.*

Notice that the above Theorem gives a local classification of generic families of systems. Indeed, families satisfying (GF1)-(GF4) at any $p = (z, \epsilon) \in X \times I$ are generic, that is, form a countable intersection of open and dense subsets in the $C^\infty$ Whitney topology of the space of all pairs $(f, g)$ of parameterized vector fields defined on $X \times I$ (see Hirsch, 1976, for properties of the Whitney topology). This is proved in Jakubczyk and Respondek (2005) by showing that (GF1)-(GF4) are equivalent to a set of conditions (G1)-(G6) that are generic.

*Proof.* The above classification theorem can be deduced from Theorem 3.3 in Jakubczyk and Respondek (2005), Section 3.2, by showing that any system satisfying the assumptions of the above theorem fulfils the corresponding conditions of the classification given in Theorem 3.3. We show this below. In particular, we prove that the conditions characterizing each equivalence class are the same in the theorem above and in Theorem 3.3 mentioned above. We will denote the partial derivatives of $h = h(x, y, \epsilon)$ by $h_x$, $h_y$, $h_\epsilon$. Our calculations are done for $\Sigma$ in the prenormal form $\Sigma_{pre}$ and then $c = e_y$ and $d = c_y = e_{yy}$.

Assume that (GF1) holds with $e(p) \neq 0$ at $p$. Then (GF2) satisfied means one of the conditions $c = *$, $(c, d) = (0, *)$, $(c, d, j(c, d)) = (0, 0, *)$, or $(c, d, j(c, d)) = (0, 0, 0)$ and $J(c, d, j(c, d)) = *$. The first three cases lead, respectively, to the conditions characterizing the normal forms $(O)$, $(C)$, and $(C\mathcal{G})$ (in both classification theorems). So we consider the case $c = d = j(c, d) = 0$ at $p$. Note that

$j(c,d)) = c_x d_y - c_y d_x$. Since $c_y(p) = d(p) = 0$, we have $j(c,d)(p) = c_x(p) \cdot d_y(p) = 0$. If $c_x(p=0$, then $c_\epsilon$ is the only possible nonzero partial derivative of $c$ at $p$. We also have $j_y(c,d)(p) = (c_{xy}d_y - c_{yy}d_x)(p) = (c_{xy}c_{yy} - c_{yy}c_{xy})(p) = 0$, which gives

$$J(c,d,j(c,d))(p) = c_\epsilon(p) \cdot d_y(p) \cdot \text{hess}\,(c)(p) \neq 0,$$

where $\text{hess}\,(c) = c_{xx}c_{yy} - c_{xy}^2$ and $\text{hess}\,(c)(p) = j_x(c,d)(p)$. This yields the conditions describing the form $(C_{bif})$. If $c_x(p) \neq 0$, then $c_{yy}(p) = d_y(p) = L_g d(p) = 0$ which, together with the condition $c_y(p) = d(p) = 0$, yield

$$J(c,d,j(c,d))(p) = c_x(p) \cdot d_{yy}(p) \cdot (c_x d_\epsilon - d_x c_\epsilon)(p) \neq 0,$$

which, in particular, implies independence of $\text{Grad}\, c(p)$ and $\text{Grad}\, d(p)$ (denoted in Jakubczyk and Respondek, 2005) by $Dc(p)$ and $Dd(p)$, respectively). Since we are in the case $c = d = L_g d = 0$ at $p$, the assumption (GF4) says that $\text{grad}\, d_{mod}(p) \neq 0$. This forms the set of conditions describing $(C\mathcal{G}_{bif})$ in both classification theorems.

Now we consider (GF1) with $e(p) = 0$. If at $p$, $c = *$ or $(c, j(e,c)) = (0, *)$, then the family is respectively equivalent to $(E)$ or $(EC)$. So we consider the case $e = c = j(e,c) = 0$ at $p$. Since $e_y(p) = c(p) = 0$, we have $j(e,c)(p) = e_x(p) \cdot d(p) = 0$. If $e_x(p) = 0$, then putting $c = e_y$ yields

$$J(e,c,j(e,c))(p) = -e_\epsilon(p) \cdot d(p) \cdot \text{hess}\,(e)(p) \neq 0,$$

which gives the conditions describing the form $(E_{bif})$. If $e_x(p) \neq 0$, then $d(p) = 0$ and hence

$$J(e,c,j(e,c))(p) = e_x(p) \cdot d_y(p) \cdot (\epsilon_x c_\epsilon - c_x e_\epsilon)(p) \neq 0.$$

Since in our case $(e,c,d)(p) = (0,0,0)$, the assumption (GF3) implies that $\text{grad}\, c(p) \neq 0$. By $c_y(p) = d(p) = 0$, we get $c_x(p) \neq 0$. This shows that the family is equivalent to $(E\mathcal{G}_{bif})$. ■

It can be noticed that without the assumption on $d_{mod}$, we have to replace the normal form $(C\mathcal{G}_{bif})$ by $\dot{x} = y^4 + a_2(x,\epsilon)y^2 + xy + a_0(x,\epsilon)$, $\dot{y} = v$, where $(\partial a_2/\partial\epsilon)(0,0) \neq 0$, $a_0(0,0) \neq 0$. If we drop also the assumption (GF1), then we have to replace in all normal forms (except $(O)$) the zero order terms with respect to $y$ by arbitrary functions $a(x,\epsilon)$.

## 9. Bifurcations of generic families

Observe that if a family satisfies the conditions (GS1)-(GS2) (given in Section 4), then by an appropriate orbital feedback it is equivalent to one of the five top normal forms of Theorem 8.1 which do not depend on the parameter $\epsilon$. In the four remaining forms, however, the elements of the phase portrait change qualitatively if the parameter varies: the family bifurcates! In what follows we will formalize this notion.

For a subset $\Omega \subset X \times I$ and a fixed parameter $\epsilon \in I = (a, b)$, we denote $\Omega_\epsilon = \{z \in X : (z, \epsilon) \in \Omega\}$. Assume $0 \in I$. We denote by $\Sigma_\epsilon$ the system obtained from a family $\Sigma^f$ by fixing the value of the parameter at $\epsilon$. We will say that the family $\Sigma^f$ does not bifurcate, locally at $(z_0, \epsilon_0) = (z_0, 0)$, if there exits a neighborhood $\Omega \subset X \times I$ of $(z_0, 0)$ and a family of homeomorphisms $\chi_\epsilon : \Omega_\epsilon \to \Omega_0$, continuous with respect to $(z, \epsilon)$, such that for $\Sigma_\epsilon$ restricted to $\Omega_\epsilon$ we have

$$\chi_\epsilon(E_\epsilon) = E_0, \quad \chi_\epsilon(C_\epsilon) = C_0, \quad \text{and} \quad \chi_\epsilon(\mathcal{G}_\epsilon) = \mathcal{G}_0,$$

for all $\epsilon \in I$ close enough to 0. Otherwise we say that $\Sigma^f$ *bifurcates locally* or has a *local bifurcation* at $(z, \epsilon) = (z_0, 0)$.

Analogous definition applies to bifurcations at arbitrary $(z_0, \epsilon_0)$. Strictly speaking, in our definition we should say that the triple $(E_\epsilon, C_\epsilon, \mathcal{G}_\epsilon)$ bifurcates or that $\Sigma^f$ bifurcates with respect to $(E_\epsilon, C_\epsilon, \mathcal{G}_\epsilon)$. The same definition will be used for any subset of the triplet $(E_\epsilon, C_\epsilon, \mathcal{G}_\epsilon)$. In particular, we define bifurcations of the equilibrium set $E_\epsilon$, of the critical set $C_\epsilon$, and of the pairs $(E_\epsilon, C_\epsilon)$, $(E_\epsilon, \mathcal{G}_\epsilon)$, and $(C_\epsilon, \mathcal{G}_\epsilon)$. We say that two (local) bifurcations of $\Sigma_\epsilon$ and $\tilde{\Sigma}_\epsilon$ are locally equivalent if there is a local, smooth, invertible transformation $(\phi(z, \epsilon), \eta(\epsilon))$ which transforms the triple $(E_\epsilon, C_\epsilon, \mathcal{G}_\epsilon)$ into the triple $(\tilde{E}_{\tilde{\epsilon}}, \tilde{C}_{\tilde{\epsilon}}, \tilde{\mathcal{G}}_{\tilde{\epsilon}})$.

An analysis of the classification Theorem 8.1 leads to the following conclusions. If a family $\Sigma^f$ is equivalent to one of the first five normal forms, which do not depend on $\epsilon$, then clearly it does not bifurcate. If the family is equivalent to the normal form $(E_{bif})$, then it undergoes an $E$-bifurcation which can be of two types: a birth of equilibria or a cross of equilibria (see Figs. 3 and 4 below). If the family is equivalent to the normal form $(C_{bif})$, then it undergoes a $C$-bifurcation which can be of two types: a birth or a cross of the critical curve (see Figs. 5 and 6 below). If the family is equivalent to the normal form $(C\mathcal{G}_{bif})$, then it undergoes a $C\mathcal{G}$-bifurcation (see Fig. 7 below). If the family is equivalent to the normal form $(E\mathcal{G}_{bif})$, then it undergoes a $E\mathcal{G}$-bifurcation which is also a $EC$-bifurcation-bifurcation (see Fig. 7 below). The above list exhausts all bifurcations of generic families of systems (which are defined as families satisfying the conditions (GF1)-(GF4)) at control-regular points. This result is proved in Jakubczyk and Respondek (2005), where all generic bifurcations are discussed in detail. For a classification of bifurcations at points where $g$ vanishes see Rupniewski (2005).

Notice that in our definition of bifurcations we require that a family of homeomorphism $\chi_\epsilon$ conjugates the triple $(E_\epsilon, C_\epsilon, \mathcal{G}_\epsilon)$ of fundamental equivariants of the system $\Sigma_\epsilon$ to that of the nominal system $\Sigma_0$. Another possibility would be to consider the whole phase portrait $\mathcal{PP}$ and to require (in order that a family does not bifurcate) that $\chi_\epsilon$ transforms the phase portrait $\mathcal{PP}_\epsilon$ of $\Sigma_\epsilon$ into the phase portrait $\mathcal{PP}_0$ of $\Sigma_0$. In general, such a definition would lead to more non equivalent bifurcations distinguished by discrete equivariants of the phase portrait: the optimality index $\tau$ and the drift direction $\mathcal{DD}$.

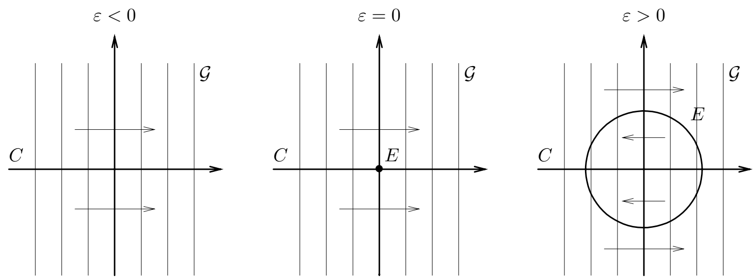


Figure 3. $E$-bifurcation (birth of equilibria)

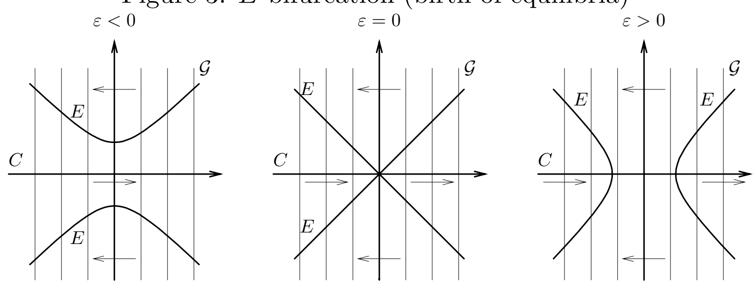


Figure 4. $E$-bifurcation (cross of equilibria)

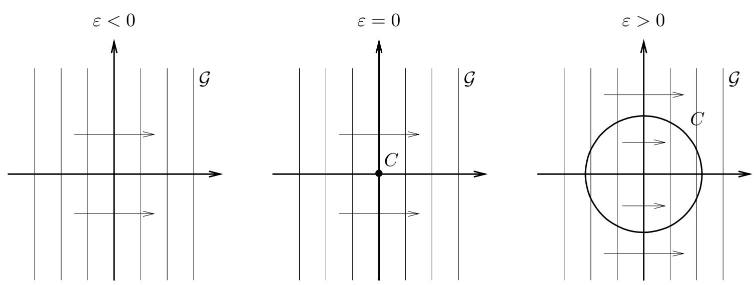


Figure 5. $C$-bifurcation (birth of critical curve)

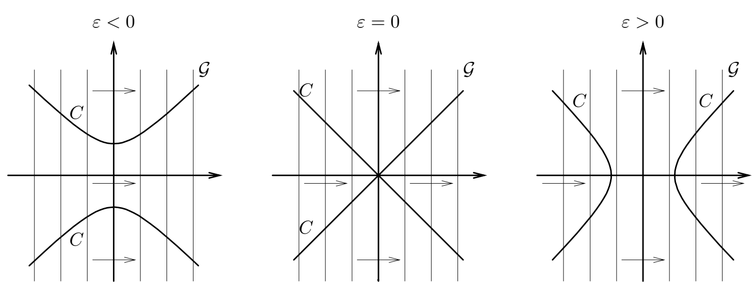


Figure 6. $C$-bifurcation (cross of critical curves)

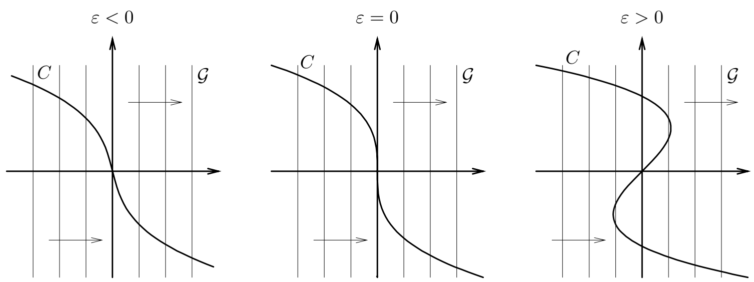


Figure 7. $C\mathcal{G}$-bifurcation, $a(0,0) > 0$

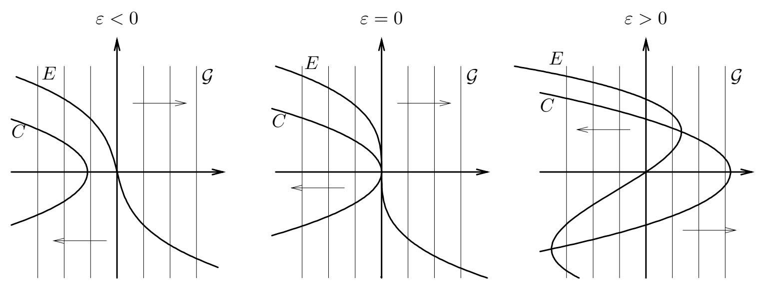


Figure 8. $E\mathcal{G}$-bifurcation, $\gamma = 1$

If we consider the bifurcations of the whole phase portrait $\mathcal{PP}$ (and not only of the triple of fundamental equivariants), then the $E$- and $C$-bifurcations remain the same. The $C\mathcal{G}$-bifurcation splits, however, into four nonequivalent ones corresponding to various signs of $\theta$ and $\kappa$ which give different drift directions and different optimality properties of the critical curves.

Also the $E\mathcal{G}$-bifurcation splits into two non equivalent ones if we consider bifurcations of the whole phase portrait. It is the richest bifurcation so we will give some comments on it. The pair $(E, \mathcal{G})$ bifurcates because the number of points of the intersection of $E$ with the leaves of $\mathcal{G}$ changes when the parameter $\epsilon$ varies. Also the pair $(E, C)$ bifurcates for an analogous reason. Now if we consider the bifurcations of the whole phase portrait $\mathcal{PP}$ (and not only of the triple of fundamental equivariants), then this bifurcation splits into two nonequivalent ones corresponding to various signs of $\gamma$. To see this, observe that the transformation $y \mapsto -y$ conjugates the triples $(E_\epsilon, C_\epsilon, \mathcal{G}_\epsilon)$ of the fundamental invariants of the form $(E\mathcal{G}_{bif})$ for $\gamma = 1$ and $\gamma = -1$. Moreover, this transformation conjugates the optimality indices $\tau$ of both normal forms (proving that they have the same optimality properties) but it fails to conjugate their drift directions $\mathcal{DD}$. This illustrate the role of the latter in distinguishing nonequivalent systems and nonequivalent bifurcations.

## 10. Appendix: Equivalence of deformations

We will prove a theorem establishing equivalence of smooth functions, with parameters, whose critical points and values coincide (for a more general result, see Jakubczyk, 2005). The result is used in proving Theorems 6.1 and 7.1.

Consider a smooth function $F(y,w)$ of $y \in \mathbb{R}$ and $w \in \mathbb{R}^r$, defined in a neighborhood of $(0,0) \in \mathbb{R} \times \mathbb{R}^r$. The variables $w = (w_1, \ldots, w_r)$ can be treated as parameters and $F$ as a deformation of the function $f(y) = F(y,0)$. We denote $w_0 = y$ and $\tilde{w} = (w_0, w_1, \ldots, w_r)$.

Two such functions (deformations) $F_0$ and $F_1$ will be called *strongly equivalent* if there exists a local diffeomorphism $\chi$, preserving $(0,0)$, of the form

$$(\tilde{y}, \tilde{w}) = \chi(y,w) = (\psi(y,w), w) \qquad (\star)$$

such that

$$F_1(\psi(y,w), w) = F_0(y,w)$$

holds in a neighborhood of $(0,0)$. Denote

$$F' = \partial F/\partial y \quad \text{and} \quad F'' = \partial^2 F/\partial y^2.$$

We define the *critical set* and the *discriminant set* of $F$ as

$$\mathcal{C} = \mathcal{C}(F) = \{\, (y,w) \,:\, F'(y,w) = 0 \,\},$$
$$\mathcal{D} = \mathcal{D}(F) = \{\, (y,w) \,:\, F'(y,w) = F''(y,w) = 0 \,\}.$$

Let $I_F = I(F')$ denote the ideal generated by $F'$, in the ring $C_0^\infty(\mathbb{R}^{r+1}, \mathbb{R})$ of smooth function germs at 0. Then $\mathcal{C} = \{\, F' = 0 \,\}$ and $\mathcal{C}$ is the set of zeros of $I_F$ (more precisely, we take a representant of the set germ $\mathcal{C}$). At the points $(y,w) \in \mathcal{C} \setminus \mathcal{D}$ we define the *signature*

$$s_F(y,w) = sgn\, F''(y,w).$$

We say that an ideal $I = I(f)$ of functions (or function germs) of $\tilde{w}$, generated by $f$, is *structurally smooth* if $(\frac{\partial f}{\partial w_0}, \ldots, \frac{\partial f}{\partial w_r})(\tilde{w}) \neq 0$ for any $\tilde{w}$ such that $f(\tilde{w}) = 0$. This implies that the set of zeros $\{f = 0\}$ is a smooth hypersurface. Below we take $f = F'$ .

We shall assume that:

(A1) $I(F')$ is structurally smooth.

(A2) $\mathcal{D}$ is nowhere dense in $\mathcal{C}$.

Denote $\mathcal{C}_i = \mathcal{C}(F_i)$, $\mathcal{D}_i = \mathcal{D}(F_i)$, and $s_i = s_{F_i}$, for $i = 0, 1$.

Theorem 10.1 *Two local functions $F_0$ and $F_1$ which satisfy (A1) and (A2), and such that $\mathcal{C}_0$ and $\mathcal{C}_1$ contain the point $0 \in \mathbb{R}^{r+1}$ are strongly equivalent if* *the following conditions hold:*

$(i)\quad \mathcal{C}_0 = \mathcal{C}_1 =: \mathcal{C}$,

$(ii)\quad F_0|_{\mathcal{C}} = F_1|_{\mathcal{C}}$.

$(iii)\quad s_0 = s_1 \quad on \quad \mathcal{C} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$.

*Proof.* Consider the family of functions

$$F_t(y,w) = (1-t)F_0(y,w) + tF_1(y,w), \qquad t \in [0,1].$$

Denote $\mathcal{C}_t = \mathcal{C}(F_t)$. We shall prove a stronger result, namely we shall find a family of diffeomorphisms $(y,w) \mapsto (\psi_t(y,w),w)$ satisfying

$$F_t(\psi_t(y,w),w) = F_0(y,w).$$

Differentiating this equality with respect to $t$ and denoting $\chi_t(y,w) = (\psi_t(y,w),w)$ gives

$$\frac{\partial F_t}{\partial t} \circ \chi_t + \left(\frac{\partial F_t}{\partial y}\, X_t\right) \circ \chi_t = 0,$$

where $X_t$ is a local family of functions on $\mathbb{R}^{r+1}$ defined by the equality

$$\frac{\partial \psi_t}{\partial t}(y,w) = X_t(\psi_t(y,w),w).$$

We obtain the *homotopy equation*

$$F_t'\, X_t = -\frac{\partial F_t}{\partial t}. \qquad (HE)$$

Conversely, if we find a smooth family of functions $X_t$ which satisfies the homotopy equation (HE) and such that $X_t|_{\mathcal{C}} = 0$, then the family of diffeomorphisms $\chi_t = (\psi_t, \phi_t)$, $\phi_t = id$, where $\psi_t$ is determined by $X_t$ via the above differential equation and the condition $\chi_0 = id$, establishes equivalence of $F_t$ and $F_0$ around $0 \in \mathbb{R}^{r+1}$. (Condition $X_t|_{\mathcal{C}} = 0$ implies that $X_t(0) = 0$, since $0 \in \mathcal{C}$, and guarantees that $\chi_t(0) = 0$.)

In order to solve (HE), we first show that the assumptions (i) and (iii) imply

$(i)' \quad \mathcal{C}_t = \mathcal{C}_0 =: \mathcal{C}, \qquad t \in [0,1].$

We shall use the obvious property that if two smooth function germs $f, g : (\mathbb{R}^m, 0) \to \mathbb{R}$ have the same zeros and the ideals $I(f)$ and $I(g)$ are structurally smooth then there is a smooth, nonvanishing function germ $h$ such that $f = hg$.

Using this property together with (A1) and (i) we get

$$F_1' = HF_0',$$

for a smooth, nonvanishing function germ $H$. By (A2), arbitrarily close to $(0,0)$ there exist points $(y,w) \in \mathcal{C}_0 = \mathcal{C}$ which do not belong to $\mathcal{D}_0 \cup \mathcal{D}_1$. Differentiating the above equality with respect to $y$ at such points we get $F_1''(y,w) = (HF_0'')(y,w)$. It follows from (iii) that at such points, $F_1''$ and $F_0''$ are of the same sign and thus $H(y,w) > 0$. By continuity we get $H(0,0) \geq 0$ and, since $H$ is nonvanishing, $H(0,0) > 0$. We can write

$$F_t' = (1-t)F_0' + tF_1' = H_t F_0',$$

where $H_t = (1+t(H-1))$. Since $H(0,0) > 0$, we have $H_t(0,0) > 0$ for $t \in [0,1]$. Thus $H_t$ does not vanish near the origin and so $\mathcal{C}(F_t) = \mathcal{C}_0$, which shows (i)'.

The following condition is an immediate consequence of (i)', (ii), and the definition of $F_t$:

$(ii)'$ $\quad F_t|_{\mathcal{C}} = F_0|_{\mathcal{C}}$.

Now we will show that (HE) is solvable. The condition (ii)' implies that $(F_t - F_0)|_{\mathcal{C}} = 0$ and so $\frac{\partial F_t}{\partial t}|_{\mathcal{C}} = 0$, for $t \in [0,1]$. The structural smoothness of the ideal $I(F_0')$ together with the equality $F_t' = H_t F_0'$, with $H_t$ nonvanishing, imply that, for some function germ $G_t$,

$$\frac{\partial F_t}{\partial t} = G_t\, F_t'.$$

Thus it is enough to take $X_t^1 = -G_t$ and the equation (HE) is solved.

It remains to show that $X_t^1|_{\mathcal{C}} = 0$ (then $X_t(0,0) = 0$ and $\chi(0,0) = (0,0)$). Let us differentiate the equation (HE) with respect to $y$. We get

$$F_t''\, X_t + F_t'\, \frac{\partial X_t}{\partial y} = -\frac{\partial^2 F_t}{\partial t \partial y}\,. \tag{$\diamondsuit$}$$

The second term vanishes on $\mathcal{C}$ since $(\partial F_t/\partial y)|_{\mathcal{C}} = F_t'|_C = 0$, by (i)'. Differentiating this equality with respect to $t$ we get $(\partial^2 F_t/\partial y \partial t)|_{\mathcal{C}} = 0$. Thus the right hand side of ($\diamondsuit$) also vanishes on $\mathcal{C}$. Since $F_t'' \neq 0$ on $\mathcal{C} \setminus \mathcal{D}$ and $\mathcal{D}$ is nowhere dense in $\mathcal{C}$, by (A2), we get $X_t|_{\mathcal{C}} = 0$. Theorem 10.1 is proved. ∎

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