Control and Cybernetics

vol. **34** (2005) No. 2

Regularity and stability of optimal controls of nonstationary Navier-Stokes equations[1]

by

Daniel Wachsmuth

Institut für Mathematik
Technische Universität Berlin
Str. des 17. Juni 136,
D-10623 Berlin, Germany

Abstract: The regularity and stability of optimal controls of nonstationary Navier-Stokes equations are investigated. Under suitable assumptions every control satisfying first-order necessary conditions is shown to be a continuous function in both space and time. Moreover, the behaviour of a locally optimal control under certain perturbations of the cost functional and the state equation is investigated. Lipschitz stability is proven provided a second-order sufficient optimality condition holds.

Keywords: optimal control, Navier-Stokes equations, control constraints, Lipschitz stability.

1. Introduction

We are considering optimal control of the nonstationary Navier-Stokes equations. As a model problem we use the minimization of the quadratic objective functional

$$\min J(y,u) = \frac{\alpha_T}{2}\int_\Omega |y(x,T)-y_T(x)|^2 dx + \frac{\alpha_Q}{2}\int_Q |y(x,t)-y_Q(x,t)|^2 dxdt \\ + \frac{\alpha_R}{2}\int_Q |\operatorname{curl} y(x,t)|^2 dxdt + \frac{\gamma}{2}\int_Q |u(x,t)|^2 dxdt \qquad (1)$$

subject to the nonstationary Navier-Stokes equations

$$\begin{aligned} y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u \quad \text{in } Q,\\ \operatorname{div} y &= 0 \quad \text{in } Q, \\ y(0) &= y_0 \quad \text{in } \Omega, \end{aligned} \qquad (2)$$

[1]This work was supported by DFG SFB 557 "Control of complex turbulent shear flows" at TU Berlin.

and the control constraints $u \in U_{ad}$ with control set defined by

$$U_{ad} = \{u \in L^2(Q)^n : \; u_{a,i}(x,t) \leq u_i(x,t) \leq u_{b,i}(x,t) \text{ a.e. on } Q,\, i = 1..2\}.$$

Here, Ω is an open bounded subset of $\mathbb{R}^2$ with C^3-boundary Γ, such that Ω is locally on one side of Γ, and Q is defined by $Q = (0,T) \times \Omega$. Further, functions $y_T \in H \subseteq L^2(\Omega)^2$, $y_Q \in L^2(Q)^2$, and $y_0 \in H \subset L^2(\Omega)^2$ are given. The parameters γ and ν are positive real numbers. The constraints u_a, u_b are required to be in $L^2(Q)^2$ with $u_{a,i}(x,t) \leq u_{b,i}(x,t)$ a.e. on Q, $i = 1, 2$.

In this article, we deal with two questions arising in the optimal control of partial differential equations:

1 How smooth is a locally optimal control?
2 Does a locally optimal control enjoy stability under perturbations of the data?

Both are related in the following sense: if the optimal control enjoys some regularity then under reasonable perturbations this regularity should be preserved.

Actually, regularity results for optimal controls can be derived from the first order necessary optimality system. It introduces some coupling between the control and the adjoint state. The adjoint state itself is solution of a partial differential equation and therefore has some regularity, which is inherited by an optimal control. We want to show that any control satisfying the first-order necessary optimality conditions of problem (1) is a continuous function in both space and time.

Since the 1980s the investigation of stability of optimal controls has attracted much interest. Once a stability result holds true, one easily can prove convergence of numerical methods such as the SQP-method for instance, Dontchev et al. (1995). The first stability results for optimal control of partial differential equations are due to Tröltzsch (1996), where a linear-quadratic control problem is studied. For the treatment of general state equations including nonstationary ones we refer to Goldberg and Tröltzsch (1998), Hinze and Kunisch (2001), Malanowski and Tröltzsch (2000), Tröltzsch (2000) and the references cited therein. For the control of the stationary Navier-Stokes system, we refer to Roubíček and Tröltzsch (2002). The stability of optimal controls of the nonstationary Navier-Stokes equations was presented in the recent research paper of Hintermüller and Hinze (2003).

The control of nonstationary Navier-Stokes flow has been studied very intensively since the pioneering work of Abergel and Temam (1990), see for instance Casas (1993), Fattorini and Sritharan (1994), Gunzburger (1995), Gunzburger and Manservisi (1999, 2000), Hintermüller and Hinze (2003), Hinze (2002), Hinze and Kunisch (2001), Sritharan (1991), Tröltzsch and Wachsmuth (2003). Stability problems were addressed in Hintermüller and Hinze (2003) to prove convergence of the SQP-method.

Since we will show that optimal controls of (1) are continuous, we will give stability results in the associated L^∞-norm. This extends the results obtained in Hintermüller and Hinze (2003), where stability of optimal controls in $L^s(Q)^2$, $s < 7/2$, was achieved. But, it requires a change in the methods, too. Using Hilbert space theory of the nonstationary Navier-Stokes equations one can prove stability of optimal controls in the space $L^s(0,T;L^\infty(\Omega)^2) \cap L^\infty(0,T;L^s(\Omega)^2)$ only for $s < \infty$. We will close the gap to $s = \infty$ by employing the L^p-solution theory due to von Wahl (1980, 1985).

The outline of the article is as follows. In Section 2, we will introduce some notation and state common results concerning solvability of the nonstationary Navier-Stokes system (2). Section 3 contains a brief summary of known facts about optimality conditions. The continuity of optimal controls is proven in Section 4. Finally, Section 5 is devoted to the study of stability of optimal controls.

2. Notations and preliminary results

Here, we will restrict ourselves to the two-dimensional case, $n = 2$. First, we introduce some notations and provide some results that we need later on.

To begin with, we define the solenoidal spaces

$$H_p := \{v \in L^p(\Omega)^2 : \operatorname{div} v = 0\}, \qquad V_p := \{v \in W_0^{1,p}(\Omega)^2 : \operatorname{div} v = 0\}.$$

Here, p denotes an arbitrary exponent $p \geq 2$. These spaces are Banach spaces with their norms denoted by $|\cdot|_p$ respectively $|\cdot|_{1,p}$. For $p = 2$, we get the frequently used solenoidal spaces $H := H_2$ and $V := V_2$, which are Hilbert spaces with scalar products $(\cdot,\cdot)_H$ respectively $(\cdot,\cdot)_V$. The dual of V with respect to the scalar product of H is denoted by V' with the duality pairing $\langle \cdot, \cdot \rangle_{V',V}$.

We shall work in the standard space of abstract functions from $[0,T]$ to a real Banach space X, $L^p(0,T;X)$, endowed with its natural norm,

$$\|y\|_{L^p(X)} := \|y\|_{L^p(0,T;X)} = \left(\int_0^T |y(t)|_X^p \, dt \right)^{1/p} \qquad 1 \leq p < \infty,$$
$$\|y\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{t \in (0,T)} |y(t)|_X .$$

In the sequel, we will identify the spaces $L^p(0,T;L^p(\Omega)^2)$ and $L^p(Q)^2$ for $1 < p < \infty$, and denote their norm by $\|u\|_p := |u|_{L^p(Q)^2}$. The usual $L^2(Q)^2$-scalar product we is denoted by $(\cdot,\cdot)_Q$ to avoid ambiguity.

In all what follows, $\|\cdot\|$ stands for norms of abstract functions, while $|\cdot|$ denotes norms of "stationary" spaces like H and V.

To deal with the time derivative in (2), we introduce the following spaces of functions y whose time derivatives y_t exist as abstract functions,

$$W^\alpha(0,T;V) := \{y \in L^2(0,T;V) : y_t \in L^\alpha(0,T;V')\}, \quad W(0,T) := W^2(0,T;V),$$

where $1 \leq \alpha < \infty$. Endowed with the norm

$$\|y\|_{W^\alpha} := \|y\|_{W^\alpha(0,T;V)} = \|y\|_{L^2(V)} + \|y_t\|_{L^\alpha(V')},$$

these spaces are Banach spaces, respectively Hilbert spaces in the case of $W(0,T)$. Every function of $W(0,T)$ is, up to changes on sets of zero measure, equivalent to a function of $C([0,T],H)$, and the imbedding $W(0,T) \hookrightarrow C([0,T],H)$ is continuous, see Adams (1978), Lions and Magenes (1972). Furthermore, we introduce the following space of abstract functions in the L^p-context:

$$W_p^{2,1} := \{y \in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p) : \; y_t \in L^p(0,T;L^p(\Omega)^2)\},$$

which is continuously imbedded in $C([0,T],W_0^{2-2/p,\,p}(\Omega)^2)$, Ladyhzenskaya, Solonnikov and Ural'tseva (1968). Here, $W_0^{2-2/p,\,p}(\Omega)^2$ denotes the space of solenoidal $W^{2-2/p,\,p}$-functions where zero boundary values are prescribed if $p \geq 4/3$. We abbreviate $H^{2,1} = W_2^{2,1}$ for $p = 2$. Note, that in this case we have $W_0^{2-2/2,\,2}(\Omega)^2 = V$. In this article, we will use exponents $p \geq 2$.

We define the trilinear form $b : V \times V \times V \mapsto \mathbb{R}$ by

$$b(u,v,w) = ((u \cdot \nabla)v, w)_2 = \int_\Omega \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i} \; w_j \; dx.$$

To specify the problem setting, we introduce a linear operator $A : L^2(0,T;V) \mapsto L^2(0,T;V')$ by

$$\int_0^T \langle (Ay)(t),\, v(t)\rangle_{V',V} dt := \int_0^T (y(t), v(t))_V \, dt,$$

and a nonlinear operator B by

$$\int_0^T \big\langle \big(B(y)\big)(t),\, v(t)\big\rangle_{V',V} \, dt := \int_0^T b(y(t), y(t), v(t))dt.$$

B is continuous for instance as operator from $W(0,T)$ to $L^2(0,T;V')$. Further properties can be found in Constantin and Foias (1988), Temam (1979), von Wahl (1985). For convenience, we will use in the sequel the notation

$$b_Q(y,v,w) = \int_0^T b(y(t),v(t),w(t))dt.$$

2.1. The state equation

We begin with the notation of weak solutions for the nonstationary Navier-Stokes equations (2) in the Hilbert space setting.

DEFINITION 2.1 (Weak solution) *Let* $f \in L^2(0,T;V')$ *and* $y_0 \in H$ *be given. A function* $y \in L^2(0,T;V)$ *with* $y_t \in L^2(0,T;V')$ *is called a weak solution of* (2) *if*

$$\begin{aligned} y_t + \nu Ay + B(y) &= f, \\ y(0) &= y_0. \end{aligned} \tag{3}$$

Results concerning the solvability of (3) are standard, see Constantin and Foias (1988), and Temam (1979) for proofs and further details.

THEOREM 2.1 (Existence and uniqueness of solutions) *For every* $f \in L^2(0,T;V')$ *and* $y_0 \in H$*, the equation* (3) *has a unique solution* $y \in W(0,T)$*. Moreover, the mapping* $(y_0, u) \mapsto y$ *is locally Lipschitz continuous from* $H \times L^2(0,T;V')$ *to* $W(0,T)$*.*

For more regular data, one expects more regular solutions. The next theorem states some well-known facts, see for instance Temam (1979) for the details and Temam (1995) for more regularity results.

THEOREM 2.2 (Regularity) *For the higher regularity of the weak solutions of* (3) *the following holds.*

(i) Let $y_0 \in V$ *and* $f \in L^2(Q)^2$ *be given. Then the weak solution of* (3) *satisfies*

$$\begin{aligned} y &\in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V), \\ y_t &\in L^2(0,T;H). \end{aligned}$$

(ii) Let additionally, $y_0 \in H^2(\Omega)^2 \cap V$ *and* $f_t \in L^2(0,T;V')$ *and* $f(0) \in L^2(\Omega)^2$ *be given. Then the weak solution* y *of* (3) *satisfies*

$$y_t \in L^2(0,T;V) \cap L^\infty(0,T;H).$$

(iii) If moreover $f \in L^\infty(0,T;L^2(\Omega)^2)$ *then*

$$y \in L^\infty(0,T;H^2(\Omega)^2).$$

The solution mapping $(f, y_0) \mapsto (y, y_t)$ *is locally Lipschitz continuous between the mentioned spaces.*

For the proofs of the three statements we refer to Temam (1979), Theorems III.3.10, 3.5, 3.6.

REMARK 2.1 (Linearized state equation) *We consider the linearized equation*

$$\begin{aligned} y_t + \nu Ay + B'(\bar{y})y &= f, \\ y(0) &= y_0, \end{aligned} \tag{4}$$

for a given state $\bar{y}$*, which is usually the solution of the nonlinear system* (3)*. Following the lines of Temam, one can proof existence and uniqueness of a weak* *solution $y \in W(0,T)$. Regularity results similar to (i)–(iii) hold, provided the state $\bar{y}$ has the same regularity as one wants to get for the solution of the linearized equation, see also Hinze and Kunisch (2001).*

Notice that result *(ii)* implies in particular $y \in C(\bar{Q})^2$. But the prerequisites are quite restrictive with respect to f. We need that its time-derivative have some regularity. Also, any other result in the Hilbert theory which leads to continuous states of class $C(\bar{Q})^2$ needs regularity of space or time derivatives of the right hand side, see Constantin and Foias (1988), Sohr (2001), Temam (1979), von Wahl (1985). In the context of optimal control this is quite problematic. We will comment on it later on, see remarks at the end of Section 5.3.

Can we gain some 'intermediate' regularity of the solution if the right-hand side is in $L^p(Q)^2$ with $p > 2$? If then the weak solution would satisfy $y \in L^p(0,T;W^{2,p}(\Omega)^2)$ and $y_t \in L^p(0,T;L^p(\Omega)^2)$, we would get immediately $y \in C([0,T];W^{1,p}(\Omega)^2) \hookrightarrow C(\bar{Q})^2$. Actually, such a result is available. At first, we have to specify the notation of a strong solution in the L^p-setting.

Definition 2.2 (Strong solution in L^p) *Let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given. A function $y \in L^p(0,T;V_p)$ with $y_t \in L^p(0,T;L^p(\Omega)^2)$ is called strong solution to the exponent $p > 2$ of (2) if*

$$-\int_0^T (y,\phi')dt + \nu\int_0^T (\nabla y, \nabla\phi)dt + \int_0^T b(y,y,\phi) = \int_0^T (f,\phi)dt + (y_0,\phi(0)) \qquad (5)$$

for all test functions $\phi \in L^q(0,T;V_q)$ with $\phi_t \in L^q(0,T;L^q(\Omega)^2)$ and $\phi(T) = 0$, where q is the dual exponent to p, $1/q + 1/p = 1$.

Here the space $W_0^{2-2/p,\,p}(\Omega)^2$ is the natural trace space. Every abstract function of $L^p(0,T;W^{2,p}(\Omega)^2)$ with time derivative in $L^p(0,T;L^p(\Omega)^2)$ is - after changes on a zero measure set - continuous with values in this space, Ladyhzenskaya, Solonnikov and Ural'tseva (1968). Obviously, every strong L^p-solution is a weak solution. For existence of L^p-solutions we have the following theorem.

Theorem 2.3 (L^p-solutions) *Let $f \in L^p(Q)^2$ and $y_0 \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \geq 2$. Then the weak solution y of (3) in the sense of Definition 2.1 is a strong solution and satisfies*

$$\begin{aligned} y &\in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p), \\ y_t &\in L^p(0,T;L^p(\Omega)^2). \end{aligned}$$

There exists a constant $c > 0$ such that

$$\|y\|_{L^p(W^{2,p})} + \|y_t\|_p + \|y\|_{L^\infty(W^{2-2/p,\,p})} \leq c\left\{|y_0|_{W^{2,p}} + \|f\|_p\right\}.$$

Moreover, the mapping $(f, y_0) \mapsto y$ is locally Lipschitz continuous, hence the strong solution y is unique.

If $p = 2$ this result reduces to Theorem **??**(i). For the non-Hilbert space case with $p > 2$, it is proven in von Wahl (1977, 1980).

3. The optimal control problem

3.1. First order necessary optimality conditions

Now we return to our optimal control problem. We briefly recall the necessary conditions for local optimality. For the proofs and further discussion see Abergel and Temam (1990), Casas (1993), Gunzburger and Manservisi (1999), Hinze (2002), Tröltzsch and Wachsmuth (2003) and the references cited therein.

Definition 3.1 (Locally optimal control) *A control $\bar{u} \in U_{ad}$ is said to be locally optimal in $L^2(Q)^2$, if there exists a constant $\rho > 0$ such that*

$$J(\bar{y}, \bar{u}) \le J(y_\rho, u_\rho)$$

holds for all $u_\rho \in U_{ad}$ with $\|\bar{u} - u_\rho\|_2 \le \rho$. Here, $\bar{y}$ and y_ρ denote the states associated with $\bar{u}$ and u_ρ, respectively.

In the following, we denote by $B'(\bar{y})^*$ the formal adjoint of $B'(\bar{y})$. For $\bar{y} \in W(0,T)$, it is a continuous linear operator from $L^2(0,T;V)$ to $L^{4/3}(0,T;V')$.

Theorem 3.1 (Necessary condition) *Let $\bar{u}$ be a locally optimal control with associated state $\bar{y} = y(\bar{u})$. Then there exists a unique solution $\bar{\lambda} \in W^{4/3}(0,T;V)$ of the adjoint equation*

$$\begin{aligned} -\bar{\lambda}_t + \nu A\bar{\lambda} + B'(\bar{y})^*\bar{\lambda} &= \alpha_Q(\bar{y} - y_Q) + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} \bar{y} \\ \bar{\lambda}(T) &= \alpha_T(\bar{y}(T) - y_T). \end{aligned} \tag{6}$$

Moreover, the variational inequality

$$(\gamma\bar{u} + \bar{\lambda}, u - \bar{u})_{L^2(Q)^2} \ge 0 \quad \forall u \in U_{ad} \tag{7}$$

is satisfied.

Proofs can be found in Gunzburger and Manservisi (1999, 2000), Tröltzsch and Wachsmuth (2003). The regularity of $\bar{\lambda}$ is proven in Hinze and Kunisch (2001).

Here, the operator $\vec{\operatorname{curl}} \operatorname{curl}$ is defined as

$$\begin{aligned} (\vec{\operatorname{curl}} \operatorname{curl} v, w) &= \int_Q \operatorname{curl} v(x,t)\ \operatorname{curl} w(x,t)\, dxdt \\ &= \int_Q \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right)\left(\frac{\partial w_1}{\partial x_2} - \frac{\partial w_2}{\partial x_1}\right) dxdt \end{aligned}$$

for $v, w \in L^2(0,T;V)$. By partial integration, the representation

$$(\vec{\operatorname{curl}} \operatorname{curl} v, w) = \int_Q \left(\frac{\partial^2 v_2}{\partial x_1 x_2} - \frac{\partial^2 v_1}{\partial x_2^2}\right) w_1 + \left(\frac{\partial^2 v_1}{\partial x_2 x_1} - \frac{\partial^2 v_2}{\partial x_1^2}\right) w_2\, dxdt$$

is obtained.

In the sequel we need the notation of the normal cone $N_{U_{ad}}(\bar u)$ of the set of admissible controls given by

$$N_{U_{ad}}(\bar u) = \begin{cases} \left\{z \in L^2(Q)^2 : (z, u - \bar u)_2 \le 0 \ \forall u \in U_{ad}\right\} & \text{if } \bar u \in U_{ad} \\ \emptyset & \text{otherwise.} \end{cases} \tag{8}$$

Then the variational inequality (7) can be written equivalently as the inclusion

$$\nu \bar u + \bar\lambda + N_{U_{ad}}(\bar u) \ni 0. \tag{9}$$

The regularity of the adjoint state depending on the regularity of the data is stated more precisely in the next lemma. It can be proven following the lines of Temam (1979), see also Hinze and Kunisch (2001), Rösch and Wachsmuth (2005). For convenience, we denote by f the right-hand side of (6), and by λ_T the initial value $\alpha_T(\bar y(T) - y_T)$.

Theorem 3.2 (Regularity of the adjoint state)

(i) *Let $\lambda_T \in H$, $f \in L^2(0,T;V')$, and $\bar y \in L^2(0,T;V) \cap L^\infty(0,T;H)$ be given. Then there exists a unique weak solution λ of (6) satisfying*

$$\begin{aligned} \lambda &\in L^2(0,T;V), \\ \lambda_t &\in L^{4/3}(0,T;V'). \end{aligned} \tag{10}$$

(ii) *Let $\lambda_T \in V$, $f \in L^2(Q)^2$, and $\bar y \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V)$ be given. Then the unique weak solution λ of (6) satisfies*

$$\begin{aligned} \lambda &\in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V), \\ \lambda_t &\in L^2(0,T;H). \end{aligned} \tag{11}$$

(iii) *Additionally, let $\lambda_T \in H^2(\Omega)^2 \cap V$, $f \in L^\infty(0,T;L^2(\Omega)^2)$, $f_t \in L^2(0,T;V')$, $\bar y_t \in L^2(0,T;V) \cap L^\infty(0,T;H)$, and $\bar y(0) \in H^2(\Omega)^2 \cap V$ be given. Then the weak solution λ of (6) satisfies*

$$\begin{aligned} \lambda &\in L^\infty(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V), \\ \lambda_t &\in L^\infty(0,T;H) \cap L^2(0,T;V). \end{aligned}$$

The mapping $(f, \lambda_T) \mapsto \lambda$ is continuous in the mentioned spaces.

The existence of L^p-solutions of the adjoint equation is the topic of the next Theorem.

Theorem 3.3 *Let $f \in L^p(Q)^2$ and $\lambda_T \in W_0^{2-2/p,\,p}(\Omega)^2$ be given with $p \ge 2$. If $\bar y \in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p)$, then the weak solution λ of (6) is a strong solution and satisfies*

$$\begin{aligned} \lambda &\in L^p(0,T;W^{2,p}(\Omega)^2 \cap V_p), \\ \lambda_t &\in L^p(0,T;L^p(\Omega)^2). \end{aligned}$$

Moreover, the mapping $(f, \lambda_T) \mapsto \lambda$ is continuous, hence the weak solution λ is unique.

The result in the case $p = 2$ is equivalent to Theorem **??**(ii). Following the lines of von Wahl (1977, 1980) one can prove the claim also for $p > 2$.

Let us introduce the Lagrange function $\mathcal{L} : W(0,T) \times L^2(Q)^2 \times W^{4/3}(0,T) \mapsto \mathbb{R}$ for the optimal control problem as follows:

$$\mathcal{L}(y, u, \lambda) = J(u, y) - \langle y_t,\, \lambda\rangle_{L^2(V'),L^2(V)} - \nu(y, \lambda)_{L^2(V)} - b_Q(y, y, \lambda) + (u, \lambda)_Q.$$

This function is twice Fréchet differentiable with respect to $(y, u) \in W(0,T) \times L^2(Q)^2$, see Tröltzsch and Wachsmuth (2003). The reader can readily verify that the necessary conditions can be expressed equivalently by

$$\mathcal{L}_y(\bar{y}, \bar{u}, \bar{\lambda})\, h = 0 \quad \forall h \in W(0,T) \text{ with } h(0) = 0,$$

and

$$\mathcal{L}_u(\bar{y}, \bar{u}, \bar{\lambda})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad}.$$

Here, $\mathcal{L}_y$, $\mathcal{L}_u$ denote the partial Fréchet derivative of $\mathcal{L}$ with respect to y and u.

In the sequel we denote the pair of state and control (y, u) by v for convenience. The second derivative of the Lagrangian $\mathcal{L}$ at $y \in W(0,T)$ with associated adjoint state λ in the directions $v_1 = (z_1, h_1)$, $v_2 = (z_2, h_2) \in W(0,T) \times L^2(Q)^2$ is given by

$$\mathcal{L}_{vv}(y, u, \lambda)[v_1, v_2] = \mathcal{L}_{yy}(y, u, \lambda)[z_1, z_2] + \mathcal{L}_{uu}(y, u, \lambda)[h_1, h_2] \tag{12}$$

with

$$\begin{aligned}\mathcal{L}_{yy}(y, u, \lambda)[z_1, z_2] = \alpha_T(z_1(T), z_2(T))_H + \alpha_Q(z_1, z_2)_Q + \alpha_R(\operatorname{curl} z_1, \operatorname{curl} z_2)_Q \\ - b_Q(z_1, z_2, \lambda) - b_Q(z_2, z_1, \lambda)\end{aligned}$$

and

$$\mathcal{L}_{uu}(y, u, \lambda)[h_1, h_2] = \gamma(h_1, h_2)_2.$$

It satisfies the estimate

$$|\mathcal{L}_{yy}(y, u, \lambda)[z_1, z_2]| \leq c\left(1 + \|\lambda\|_{L^2(V)}\right) \|z_1\|_{W(0,T)} \|z_2\|_{W(0,T)} \tag{13}$$

for all $z_1, z_2 \in W(0,T)$.

3.2. Second-order sufficient optimality conditions

Let $\bar{v} := (\bar{y}, \bar{u})$ be an admissible reference pair satisfying the first-order necessary optimality conditions. We assume further that the reference pair $\bar{v} = (\bar{y}, \bar{u})$

satisfies the following coercivity assumption on $\mathcal{L}''(\bar{v}, \bar{\lambda})$, in the sequel called second-order sufficient condition:

$$\textbf{(SSC)} \begin{cases} \textit{There exists } \delta > 0 \textit{ such that} \\ \qquad \mathcal{L}_{vv}(\bar{v}, \bar{\lambda})[(z,h)]^2 \geq \delta \, \|h\|_2^2 \\ \textit{holds for all pairs } (z,h) \in W(0,T) \times L^2(Q)^2 \textit{ with } z \in W(0,T) \textit{ being} \\ \textit{the weak solution of the linearized equation} \\ \qquad z_t + Az + B'(\bar{y})z = h \\ \qquad\qquad\qquad z(0) = 0. \end{cases}$$

THEOREM 3.4 *Let $\bar{v} = (\bar{y}, \bar{u})$ be admissible for the optimal control problem and suppose that $\bar{v}$ fulfills the first-order necessary optimality condition with associated adjoint state $\bar{\lambda}$. Assume further that (SSC) is satisfied at $\bar{v}$. Then there exist $\alpha > 0$ and $\rho > 0$ such that*

$$J(v) \geq J(\bar{v}) + \alpha \, \|u - \bar{u}\|_2^2$$

holds for all admissible pairs $v = (y, u)$ with $\|u - \bar{u}\|_\infty \leq \rho$.

For the proof we refer to Tröltzsch and Wachsmuth (2003). There, Theorem 3.4 was proven in a slightly weaker form: The space of directions in which $\mathcal{L}_{vv}$ has to be positive definite was shrinked using the concept of strongly active control constraints. Sufficiency was achieved in a L^s-neighborhood of the reference control, whereas the quadratic growth takes place in the L^q-norm with $q \leq 2 \leq s \leq \infty$, $s = q/(2-q)$. The usage of the L^s-norm with $s < \infty$ was motivated as follows: if one utilizes a L^∞-neighborhood of the reference control then jumps of the optimal control have to be known a-priorily. For general objective functionals such jumps can not be excluded. It is one goal of the present article to show that the quadratic functional given by (1) results in continuous optimal controls without jumps.

4. Regularity of extremal controls

In this section, we are going to prove continuity in space and time of extremal controls, i.e. controls satisfying the first-order necessary optimality conditions. The key tool in our analysis is the well-known projection formula

$$u(x,t) = \text{Proj}_{[u_a(x,t), u_b(x,t)]} \left(-\frac{1}{\gamma} \bar{\lambda}(x,t) \right) \qquad \text{a.e. on } Q, \tag{14}$$

which is equivalent to the variational inequality (7).

To begin with, we state the assumptions imposed on the various ingredients of the optimal control problem (1).

$$\textbf{(A1)} \begin{cases} \textit{The bounds } u_a,\, u_b \textit{ are of class } C(\bar{Q})^2. \textit{ Their time derivatives } u_{a,t},\, u_{b,t} \\ \textit{exist as functions in } L^2(Q)^2. \end{cases}$$

$$\textbf{(A2)} \begin{cases} y_0 \in H^2(\Omega)^2 \cap V. \\ \textit{Either } \alpha_T = 0 \textit{ or } y_T \in H^2(\Omega)^2 \cap V. \\ \textit{Either } \alpha_Q = 0 \textit{ or } y_Q \in L^\infty(0,T;L^2(\Omega)^2) \textit{ and } y_{Q,t} \in L^2(0,T;V'). \end{cases}$$

Assuming this allows us to prove

THEOREM 4.1 *Let $u \in U_{ad}$ satisfy the first-order necessary conditions of the optimal control problem (P). Then, u is continuous in $\bar{Q}$, i.e. $u \in C(\bar{Q})^2$.*

Proof. For convenience, we denote the right-hand side of the adjoint equation (6) by f, i.e. $f := \alpha_Q(\bar{y} - y_Q) + \alpha_R \overrightarrow{\text{curl}}\, \text{curl}\, \bar{y}$.

Since $u \in U_{ad}$ it follows that $u \in L^2(Q)^2$. Then, Theorem **??**(i) yields the regularity of the associated state $y \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V)$, $y(T) \in V$. The right-hand side f of the adjoint equation is therefore at least of class $L^2(\Omega)^2$. Additionally, the initial value $\lambda(T)$ is in V. By Theorem **??**(ii) we conclude that $\lambda \in L^2(0,T;H^2(\Omega)^2) \cap L^\infty(0,T;V)$, $\lambda_t \in L^2(0,T;H)$.

The projection formula (14) gives $u \in L^\infty(0,T;L^2(\Omega)^2)$. Using a well-known result of Kinderlehrer and Stampacchia (1980, Thm. II.3.1), we conclude that $u_t \in L^2(Q)^2$. Now, we can apply Theorem **??**(ii) and **??**(iii) to obtain $y \in L^\infty(0,T;H^2(\Omega)^2)$, $y_t \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Then the right-hand side of the adjoint equation satisfies $f \in L^\infty(0,T;L^2(\Omega)^2)$, $f_t \in L^2(0,T;V')$. The initial value $\lambda(T)$ is now of class H^2. Thus, Theorem **??**(iii) implies $\lambda \in L^\infty(0,T;H^2(\Omega)^2)$ and $\lambda_t \in L^2(0,T;V)$.

Finally, we want to prove $\lambda \in C(\bar{Q})^2$. To this end, observe that

$$\lambda \in Y = \left\{ w|\, w \in L^2(0,T;H^2(\Omega)^2),\ \frac{dw}{dt} \in L^2(0,T;V) \right\}.$$

Every function in Y is – up to changes on a zero-measure set – a continuous function with values in $[H^2(\Omega)^2, V]_{1/2}$. And the imbedding of Y in $C([0,T],[H^2(\Omega)^2, V]_{1/2})$ is linear and continuous, Lions and Magenes (1972), Thm 1.3. Here, $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor, see Triebel (2002). The interpolation identity $[H^2(\Omega)^2, V]_{1/2} = H^{3/2}(\Omega)^2$ is proven for instance in Lions and Magenes (1972), Triebel (2002). The space $H^{3/2}(\Omega)$ is continuously imbedded in $C(\bar{\Omega})$, see Adams (1978). Thus, we obtain $\lambda \in C([0,T], C(\bar{\Omega})^2) = C(\bar{Q})^2$.

Now, the projection formula (14) together with the assumptions on the box constraints in (A1) gives $u \in C(\bar{Q})^2$. ∎

As the proof shows, one can even prove H^1-regularity of the extremal controls, if the bounds are smooth enough.

(A3) *The functions u_a, u_b are of class $H^1(Q)^2$.*

Using again, Stampacchia's Theorem, we have the following

COROLLARY 4.1 *Let (A1),(A2), and (A3) be satisfied. Then every extremal solution $\bar{u} \in U_{ad}$ is in $H^1(Q)^2$.*

The projection mapping is only bounded in spaces with differentiation order less than or equal 1. That means, in the L^p-context it is possible to get $\bar{u} \in W^{1,p}(Q)^2$ for $p < \infty$. But, one cannot prove regularity higher than $W^{1,p}$ of optimal controls without further assumptions.

Let us mention that the result $\bar{u} \in C(\bar{Q})^2$ can also be obtained using the L^p-solution theory. For $2 < p < \infty$ let the following prerequisites be fulfilled:

(A1$_p$) *The bounds u_a, u_b are of class $C(\bar{Q})^2$.*

$$
\textbf{(A2}_p\textbf{)} \begin{cases} y_0 \in W_0^{2-2/p,\,p}(\Omega)^2. \\ \textit{Either } \alpha_T = 0 \textit{ or } y_T \in W_0^{2-2/p,\,p}(\Omega)^2. \\ \textit{Either } \alpha_Q = 0 \textit{ or } y_Q \in L^p(Q)^2. \end{cases}
$$

Since the proof is analogous to the previous one, we only state the result.

COROLLARY 4.2 *Let (A1$_p$) and (A2$_p$) be satisfied. Then every extremal solution $\bar{u} \in U_{ad}$ is in $C(\bar{Q})^2$. The associated state $\bar{y}$ and adjoint $\bar{\lambda}$ are at least in $W_p^{2,1}$.*

5. Local stability analysis

Finally, we are dealing with stability of a locally optimal reference triple $(\bar{y}, \bar{u}, \bar{\lambda})$ of the original problem (1). To be more specific, consider the perturbed optimal control problem with perturbation vector $z = (z_y, z_0, z_Q, z_T, z_u)$ in some function space Z

$$
\begin{aligned}
\min J(y,u,z) = \frac{\alpha_T}{2}|y(\cdot,T) - y_T|_2^2 + (z_T, y(T))_\Omega + \frac{\alpha_Q}{2}\|y - y_Q\|_2^2 + (z_Q, y)_Q \\
+ \frac{\alpha_R}{2}\|\operatorname{curl} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 - (z_u, u)_Q \qquad (15)
\end{aligned}
$$

subject to the perturbed Navier-Stokes equations

$$
\begin{aligned}
y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= u + z_y && \text{in } Q, \\
\operatorname{div} y &= 0 && \text{in } Q, \\
y(0) &= y_0 + z_0 && \text{in } \Omega,
\end{aligned} \qquad (16)
$$

and the constraint

$$u \in U_{ad}.$$

Here the natural question arises: How does the optimal triple (y, u, λ) depend on the perturbation z? This question is answered in the rest of this article.

The plan of this section is as follows: At first, we will introduce the concept of generalized equations, where we emphasize an abstract stability result due to Robinson. Secondly, the optimality system is written as a generalized equation in function spaces. Finally, we prove stability of optimal controls provided a second-order sufficient optimality condition holds. Under suitable assumptions, we get even stability of optimal controls with respect to the L^∞-norm.

5.1. Generalized equations

In the sequel, we will apply a result on generalized equations due to Robinson (1980). First, we recall some basic notations. We consider the generalized equation

$$0 \in F(x) + N(x), \tag{17}$$

where F is a C^1-mapping between two Banach spaces X and Z, while $N : X \mapsto 2^Z$ is a set-valued mapping with closed graph.

Let $\bar{x}$ be a solution of (17). The generalized equation is said to be *strongly regular* at the point $\bar{x}$, if there are open balls $B_X(\bar{x}, \rho_x)$ and $B_Z(0, \rho_z)$ such that for all $z \in B_Z(0, \rho_z)$ the linearized and perturbed equation

$$z \in F(\bar{x}) + F'(\bar{x})(x - \bar{x}) + N(x)$$

admits a unique solution $x = x(z)$ in $B_X(\bar{x}, \rho_x)$, and the mapping $z \mapsto x$ is Lipschitz continuous $B_Z(0, \rho_z)$ from to $B_X(\bar{x}, \rho_x)$. The following theorem allows to get from stability results for the perturbed linearized equation to similar results for the perturbed nonlinear problem.

Theorem 5.1 *Let $\bar{x}$ be a solution of* (17) *and assume that* (17) *is strongly regular at $\bar{x}$. Then there exist open balls $B_X(\bar{x}, \rho'_x)$ and $B_Z(0, \rho'_z)$ such that for all $z \in B_Z(0, \rho'_z)$ the perturbed equation*

$$z \in F(x) + N(x)$$

has a unique solution in $x = x(z) \in B_X(\bar{x}, \rho'_x)$, and the solution mapping $z \mapsto x(z)$ is Lipschitz continuous from $B_Z(0, \rho'_z)$ to $B_X(\bar{x}, \rho'_x)$.

5.2. The perturbed optimal control problem

Let $(\bar{y}, \bar{u}, \bar{\lambda})$ satisfy the first-order necessary optimality conditions, see Theorem 3.1, together with the second-order sufficient optimality conditions (SSC). The optimality system consisting of state equation (2), adjoint equation (6) and variational inequality (7), can be written in the condensed form

$$F(\bar{y}, \bar{u}, \bar{\lambda}) + \big(0, 0, 0, 0, N_{U_{ad}}(\bar{u})\big)^T \ni 0, \tag{18}$$

where the function F,

$$F : H^{2,1} \times L^2(Q)^2 \times H^{2,1} \to L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^2(Q)^2 \tag{19}$$

is given by

$$F(y, u, \lambda) = \begin{pmatrix} y_t + \nu A y + B(y) \\ y(0) \\ -\lambda_t + \nu A\lambda + B'(y)^*\lambda \\ \lambda(T) \\ \gamma u + \lambda \end{pmatrix} - \begin{pmatrix} u \\ y_0 \\ \alpha_Q(y - y_Q) + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} y \\ \alpha_T(y(T) - y_T) \\ 0 \end{pmatrix}. \tag{20}$$

We will apply Theorem 5.1 to the generalized equation (18). To do so, we have to show strong regularity of this equation at the reference triple $(\bar{y}, \bar{u}, \bar{\lambda})$. At first, we investigate the mapping F.

COROLLARY 5.1 *The function F defined by* (20) *is continuously differentiable in the setting* (19).

Proof. The components of F are affine linear functions except for F_1, which contains the nonlinear part $B(y)$. We derive for $y, h \in H^{2,1}$, $v \in L^2(Q)^2$

$$\begin{aligned} B(y+h)v - B(y)v &= \int_0^T b(y+h, y+h, v) - b(y, y, v)dt \\ &= \int_0^T b(y, h, v) + b(h, y, v) + b(h, h, v)dt. \end{aligned}$$

This gives immediately the directional derivative of B in direction h as $B'(y)h = \int_0^T b(y, h, v) + b(h, y, v)dt$. We proceed with

$$\begin{aligned} \|B(y+h) - B(y) - B'(y)h\|_2 &= \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_2^{-1} \int_0^T b(h, h, v)dt \\ &\leq \sup_{v \in L^2(Q)^2 \setminus \{0\}} \|v\|_2^{-1} c \, \|h\|_{L^4(W^{1,4})} \|h\|_4 \|v\|_2 \leq c \, \|h\|_{H^{2,1}}^2, \end{aligned}$$

which proves Frechét-differentiability of $B(y)$. To prove continuous differentiability we take $y_1, y_2 \in H^{2,1}$. Then for any direction $h \in H^{2,1}$ and element $v \in L^2(Q)^2$ we obtain

$$\begin{aligned} \left|(B'(y_1)h - B'(y_2)h)v\right| &= \left| \int_0^T b(y_1 - y_2, h, v) + b(h, y_1 - y_2, v)dt \right| \\ &\leq c\|y_1 - y_2\|_{H^{2,1}} \|h\|_{H^{2,1}} \|v\|_2, \end{aligned}$$

which shows that the mapping $y \mapsto B'(y)$ is even Lipschitz continuous from $H^{2,1}$ in the space $\mathcal{L}(H^{2,1}, L^2(Q)^2)$. ■

For convenience, we introduce the space of perturbation vectors Z as

$$Z := L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^2(Q)^2 \tag{21}$$

equipped with the norm $\|z\|_Z = \|z_y\|_2 + |z_0|_V + \|z_Q\|_2 + |z_T|_V + \|z_u\|_2$.

The optimality system of the perturbed problem (15) is equivalent to the generalized equation

$$F(y, u, \lambda) + \big(0, 0, 0, 0, N_{U_{ad}}(u)\big)^T \ni z, \tag{22}$$

where $z = (z_y, z_0, z_Q, z_T, z_u) \in Z$. The components one to four of this inclusion are in fact equations.

The next step in proving strong regularity of (18) is the investigation of the linearized version of the inclusion (22)

$$F(\bar{y},\bar{u},\bar{\lambda}) + F'(\bar{y},\bar{u},\bar{\lambda})(y-\bar{y},u-\bar{u},\lambda-\bar{\lambda}) + \big(0,0,0,0,N_{U_{ad}}(u)\big) \ni z.$$

This generalized equation corresponds to the following system. It consists of the state equations

$$\begin{aligned} y_t + \nu A y + B'(\bar{y})y &= u + B(\bar{y}) + z_y \\ y(0) &= y_0 + z_0, \end{aligned}$$

the adjoint equations

$$\begin{aligned} -\lambda_t + \nu A\lambda + B'(\bar{y})^*\lambda &= -B'(y-\bar{y})^*\bar{\lambda} + \alpha_Q(y-y_Q) + \alpha_R \vec{\operatorname{curl}}\operatorname{curl} y + z_Q \\ \lambda(T) &= \alpha_T(y(T)-y_T) + z_T, \end{aligned}$$

and the variational inequality

$$\gamma u + \lambda + N_{U_{ad}}(u) \ni z_u.$$

This altogether builds up the optimality system of the perturbed linear-quadratic optimization problem given by

$$\begin{aligned} \min J^{(z)}(y,u) = \frac{\alpha_T}{2}|y(T)-y_d|_H^2 + \frac{\alpha_Q}{2}\|y-y_Q\|_2^2 + \frac{\alpha_R}{2}\|\operatorname{curl} y\|_2^2 + \frac{\gamma}{2}\|u\|_2^2 \\ + (z_Q,y)_Q + (z_T,y(T))_\Omega - (z_u,u)_Q - b_Q(y-\bar{y},y-\bar{y},\bar{\lambda}) \qquad (23) \end{aligned}$$

subject to the linearized state equation

$$\begin{aligned} y_t + \nu A y + B'(\bar{y})y &= u + B(\bar{y}) + z_y \\ y(0) &= y_0 + z_0 \end{aligned}$$

and the control constraint

$$u \in U_{ad}.$$

The existence of a unique optimal control of the problem (23) is an easy consequence of the coercivity assumption (SSC). Let us denote the Lagrangian associated to (23) by $\mathcal{L}^{(z)}$. Then it holds for all y,u,λ that

$$\mathcal{L}^{(z)}_{vv}(y,u,\lambda) = \mathcal{L}_{vv}(\bar{y},\bar{u},\bar{\lambda}).$$

Hence, the second-order sufficient condition yields unique solvability of (23) as a linear-quadratic optimization problem with strong convex objective functional. We denote its unique solution of (23) by $u_z = u(z)$ with associated state y_z and adjoint state λ_z. For a more detailed discussion of those aspects we refer to Roubíček and Tröltzsch (2002), where the stability analysis is made for the stationary Navier-Stokes system.

5.3. Stability of optimal controls in $L^2(Q)^2$

Now, we are ready to prove stability of optimal controls in the setting given in the last section. To verify strong regularity we have to prove Lipschitz continuity of the solution mapping $z \mapsto (y_z, u_z, \lambda_z)$ of the perturbed linearized problem (23).

THEOREM 5.2 *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Let additionally y_0, $y_T \in V$, $y_Q \in L^2(Q)^2$ be given. Then the mapping $z \to (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z to $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$.*

Proof. Let $z_1, z_2 \in Z$ be given. Denote the optimal controls of the perturbed problem by $u_i := u_{z_i}$ with associated states y_i and adjoints λ_i, $i = 1, 2$. Denote the differences by $z = z_1 - z_2$, $u := u_1 - u_2$, $y = y_1 - y_2$, and $\lambda = \lambda_1 - \lambda_2$.

Throughout the proof we abbreviate $(\cdot,\cdot) := (\cdot,\cdot)_Q$.

At first, we consider the variational inequality connected with the constraint $u_i \in U_{ad}$,

$$(\gamma u_i + \lambda_i - z_{u,i},\, u - u_i) \geq 0 \qquad \forall u \in U_{ad}.$$

Testing the inequality for u_i, $i = 1, 2$ with u_j, $j = 2 - i$, and adding the resulting two inequalities, we find

$$-(\lambda, u) + (u, z_u) \geq \gamma \|u\|_2^2. \tag{24}$$

Secondly, we consider the state equation. The difference y is the weak solution of

$$\begin{aligned} y_t + \nu A y + B'(\bar{y})y &= u + z_y \\ y(0) &= z_0. \end{aligned} \tag{25}$$

We test this equation by $\lambda = \lambda_1 - \lambda_2$ to obtain

$$(y_t, \lambda) + \nu(y, \lambda)_{L^2(V)} + b_Q(\bar{y}, y, \lambda) + b_Q(y, \bar{y}, \lambda) = (u, \lambda) + (z_y, \lambda). \tag{26}$$

And thirdly, we investigate the adjoint equations. The difference λ of the adjoint states satisfies

$$\begin{aligned} -\lambda_t + \nu A \lambda + B'(\bar{y})^* \lambda &= -B'(y)^* \bar{\lambda} + \alpha_Q y + \alpha_R \vec{\operatorname{curl}} \operatorname{curl} y + z_Q \\ \lambda(T) &= \alpha_T y(T) + z_T. \end{aligned} \tag{27}$$

Testing this equation by $y = y_1 - y_2$ yields

$$\begin{aligned} &- (\lambda_t, y) + \nu(\lambda, y)_{L^2(V)} + b_Q(\bar{y}, y, \lambda) + b_Q(y, \bar{y}, \lambda) = \\ &\qquad - 2b_Q(y, y, \bar{\lambda}) + \alpha_Q \|y\|_2^2 + \alpha_R \|\operatorname{curl} y\|_2^2 + (z_Q, y). \end{aligned} \tag{28}$$

By integration by parts we find

$$-(\lambda_t, y) = (y_t, \lambda) - (\lambda(T), y(T))_H + (\lambda(0), y(0))_H \\ = (y_t, \lambda) - \alpha_T |y(T)|_H^2 - (z_T, y(T))_H + (\lambda(0), z_0)_H. \tag{29}$$

From the combination of (26), (28), and (29), the equation

$$(u, \lambda) + (z_y, \lambda) = \alpha_T \|y(T)|_H^2 + (z_T, y(T))_H - (\lambda(0), z_0)_H \\ - 2b_Q(y, y, \bar\lambda) + \alpha_Q \|y\|_2^2 + \alpha_R \|\operatorname{curl} y\|_2^2 + (z_Q, y) \tag{30}$$

results.

We introduce the auxiliary function $\tilde y$ as the weak solution of (25) with $u = 0$. Now, the coercivity assumption of $\mathcal{L}_{vv}$ comes into play. The pair $(y - \tilde y, u)$ fits in the assumptions of (SSC). With $\mathcal{L}_{vv}$ given by (12), we derive

$$\delta \|u\|_2^2 \le \mathcal{L}_{vv}(\bar v, \bar\lambda)[(y - \tilde y, u)]^2 \\ = \mathcal{L}_{uu}(\bar v, \bar\lambda)[u]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[y]^2 + 2\mathcal{L}_{yy}(\bar v, \bar\lambda)[y, \tilde y]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[\tilde y]^2. \tag{31}$$

The first and second addend we write according to (12) as

$$\mathcal{L}_{uu}(\bar v, \bar\lambda)[u]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[y]^2 = \alpha_T |y(T)|_H^2 + \alpha_Q \|y\|_2^2 + \alpha_R \|\operatorname{curl} y\|_2^2 \\ - 2b_Q(y, y, \bar\lambda) + \gamma \|u\|^2.$$

Using (30) and inequality (24), we proceed with

$$\mathcal{L}_{uu}(\bar v, \bar\lambda)[u]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[y]^2 \\ = (u, \lambda) + (z_y, \lambda) - (z_T, y(T))_H + (\lambda(0), z_0)_H - (z_Q, y) + \gamma \|u\|^2 \\ = (z_u, u) + (z_y, \lambda) - (z_T, y(T))_H + (\lambda(0), z_0)_H - (z_Q, y) \\ \le c \|z\|_Z \{\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}}\} + (z_u, u). \tag{32}$$

Since $\tilde y$ is the weak solution of a linearized equation, we can conclude

$$\|\tilde y\|_{W(0,T)} \le c \|\tilde y\|_{H^{2,1}} \le c \{\|z_y\|_2 + |z_0|_V\} \le c \|z\|_Z.$$

Applying (13), we can estimate the third and fourth addend in (31) by

$$2\mathcal{L}_{yy}(\bar v, \bar\lambda)[y, \tilde y]^2 + \mathcal{L}_{yy}(\bar v, \bar\lambda)[\tilde y]^2 \le c\big\{\big(\|y\|_{L^\infty(H)} + \|y\|_{L^2(V)}\big) \|\tilde y\|_{W(0,T)} \\ + \|\tilde y\|_{W(0,T)}^2\big\} \\ \le c\big\{\|y\|_{W(0,T)} \|z\|_Z + \|z\|_Z^2\big\}. \tag{33}$$

Collecting (31)–(33), we find

$$\delta \|u\|_2^2 \le c \left(\|z\|_Z^2 + \|z\|_Z \{\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}}\}\right) + (z_u, u). \tag{34}$$

By Theorems **??**(i) and Lemma **??**(ii), we estimate the differences of the states and adjoints as weak solutions of (25) and (27)

$$\begin{aligned}\|y\|_{H^{2,1}} &\le c\left\{\|u\|_2 + \|z_y\|_2 + |z_0|_V\right\}\\ \|\lambda\|_{H^{2,1}} &\le c\left\{\|y\|_{L^\infty(H)} + \|y\|_{L^2(H^2)} + \|z_Q\|_2 + |z_T|_V\right\},\end{aligned}$$

which gives immediately

$$\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}} \le c\left\{\|u\|_2 + \|z\|_Z\right\}. \tag{35}$$

Combining (34) and (35) we get

$$\begin{aligned}\delta\|u\|_2^2 &\le c\|z\|_Z\left\{\|y\|_{H^{2,1}} + \|\lambda\|_{H^{2,1}} + \|z\|_Z\right\} + (z_u, u)\\ &\le c\|z\|_Z\left\{\|u\|_2 + \|z\|_Z\right\}\\ &\le c\|z\|_Z^2 + \frac{\delta}{2}\|u\|_2^2,\end{aligned}$$

and the claim is proven. ■

So far, we provided all prerequisites to prove the L^2-stability theorem.

THEOREM 5.3 *Let (SSC) be satisfied for the reference solution* $\bar{v}$ *with adjoint state* $\bar{\lambda}$. *Let in addition* y_0, $y_T \in V$, $y_Q \in L^2(Q)^2$ *be given. Then there exists* $\rho > 0$, *such that for all* $z \in Z$ *with* $\|z\|_Z \le \rho$, *the perturbed optimal control problem* (15) *admits a unique solution* (y_z, u_z, λ_z). *Moreover, the mapping* $z \mapsto (y_z, u_z, \lambda_z)$ *is Lipschitz continuous from* Z *to* $H^{2,1} \times L^2(Q)^2 \times H^{2,1}$.

Proof. Theorem 5.2 yields strong regularity of the equation (18) at the point $(\bar{y}, \bar{u}, \bar{\lambda})$. So we can apply Theorem 5.1 which finishes the proof. ■

If the vector of perturbations z is slightly more regular than stated in (21), say

$$z \in \tilde{Z} = L^2(Q)^2 \times V \times L^2(Q)^2 \times V \times L^s(Q)^2$$

for some $2 < s < \infty$ equipped with norm $\|z\|_{\tilde{Z}} = \|z_y\|_2 + |z_0|_V + \|z_Q\|_2 + |z_T|_V + \|z_u\|_s$, then one can show the following

THEOREM 5.4 *Let (SSC) be satisfied for the reference solution* $\bar{v}$ *with adjoint state* $\bar{\lambda}$. *Let additionally* $y_0, y_T \in V$, $y_Q \in L^2(Q)^2$ *be given. Then there exists* $\rho > 0$, *such that for all* $z \in \tilde{Z}$ *with* $\|z\|_{\tilde{Z}} \le \rho$, *the perturbed optimal control problem* (15) *admits a unique solution* (y_z, u_z, λ_z). *Moreover, the mapping* $z \mapsto (y_z, u_z, \lambda_z)$ *is Lipschitz continuous from* $\tilde{Z}$ *to* $H^{2,1} \times L^s(Q)^2 \times H^{2,1}$.

Proof. The proof is very similar to the proof of Theorem 5.5, see below. It uses the stability result of the previous Theorem 5.3, the projection formula (14), and the imbedding $H^{2,1} \hookrightarrow L^s(Q)^2$ for $s < \infty$. ■

However, this result is maximal in the following sense. Stability of optimal controls in $C(\bar{Q})^2$ cannot be achieved using Hilbert space results, since it is not possible to derive a Lipschitz estimate for the time derivatives of the controls, for which it would be necessary to employ Theorem **??**(ii). To this end consider the following example.

Example 5.1 *Let $\lambda_1, \lambda_2 \in C^1[0,T]$ be given by $\lambda_1(t) = \sin(nt) + 2$ and $\lambda_2(t) = \sin(nt) - 2$. Then we have $\lambda_1(t) - \lambda_2(t) = 4$ and $\frac{d}{dt}(\lambda_1(t) - \lambda_2(t)) = 0$. With $u_a(t) = 0$, $u_b(t) = +\infty$, and $\gamma = 1$ we get*

$$\mathrm{Proj}_{[0,+\infty)}(-\lambda_1(t)) - \mathrm{Proj}_{[0,+\infty)}(-\lambda_2(t)) = 0 - (2 - \sin(nt)) = \sin(nt) - 2,$$

hence,

$$\frac{d}{dt}\Big(\mathrm{Proj}_{[0,+\infty)}(-\lambda_1(t)) - \mathrm{Proj}_{[0,+\infty)}(-\lambda_2(t))\Big) = n\cos(nt) \neq 0,$$

which proves that we cannot show Lipschitz dependence of the time derivatives of the projected adjoint states λ_i.

At this point, we have to use L^p-methods to derive a stability result in the $C(\bar{Q})^2$-norm.

Remark 5.1 *Obviously, these difficulties do not appear for the unconstrained problem $U_{ad} = L^2(Q)^2$, where the variational inequality is equivalent to $u = -\frac{1}{\gamma}\lambda$. Then, any extremal control $\bar{u}$ is as smooth as the associated adjoint $\bar{\lambda}$ and admits almost the same stability properties, i.e. $z \mapsto u_z$ is Lipschitz from Z to $H^{2,1}$.*

5.4. L^∞-stability of optimal controls

Here, we give the stability result of optimal controls in norms adequate to the regularity achieved in Section 4. Again, we are considering the inclusion (18) and the linearized and perturbed problem (23). Now, we regard F to be a function in the setting

$$F : W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1} \to \\ L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^\infty(Q)^2. \quad (36)$$

Again, F is continuously differentiable with respect to the spaces given by (36).

Accordingly, the perturbation vector z has to be in the smaller space of perturbations Z_p,

$$Z_p := L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^p(Q)^2 \times W_0^{2-2/p,\,p}(\Omega)^2 \times L^\infty(Q)^2$$

which we endow with the norm

$$\|z\|_{Z_p} = \|(z_y, z_0, z_Q, z_T, z_u)\|_{Z_p} := \|z_y\|_p + |z_0|_{W^{2-2/p,\,p}} \\ + \|z_Q\|_p + |z_T|_{W^{2-2/p,\,p}} + \|z_u\|_\infty.$$

Finally, we have to modify the definition of the normal cone $N_{U_{ad}}$ in (8). Here, this set has to be a subset of $L^\infty(Q)^2$,

$$\tilde N_{U_{ad}}(\bar u) := \begin{cases} \{z \in L^\infty(Q)^2 : (z, u-\bar u)_2 \le 0 \ \forall u \in U_{ad}\} & \text{if } \bar u \in U_{ad} \\ \emptyset & \text{otherwise.} \end{cases} \tag{37}$$

Observe that $\tilde N_{U_{ad}}(u)$ is a non-empty, closed and convex subset of $L^\infty(Q)^2$.

THEOREM 5.5 *Let (SSC) be satisfied for the reference solution $\bar v = (\bar y, \bar u)$ with adjoint state $\bar\lambda$. Moreover, assume that $y_0, y_T \in W_0^{2-2/p,\,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$. Then the solution mapping $z \to (y_z, u_z, \lambda_z)$ associated to (23) is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.*

Proof. To begin with, notice that the assumptions imply $\bar u \in L^p(Q)^2$. Thus $\bar y$ as well as $\bar\lambda$ are strong solutions of the respective equations, i.e. $\bar y$, $\bar\lambda \in W_p^{2,1}$, see Theorems 2.3 and 3.3.

Let $z_1, z_2 \in Z_p$ be given. Denote the optimal controls of the perturbed problem by $u_i := u_{z_i}$ with associated states y_i and adjoints λ_i, $i = 1, 2$.

At first, Theorem 5.2 yields stability of control, state, and adjoint in $L^2(Q)^2 \times H^{2,1} \times H^{2,1}$,

$$\|u_1 - u_2\|_2 + \|y_1 - y_2\|_{H^{2,1}} + \|\lambda_1 - \lambda_2\|_{H^{2,1}} \le c\,\|z_1 - z_2\|_{Z_2}.$$

By imbedding arguments, we have

$$\|\lambda_1 - \lambda_2\|_p \le c\|\lambda_1 - \lambda_2\|_{L^\infty(V)} \le c\|z_1 - z_2\|_{Z_2}.$$

The projection formula (14) yields,

$$\|u_1 - u_2\|_p \le c\,\{\|\lambda_1 - \lambda_2\|_p + \|z_{u,1} - z_{u,2}\|_p\} \le c\,\|z_1 - z_2\|_{Z_p}.$$

By Theorem 2.3 the weak solution $y_1 - y_2$ of (25) is also a strong solution and satisfies

$$\begin{aligned} &\|y_1 - y_2\|_{L^p(W^{2,p})} + \|y_1 - y_2\|_{L^\infty(W^{2-2/p,\,p})} + \|y_{1,t} - y_{2,t}\|_p \\ &\le c\,\{|z_{0,1} - z_{0,2}|_{W^{2-2/p,\,p}} + \|z_{y,1} - z_{y,2}\|_p + \|u_1 - u_2\|_p\} \le c\,\|z_1 - z_2\|_{Z_p}. \end{aligned}$$

A similar estimate is valid also for the adjoint states, cf. Theorem 3.3,

$$\begin{aligned} \|\lambda_1 - \lambda_2\|_{L^p(W^{2,p})} &+ \|\lambda_1 - \lambda_2\|_{L^\infty(W^{2-2/p,\,p})} + \|\lambda_{1,t} - \lambda_{2,t}\|_p \\ &\le c\,\left\{\|z_1 - z_2\|_{Z_p} + \|y_1 - y_2\|_{L^p(W^{2,p})} + \|y_1 - y_2\|_{L^\infty(W^{2-2/p,\,p})}\right\} \\ &\le c\,\|z_1 - z_2\|_{Z_p}. \end{aligned}$$

This actually means that the mapping $z \mapsto \lambda$ is Lipschitz form Z_p to $W_p^{2,1}$. The space $W_p^{2,1}$ is continuously imbedded in $L^\infty(Q)^2$. Hence, it follows using the projection formula a last time

$$\|u_1 - u_2\|_\infty \le c\,\left\{\|\lambda_1 - \lambda_2\|_{L^\infty(Q)^2} + \|z_{u,1} - z_{u,2}\|_\infty\right\} \le c\,\|z_1 - z_2\|_{Z_p}.$$

■

Thus, we proved strong regularity of the equation (18) in the stronger setting (36), and Theorem 5.1 is applicable.

THEOREM 5.6 *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Additionally, assume that $y_0, y_T \in W_0^{2-2/p,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$. Then there exist $\rho_z > 0$ and $\rho_u > 0$ such that for all $z \in B_{Z_p}(0, \rho_z)$ the following holds: the perturbed inclusion* (22) *has in $B_{L^\infty}(\bar{u}, \rho_u)$ a unique solution u_z. Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.*

Proof. Theorem 5.5 yields strong regularity of the equation (18) at the point $(\bar{y}, \bar{u}, \bar{\lambda})$. So, we can apply Theorem 5.1. Since the state and adjoint equations are uniquely solvable in general, we need not to restrict y_z and λ_z to neighborhoods of $\bar{y}$ and $\bar{\lambda}$, and the claim follows immediately. ■

Using the estimates of $\mathcal{L}''$, see (13), it is easy to show that u_z fulfills a sufficient second-order optimality condition for small perturbations. Hence, u_z is a locally optimal control of the problem (15).

COROLLARY 5.2 *Let (SSC) be satisfied for the reference solution $\bar{v}$ with adjoint state $\bar{\lambda}$. Additionally, assume that $y_0, y_T \in W_0^{2-2/p,p}(\Omega)^2$, $y_Q \in L^p(Q)^2$ for some p satisfying $2 < p < \infty$, and $u_a, u_b \in L^\infty(Q)^2$.*

Then there exist $\rho_z' > 0$ and $\rho_u' > 0$ such that for all $z \in B_{Z_p}(0, \rho_z')$ the following holds: the perturbed optimal control problem (15) *has in $B_{L^\infty}(\bar{u}, \rho_u')$ a unique solution u_z. Moreover, the mapping $z \mapsto (y_z, u_z, \lambda_z)$ is Lipschitz continuous from Z_p to $W_p^{2,1} \times L^\infty(Q)^2 \times W_p^{2,1}$.*

As already mentioned in Remark 5.1, it is not possible to derive stability results for *bounded* optimal controls in W_p^1-norms, $1 \le p \le \infty$. So the result of Theorem 5.6 cannot be improved in this direction.

Acknowledgement

The author is very grateful to Wolf von Wahl for pointing out the results in von Wahl (1977, 1980).

References

ABERGEL, F. and TEMAM, R. (1990) On some control problems in fluid mechanics. *Theoret. Comput. Fluid Dynam.* **1**, 303–325.

Adams, R.A. (1978) *Sobolev Spaces.* Academic Press, San Diego.

Casas, E. (1993) An optimal control problem governed by the evolution Navier-Stokes equations. In: S. S. Sritharan, ed., *Optimal Control of Viscous Flows*, Frontiers in applied mathematics. SIAM, Philadelphia.

Constantin, P. and Foias, C. (1988) *Navier-Stokes Equations.* The University of Chicago Press, Chicago.

Dontchev, A.L., Hager, W.W., Poore, A.B. and Yang, B. (1995) Optimality, stability, and convergence in optimal control. *Appl. Math. Optim.* **31**, 297–326.

Fattorini, H.O. and Sritharan, S. (1994) Necessary and sufficient conditions for optimal controls in viscous flow problems. *Proc. Royal Soc. of Edinburgh* **124**, 211–251.

Goldberg, H. and Tröltzsch, F. (1998) On a Lagrange-Newton method for a nonlinear parabolic boundary control problem. *Optimization Methods and Software* **8**, 225–247.

Gunzburger, M.D., editor (1995) *Flow Control.* Springer, New York.

Gunzburger, M.D. and Manservisi, S. (1999) The velocity tracking problem for Navier-Stokes flows with bounded distributed controls. *SIAM J. Control Optim.* **37**, 1913–1945.

Gunzburger, M.D. and Manservisi, S. (2000) Analysis and approximation of the velocity tracking problem for Navier-Stokes flows with distributed control. *SIAM J. Numer. Anal.* **37**, 1481–1512.

Hintermüller, M. and Hinze, M. (2003) A SQP-semi-smooth Newton-type algorithm applied to control of the instationary Navier-Stokes system subject to control constraints. Technical Report TR03-11, Department of Computational and Applied Mathematics, Rice University, submitted.

Hinze, M. (2002) *Optimal and Instantaneous Control of the Instationary Navier-Stokes Equations.* Habilitation, TU Berlin.

Hinze, M. and Kunisch, K. (2001) Second-order methods for optimal control of time-dependent fluid flow. *SIAM J. Control Optim.* **40**, 925–946.

Kinderlehrer, D. and Stampacchia, G. (1980) *An Introduction to Variational Inequalities and Their Applications.* Academic Press, New York.

Ladyzhenskaya, O.A., Solonnikov, V.A. and Ural'tseva, N.N. (1968) *Linear and Quasilinear Equations of Parabolic Type.* AMS Publications, Providence.

Lions, J.L. and Magenes, E. (1972) *Non-homogeneous Boundary Value Problems and Applications*, volume I. Springer, Berlin.

Malanowski, K. and Tröltzsch, F. (2000) Lipschitz stability of solutions to parametric optimal control for elliptic equations. *Control and Cybernetics*, **29**, 237–256.

Robinson, S.M. (1980) Strongly regular generalized equations. *Mathematics of Operation Research* **5**, 43–62.

Rösch, A. and Wachsmuth, D. (2005) Imbeddings of abstract functions with application to an optimal control problem. *Journal for Analysis and its*

Applications **24**, 103–116.

Roubíček, T. and Tröltzsch, F. (2002) Lipschitz stability of optimal controls for the steady-state Navier-Stokes equations. *Control and Cybernetics* **32** (3), 683–705.

Sohr, H. (2001) *The Navier-Stokes Equations.* Birkhäuser, Basel.

Sritharan, S. (1991) Dynamic programming of the Navier-Stokes equations. *Systems & Control Letters*, **16**, 299–307.

Temam, R. (1979) *Navier-Stokes Equations.* North Holland, Amsterdam.

Temam, R. (1995) *Navier-Stokes Equations and Nonlinear Functional Analysis.* SIAM, Philadelphia, 2nd edition.

Triebel, H. (2002) Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers. *Revista Matemática Complutense* **15**, 475–524.

Tröltzsch, F. (1996) A stability theorem for linear-quadratic parabolic control problems. In: E. Casas, ed., *Control of Partial Differential Equations and Applications*, *Lecture Notes in Pure and Appl. Math.* **174**, Dekker, New York, 287–296.

Tröltzsch, F. (2000) Lipschitz stability of solutions of linear-quadratic parabolic control problems with respect to perturbations. *Dyn. Contin. Discrete Impulsive Syst.* **7**, 289–306.

Tröltzsch, F. and Wachsmuth, D. (2004) Second-order sufficient optimality conditions for the optimal control of Navier-Stokes equations. *ESAIM: COCV*, to appear.

von Wahl, W. (1977) Instationary Navier-Stokes equations and parabolic systems. *Pacific J. Math.* **72** (2), 557–569.

von Wahl, W. (1980) Regularitätsfragen für die instationären Navier-Stokesschen Gleichungen in höheren Dimensionen. *J. Math. Soc. Japan* **32** (2), 263–283.

von Wahl, W. (1985) *The Equations of Navier-Stokes and Abstract Parabolic Equations.* Vieweg, Braunschweig.