Control and Cybernetics

vol. **34** (2005) No. 2

On global maxima in multiphase queues

by

Saulius Minkevičius[1] and Stasys Steišūnas[2]

[1,2] Institute of Mathematics and Informatics
Akademijos 4, 2600 Vilnius, Lithuania
e-mail: stst@ktl.mii.lt
and
Vilnius Gediminas Technical University
Sauletekio 11, 2040 Vilnius, Lithuania

Abstract: The target of this research in the queueing theory is to prove the law of the iterated logarithm (LIL) under the conditions of heavy traffic in multiphase queueing systems. In this paper, the LIL for global maxima is proved in the phases of a queueing system studied for an important probability characteristic of the system (total waiting time of a customer and waiting time of a customer).

Keywords: queueing systems, multiphase queue, heavy traffic, global maxima, total waiting time of a customer, waiting time of a customer, the law of the iterated logarithm.

1. Introduction - an historical perspective

The paper deals with the LIL on extreme values for the total waiting time of a customer and the waiting time of a customer in the phases of a multiphase queueing system (MQS). The MQS is a queueing system when a customer does not visit the same queueing node twice (see, for example, Karpelevich and Kreinin, 1994). The multiphase queueing system is a special case of the open Jackson network.

Limit theorems (diffusion approximations) and the LIL for the queueing system under the conditions of heavy traffic are closely connected (they belong to the same field of research, i.e. investigations on the theory of queueing systems in heavy traffic). Therefore, first we shall try to trace the development of research on the general theory of a queueing system in heavy traffic.

One of the main trends of research in the queueing theory is related with the asymptotic analysis of explicit formulas or equations that describe the distribution of one or more characteristics of a queueing system. To make an analysis of this kind we certainly assume the existence itself of such explicit formulas or equations, and, in addition, an unrestricted approximation of a queueing system to some limit. It was namely in this field that in 1962 J. Kingman obtained the first results on the behaviour of single-server queueing systems in heavy traffic (see Kingman, 1962a, b). The single-server queue case, where intervals between the arrival time of customers to the system are independent nonnegative identically distributed random variables and there is only one server that works independently of external flow under the conditions of heavy traffic, has been studied in detail in (Kendall, 1961; Prohorov, 1963; etc.). Later on, there appeared many works designated to the various aspects of diffusion approximations of the models of queueing theory (see the survey paper of Whitt, 1974, and books of Borovkov, 1972 and 1980; Karpelevich and Kreinin, 1994). The authors of respective works (Harrison, 1985; Harrison and Nguyen, 1993; Kobyashi, 1974; Reiman, 1984; etc.) laid the basis for investigations on the diffusion approximation of queueing networks.

Due to technical difficulties, intermediate models – multiphase queues – are considered infrequently. Let us consider the works on the diffusion approximation of multiphase queues in greater detail. In Harrison (1978), it has been proved that a stationary distribution of waiting time in a two-phase queueing system is approximated by a limit distribution of a two-dimensional diffusion process with reflection. In Grigelionis and Mikulevičius (1987), it has been proved that the limit processes for the waiting time of a customer in a heavy traffic queueing system can also have discontinuous trojectories. The book of Karpelevich and Kreinin (1994) deals mainly with a multiphase queueing system with identical service in the phases of the system and also states the general theory of diffusion processes with reflection. In Minkevičius (1986, 1997), functional limit theorems in a multiphase queueing system for important probability characteristics (waiting time of a customer and queue length of customers) have been proved.

The works on extreme values of queueing systems in heavy traffic are also sparse. In one of the papers of this kind by Iglehart (1972), limit theorems for extreme values of the queue length of customers in a single-server queue are proved. Kuo-Hwa Chang (1977) proved that the distribution of the maximum queue length in a random time interval for queueing systems in heavy traffic converges to a novel extreme value distribution. The papers on the LIL of queueing systems in heavy traffic are also not so numerous. In Iglehart (1971), the LIL for multiple channel queues is studied. In Minkevičius (1995, 1997), the LIL for the queue length of customers and the virtual waiting time of a customer in multiphase queues are proved. Sakalauskas and Minkevičius (2000) also present the proof of a theorem on the LIL for the virtual waiting time of an open Jackson network in heavy traffic.

Note that various variants on the LIL in different domains of applications can be found in the survey by Bingham (1986).

2. Notations

Let **D** be a space of real-valued right-continuous functions in $[0,1]$ having left limits and endowed with the Skorokhod topology induced by the metric d (under which **D** is complete and separable).

In this paper, theorems on the LIL for extreme values of phases for the total waiting time of a customer and the waiting time of a customer in multiphase queues in heavy traffic are proved. The main tool for the analysis of multiphase queues in heavy traffic is a LIL for partial sums of independent identically distributed random variables (the proof can be found in Strassen, 1964).

We investigate here a k-phase queue (i.e., when a customer is served at the j-th phase of the queue, he goes to the $j+1$-st phase of the queue and after the customer has been served in the k-th phase of the queue, he leaves the queue). Let us denote by t_n the time of arrival of the n-th customer; by $S_n^{(j)}$ - the service time of the n-th customer in the j-th phase; $z_n = t_{n+1} - t_n$; by $\tau_{j,n+j}$ - the departure of the n-th customer from the j-th phase of the queue, $j = 1, 2, \ldots, k$.

Let interarrival times (z_n) at the multiphase queue and service times $(S_n^{(j)})$ at each phase of the queue for $j = 1, 2, \ldots, k$ be mutually independent identically distributed random variables.

Next, denote by $W_n^{(j)}$ the waiting time of the n-th customer at the j-th phase of the queue,$Y_n^{(j)} = \sum_{i=1}^{j} W_n^{(i)}$ stands for the total waiting time of the n-th customer up to the j-th phase of the queue, $j = 1, 2, \ldots, k$.

Suppose that the waiting time of a customer at each phase of the multiphase queue is unbounded, the service principle of customers is "first come, first served". All random variables are defined on the common probability space $(\Omega, F, \mathbf{P})$.

We form such modified MQS in which $W_n^{(j)} = 0,\ j = 1, 2, \ldots, k,\ n < k$. Limit distributions for modified MQS and usual MQS which work in heavy traffic conditions are the same (see, for example, Iglehart, 1973). Later on we can investigate only modified MQS and admit that $n \geq k$.

Denote by **M**X and **D**X the mean and variance of the random variable X,

$$\delta_{j,n} = S_n^{(j)} - z_n, \quad S_{j,n} = \sum_{i=1}^{n-1} \delta_{j,i}, \quad S_{0,n} \equiv 0, \quad \hat{S}_{j,n} = S_{j-1,n} - S_{j,n},$$

$$x_{j,n} = \tau_{j,n} - t_n, \quad x_{0,n} \equiv 0, \quad \hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1}, \quad \hat{x}_{0,n} \equiv 0,$$

$$\alpha_j = \mathbf{M}\delta_{j,1}, \quad \alpha_0 \equiv 0, \quad j = 1, 2, \ldots k,$$

$$\mathbf{D}z_n = \sigma_0^2 > 0, \quad \mathbf{D}S_n^{(1)} = \sigma_1^2 > 0, \quad \mathbf{D}S_n^{(k-1)} = \sigma_{k-1}^2,$$

$$\mathbf{D}S_n^{(k)} = \sigma_k^2, \quad \tilde{\sigma}_k^2 = \sigma_k^2 + \sigma_0^2, \quad \hat{\sigma}_k^2 = \sigma_k^2 + \sigma_{k-1}^2, \quad a(n) = \sqrt{2n \ln \ln n},$$

$[x]$ - is the integer part of the number x. Let $S_{j,0} \equiv 0, j = 1, 2, \ldots, k$.

We assume here that the following conditions are fulfilled:

there exists a constant $\gamma > 0$ such that

$$\sup_{n\geq 1} \mathbf{M}|\delta_{j,n}|^{4+\gamma} < \infty,\ j = 1, 2, \ldots, k \tag{1}$$

and

$$\alpha_k > \alpha_{k-1} > \cdots > \alpha_2 > \alpha_1 > 0. \tag{2}$$

3. The results

One of the main results of the work is a theorem on the LIL for the global maxima of the total waiting time of a customer.

THEOREM 3.1 *If conditions (1) and (2) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{1\leq j\leq k} \max\limits_{0\leq l\leq n} Y_l^{(j)} - \alpha_k \cdot n}{\tilde{\sigma}_k \cdot a(n)} = 1 \right) = 1$$

and

$$\boldsymbol{P}\left(\underline{\lim_{n\to\infty}} \frac{\max\limits_{1\leq j\leq k} \max\limits_{0\leq l\leq n} Y_l^{(j)} - \alpha_k \cdot n}{\tilde{\sigma}_k \cdot a(n)} = -1 \right) = 1.$$

Proof. Denote random functions in $\mathbf{D}$ as follows

$$\tilde{Y}^n(t) = \frac{\max\limits_{1\leq j\leq k} \max\limits_{0\leq l\leq [nt]} Y_l^{(j)} - \alpha_k \cdot [nt]}{\tilde{\sigma}_k \cdot a(n)},$$

$$\tilde{X}^n(t) = \frac{\max\limits_{1\leq j\leq k} \max\limits_{0\leq l\leq [nt]} \hat{x}_{j,l} - \alpha_k \cdot [nt]}{\tilde{\sigma}_k \cdot a(n)},$$

$$\tilde{S}^n(t) = \frac{\max\limits_{1\leq j\leq k} \max\limits_{0\leq l\leq [nt]} S_{j,l} - \alpha_k \cdot [nt]}{\tilde{\sigma}_k \cdot a(n)},$$

$$\tilde{S}_k^n(t) = \frac{\max\limits_{0\leq l\leq [nt]} S_{k,l} - \alpha_k \cdot [nt]}{\tilde{\sigma}_k \cdot a(n)},$$

$$S_k^n(t) = \frac{S_{k,[nt]} - \alpha_k \cdot [nt]}{\tilde{\sigma}_k \cdot a(n)},\quad t \in [0, 1].$$

First we prove that, if conditions (1) are fulfilled, then for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{Y}^n, \tilde{X}^n) > \varepsilon \right) = 0. \tag{3}$$

In Minkevičius (1986), p. 718, it is proved that, for $j = 1, 2, \ldots, k$ and $n \geq k$,

$$\hat{x}_{j,n} = \max_{0 \leq l \leq n} (\hat{x}_{j-1,l} - S_{j,l}) + S_{j,n}, \quad \hat{x}_{0,n} \equiv 0,$$
$$x_{j,n} = \max(x_{j-1,n-1} + \delta_{j,n};\ x_{j,n-1} + \delta_{j,n}),\ x_{0,n} \equiv 0. \tag{4}$$

Note that $W^{(j)}_{n-(j-1)} = \max(\tau_{j,n} - \tau_{j-1,n}; 0)$, $j = 1, 2, \ldots, k$ and $n \geq k$. Therefore, (see (4))

$$\begin{aligned} W^{(j)}_n &= \max(\tau_{j,n+(j-1)} - \tau_{j-1,n+(j-1)};\ 0) \\ &= \max(x_{j,n+(j-1)} - x_{j-1,n+(j-1)};\ 0) \\ &= x_{j,n+j} - x_{j-1,n+(j-1)} - \delta_{j,n+(j-1)} \\ &= \hat{x}_{j,n+(j-1)} + \delta_{j,n+(j-1)} - \hat{x}_{j-1,n-(j-2)} - \delta_{j-1,n+(j-2)} - \delta_{j,n+(j-1)} \\ &= \hat{x}_{j,n+(j-1)} - \hat{x}_{j-1,n-(j-2)} - \delta_{j-1,n+(j-2)} \\ &= \{\hat{x}_{j,n} - \hat{x}_{j-1,n}\} + \{\hat{x}_{j,n+(j-1)} - \hat{x}_{j,n}\} - \{\hat{x}_{j-1,n+(j-2)} - \hat{x}_{j-1,n}\} \\ &\quad - \delta_{j-1,n+(j-2)},\ j = 1, 2, \ldots, k. \end{aligned} \tag{5}$$

Also, denote

$$y_{j,n} = \hat{x}_{j,n} - \hat{x}_{j-1,n}, \quad \hat{\delta}_n = \max_{1 \leq j \leq k} \max_{0 \leq l \leq 2n} |\delta_{j,l}|,$$
$$\gamma_{j,n} = \{\hat{x}_{j,n+(j-1)} - \hat{x}_{j,n}\} - \{\hat{x}_{j-1,n+(j-2)} - \hat{x}_{j-1,n}\} - \delta_{j-1,n+(j-2)},$$
$$\hat{\gamma}_{j,n} = \sum_{i=1}^{j} \gamma_{i,n}, \quad c_{j,n} = \max_{1 \leq l \leq n} |\hat{\gamma}_{j,l}|, \quad j = 1, 2, \ldots, k,$$
$$\hat{c}_n = \max_{1 \leq j \leq k} \max_{0 \leq l \leq n} |\hat{\gamma}_{j,l}|.$$

Note that (4) implies:

$$\hat{x}_{j,n} - \hat{x}_{j-1,n} = \max_{0 \leq l \leq n} (\hat{x}_{j-1,l} - S_{j,l}) - (\hat{x}_{j-1,n} - S_{j,n}) \geq 0,\ j = 1, 2, \ldots, k.$$

So, $y_{j,n} \geq 0$, $j = 1, 2, \ldots, k$.

Then

$$W^{(j)}_n = y_{j,n} + \gamma_{j,n}, \quad y_{j,n} \geq 0,\ j = 1, 2, \ldots, k. \tag{6}$$

By summing up equality (6), we obtain $Y^{(j)}_n = \hat{x}_{j,n} + \hat{\gamma}_{j,n}$, $\hat{x}_{j,n} \geq 0$, $j = 1, 2, \ldots, k$. Therefore,

$$\max_{0 \leq l \leq n} Y^{(j)}_l \leq \max_{0 \leq l \leq n} \hat{x}_{j,l} + \max_{0 \leq l \leq n} |\hat{\gamma}_{j,l}| = \max_{0 \leq l \leq n} \hat{x}_{j,l} + c_{j,n} \leq \max_{0 \leq l \leq n} \hat{x}_{j,l} + \hat{c}_n,$$
$$j = 1, 2, \ldots, k.$$

So

$$\max_{1\le j\le k}\max_{0\le l\le n} Y_n^{(j)} \le \max_{1\le j\le k}\big(\max_{0\le l\le n}\hat{x}_{j,l}+\hat{c}_n\big)=\max_{1\le j\le k}\max_{0\le l\le n}\hat{x}_{j,l}+\hat{c}_n. \tag{7}$$

We also have that for $j=1,2,\dots,k$

$$\begin{aligned}\max_{0\le l\le n} Y_l^{(j)} &\ge \max_{0\le l\le n}\big(\hat{x}_{j,l}+\min_{0\le l\le n}\hat{\gamma}_{j,l}\big)=\max_{0\le l\le n}\hat{x}_{j,l}+\min_{0\le l\le n}\hat{\gamma}_{j,l}\\ &=\max_{0\le l\le n}\hat{x}_{j,l}-\max_{0\le l\le n}(-\hat{\gamma}_{j,l})\ge \max_{0\le l\le n}\hat{x}_{j,l}-c_{j,n}\ge \max_{0\le l\le n}\hat{x}_{j,l}-\hat{c}_n.\end{aligned} \tag{8}$$

So (see (8))

$$\max_{1\le j\le k}\max_{0\le l\le n} Y_l^{(j)} \ge \max_{1\le j\le k}\big(\max_{0\le l\le n}\hat{x}_{j,l}-\hat{c}_n\big)=\max_{1\le j\le k}\max_{0\le l\le n}\hat{x}_{j,l}-\hat{c}_n. \tag{9}$$

(7) and (9) yield

$$\big|\max_{1\le j\le k}\max_{0\le l\le n} Y_l^{(j)}-\max_{1\le j\le k}\max_{0\le l\le n}\hat{x}_{j,l}\big|\le \hat{c}_n. \tag{10}$$

Let us try to evaluate $\hat{c}_n$. Therefore, (see again (4))

$$\begin{aligned}&|\hat{x}_{j,n+1}-\hat{x}_{j,n}|=|x_{j,n+2}-\delta_{j,n+1}-x_{j,n+1}-\delta_{j-1,n}|\\ &\le |x_{j,n+2}-x_{j,n+1}|+|\delta_{j,n+1}|+|\delta_{j-1,n}|\le |x_{j,n+2}-x_{j,n+1}|+2\hat{\delta}_n\\ &\le 3\hat{\delta}_n+|\max(x_{j-1,n}-x_{j-1,n-1}+x_{j-1,n-1}\\ &\qquad -\max_{1\le l\le n-1}(x_{j-1,l}-S_{j,l})-S_{j,l};\ 0)|\\ &\le 3\hat{\delta}_n+|\max(x_{j-1,n}-x_{j-1,n-1}-\delta_{j,n};\ 0)|\\ &\le 3\hat{\delta}_n+|\max(x_{j-1,n}-x_{j-1,n-1};\ 0)|+\hat{\delta}_n\\ &\le 4\hat{\delta}_n+|x_{j-1,n}-x_{j-1,n-1}|\le\dots\le 4j\hat{\delta}_n\le 4k\hat{\delta}_n,\ j=1,2,\dots,k.\end{aligned} \tag{11}$$

Thus, (see (3))

$$\begin{aligned}&|\gamma_{j,n}|\le|\hat{x}_{j,n+(j-1)}-\hat{x}_{j,n}|+|\hat{x}_{j,n+(j-2)}-\hat{x}_{j,n}|+|\delta_{j-1,n+(j-2)}|\\ &\le\sum_{i=1}^{j-1}\{|\hat{x}_{j,n+i}-\hat{x}_{j,n+(i-1)}|\}+\sum_{i=1}^{j-2}\{|\hat{x}_{j-1,n+i}-\hat{x}_{j-1,n+(i-1)}|\}+\hat{\delta}_n\\ &\le (j-1)4k\hat{\delta}_n+(j-2)4k\hat{\delta}_n+\hat{\delta}_n\le 8k^2\hat{\delta}_n,\quad j=1,2,\dots,k.\end{aligned} \tag{12}$$

We obtain from (12) that

$$\begin{aligned}\hat{c}_n&\le\max_{1\le j\le k}\Big(\sum_{i=1}^{j}\max_{0\le l\le n}|\gamma_{i,l}|\Big)\le\max_{1\le j\le k}\Big(\sum_{i=1}^{k}\max_{0\le l\le n}|\gamma_{i,l}|\Big)\\ &\le k\cdot\max_{0\le j\le k}\max_{0\le l\le n}|\gamma_{j,l}|\le 8k^3\max_{0\le l\le n}|\hat{\delta}_l|\le 8k^3\hat{\delta}_n.\end{aligned} \tag{13}$$

From (13) and (10) it follows that

$$|\max_{1\le j\le k}\max_{0\le l\le n} Y_l^{(j)} - \max_{1\le j\le k}\max_{0\le l\le n}\hat{x}_{j,l}| \le c\cdot\hat{\delta}_n, \tag{14}$$

where $c = 8k^3$, $\hat{\delta}_n = \max_{1\le j\le k}\max_{0\le l\le 2n}|\delta_{j,l}|$.

Denote $R = \frac{\varepsilon}{c}$, $\delta = \max_{1\le j\le k}\sup_{n\ge 1} M|\delta_{j,n}|^{4+\gamma} < \infty$. Note that $a(n) \ge \sqrt{n}$, if $n \ge e^e$. Therefore, (see (14)) for each fixed $\varepsilon > 0$,

$$\begin{aligned}
&\mathbf{P}\left(d(\tilde{Y}^n, \tilde{X}^n) > \varepsilon\right)\\
&\le \mathbf{P}\left(\max_{0\le m\le n}|\max_{1\le j\le k}\max_{0\le l\le m} Y_l^{(j)} - \max_{1\le j\le k}\max_{0\le l\le m}\hat{x}_{j,l}| > \varepsilon\cdot a(n)\right)\\
&\le \mathbf{P}\left(\max_{0\le m\le n}\hat{\delta}_m > \varepsilon\cdot a(n)\right) \le \sum_{j=1}^{k}\sum_{l=1}^{2n}\frac{\mathbf{M}|\delta_{j,n}|^{4+\gamma}}{R^{4+\gamma}(\sqrt{n})^{4+\gamma}} \qquad (15)\\
&= \sum_{j=1}^{k}\frac{2n\cdot\mathbf{M}|\delta_{j,n}|^{4+\gamma}}{R^{4+\gamma}(\sqrt{n})^{4+\gamma}} \le \frac{2k}{R^{4+\gamma}}\frac{\max_{1\le j\le k}\mathbf{M}|\delta_{j,n}|^{4+\gamma}}{n^{1+\frac{\gamma}{2}}}\\
&= \frac{2k\cdot c^{4+\gamma}}{\varepsilon^{4+\gamma}}\frac{\delta}{n^{1+\frac{\gamma}{2}}} = \frac{\tilde{c}}{n^{1+\frac{\gamma}{2}}},
\end{aligned}$$

where $\tilde{c} = \tilde{c}(\varepsilon) = \dfrac{2k\cdot(3k)^{12+3\gamma}}{\varepsilon^{4+\gamma}\cdot n^{1+\frac{\gamma}{2}}}\cdot\delta < \infty$.

It follows from (15) that for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(d(\tilde{Y}^n, \tilde{X}^n) > \varepsilon\right) \le \frac{\tilde{c}}{n^{1+\frac{\gamma}{2}}}. \tag{16}$$

From (16) and the Boreli - Cantelli lemma we can derive (3).

Let us prove now that for each fixed $\varepsilon > 0$,

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} d(\tilde{X}^n, \tilde{S}^n) > \varepsilon\right) = 0. \tag{17}$$

Hence, (see again (4))

$$\hat{x}_{j,n} - S_{j,n} = \max_{0\le l\le n}(\hat{x}_{j-1,l} - S_{j,l}) \ge 0, \quad j = 1, 2, \ldots, k, \;\; \hat{x}_{0,n} \equiv 0.$$

So

$$\hat{x}_{j,n} \ge S_{j,n}, \quad j = 1, 2, \ldots, k. \tag{18}$$

Thus, (see (18)),

$$\max_{1\le j\le k}\max_{0\le l\le n}\hat{x}_{j,l} \ge \max_{1\le j\le k}\max_{0\le l\le n} S_{j,l}. \tag{19}$$

Hence, (see again (4)) for $j = 1, 2, \dots, k$,

$$\hat{x}_{j,n} \le \max_{0\le l\le n} (\hat{x}_{j-2,l} - \hat{S}_{j-1,l})) + \max_{0\le l\le n} \hat{S}_{j,l} + S_{j,n}$$

$$\le \hat{x}_{j,n} + \max_{0\le l\le n} \hat{S}_{j,l} - \hat{S}_{j,n} \le \dots \le \sum_{i=1}^{j} \{\max_{0\le l\le n} \hat{S}_{i,l} - \hat{S}_{i,n}\}$$

$$= \sum_{i=1}^{j} \{\max_{0\le l\le n} \hat{S}_{i,l}\} + S_{j,n}. \tag{20}$$

So according to (20), we have

$$\hat{x}_{j,n} \le \sum_{i=1}^{j} \left\{\max_{0\le l\le n} \hat{S}_{i,l}\right\} + S_{j,n}, \quad j = 1, 2, \dots, k. \tag{21}$$

In view of (21), we obtain

$$\max_{0\le l\le n} \hat{x}_{j,l} \le \max_{0\le l\le n} \left(\sum_{i=1}^{j} \{\max_{0\le p\le l} \hat{S}_{i,p}\} + S_{j,l} \right)$$

$$\le \sum_{i=1}^{k} \{\max_{0\le l\le n} \hat{S}_{i,l}\} + \max_{0\le l\le n} S_{j,l} \le \max_{0\le l\le n} S_{j,l} + k \cdot \max_{1\le j\le k} \max_{0\le l\le n} \hat{S}_{j,l},$$

$$j = 1, 2, \dots, k.$$

Consequently,

$$\max_{1\le j\le k} \max_{0\le l\le n} \hat{x}_{j,l} \le \max_{1\le j\le k} \max_{0\le l\le n} S_{j,l} + k \cdot \max_{1\le j\le k} \max_{0\le l\le n} \hat{S}_{j,l}. \tag{22}$$

Denote $c_n = \max_{1\le j\le k} \max_{0\le l\le n} \hat{S}_{j,l}$. We assume that $S_{j,0} \equiv 0$, $j = 1, 2, \dots, k$. So, $c_n \ge 0$.

From (19) and (22) we get

$$| \max_{1\le j\le k} \max_{0\le l\le n} \hat{x}_{j,l} - \max_{1\le j\le k} \max_{0\le l\le n} S_{j,l}| \le k \cdot c_n. \tag{23}$$

Therefore, using (23) we get, for each fixed $\varepsilon > 0$, that

$$\mathbf{P}(d(\tilde{X}^n, \tilde{S}^n) > \varepsilon)$$

$$\le \mathbf{P}\left(\max_{0\le m\le n} | \max_{1\le j\le k} \max_{0\le l\le m} \hat{x}_{j,l} - \max_{1\le j\le k} \max_{0\le l\le m} S_{j,l}| > \varepsilon \cdot a(n) \right)$$

$$\le \mathbf{P}\left(\max_{0\le m\le n} c_m > \frac{\varepsilon}{k} \cdot a(n) \right) \le \mathbf{P}\left(\max_{0\le m\le n} (\max_{1\le j\le k} \max_{0\le l\le m} \hat{S}_{j,l}) > \frac{\varepsilon}{k} \cdot a(n) \right)$$

$$\le \mathbf{P}\left(\max_{1\le j\le k} \max_{0\le l\le n} \hat{S}_{j,l} > \frac{\varepsilon}{k} \cdot a(n) \right) \le \sum_{j=1}^{k} \mathbf{P}\left(\frac{\max_{0\le l\le n} \hat{S}_{j,l}}{a(n)} > \frac{\varepsilon}{k} \right). \tag{24}$$

We shall prove that, if conditions (2) are satisfied, then for each fixed $\varepsilon > 0$

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} \hat{S}_{j,l}}{a(n)} > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \tag{25}$$

Functions $\{S_{j,n}, n \geq 1, j = 1, 2, \ldots, k\}$ have a negative drift (see conditions (2)). Thus, (25) follows from the strong law for $\{S_{j,n}, n \geq 1, j = 1, 2, \ldots, k\}$ (see Iglehart, 1973; p. 583). So, (17) is proved (see (24) and (25)).

Let us prove that, if conditions (2) are satisfied, then for each fixed $\varepsilon > 0$

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} d(\tilde{S}^n, \tilde{S}^n_k) > \varepsilon\right) = 0. \tag{26}$$

But

$$\begin{aligned}
&\mathbf{P}\left(d(\tilde{S}^n, \tilde{S}^n_k) > \varepsilon\right)\\
&\leq \mathbf{P}\left(\max_{0\le m\le n} |\max_{1\le j\le k} \max_{0\le l\le m} S_{j,l} - \max_{0\le l\le m} S_{k,l}| > \varepsilon \cdot a(n)\right)\\
&= \mathbf{P}\left(\max_{0\le m\le n} (\max_{1\le j\le k} \max_{0\le l\le m} S_{j,l} - \max_{0\le l\le m} S_{k,l}) > \varepsilon \cdot a(n)\right)\\
&\leq \mathbf{P}\left(\max_{0\le m\le n} (\max_{0\le l\le m} \max_{1\le j\le k} S_{j,l} - \max_{0\le l\le m} S_{k,l}) > \varepsilon \cdot a(n)\right)\\
&\leq \mathbf{P}\left(\max_{0\le l\le n} (\max_{1\le j\le k-1} (S_{j,l} - S_{k,l})) > \varepsilon \cdot a(n)\right)\\
&= \mathbf{P}\left(\max_{0\le l\le n} (\max_{1\le j\le k-1} (\sum_{i=j}^{k} \hat{S}_{i,l})) > \varepsilon \cdot a(n)\right)\\
&\leq \mathbf{P}\left(\max_{1\le j\le k-1} (\max_{0\le l\le n} (\sum_{i=j}^{k} \hat{S}_{i,l})) > \varepsilon \cdot a(n)\right)\\
&\leq \mathbf{P}\left(\max_{1\le j\le k-1} (\sum_{i=j}^{k} \max_{0\le l\le n} \hat{S}_{i,l}) > \varepsilon \cdot a(n)\right)\\
&\leq \mathbf{P}\left(\sum_{i=1}^{k} \max_{0\le l\le n} \hat{S}_{i,l} > \varepsilon \cdot a(n)\right) \leq \sum_{i=1}^{k} \mathbf{P}\left(\frac{\max\limits_{0\le l\le n} \hat{S}_{i,l}}{a(n)} > \frac{\varepsilon}{k}\right).
\end{aligned} \tag{27}$$

Also, using conditions (2), just like in (25), we obtain for each fixed $\varepsilon > 0$

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{0\le l\le n} \hat{S}_{j,l}}{a(n)} > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \tag{28}$$

So, (26) is proved (see (27) and (28)).

Let us prove now that, if conditions (2) are satisfied, then for each fixed $\varepsilon > 0$

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{S}_k^n, S_k^n) > \varepsilon\right) = 0, \quad j = 1, 2, \ldots, k. \tag{29}$$

Hence, for $j = 1, 2, \ldots, k$

$$\begin{aligned}
&\mathbf{P}\left(d(\tilde{S}_k^n, S_k^n) > \varepsilon\right) \le \mathbf{P}\left(\max_{0\le m\le n} |\max_{0\le l\le m} S_{k,l} - S_{k,m}| > \varepsilon \cdot a(n)\right) \\
&= \mathbf{P}\left(\max_{0\le m\le n} (\max_{0\le l\le m} S_{k,l} - S_{k,m}) > \varepsilon \cdot a(n)\right) \\
&= \mathbf{P}\left(\max_{0\le m\le n} (\max_{0\le l\le m} (-S_{k,m-l})) > \varepsilon \cdot a(n)\right) \\
&= \mathbf{P}\left(\max_{0\le m\le n} \max_{0\le l\le m} (-S_{k,l}) > \varepsilon \cdot a(n)\right) \le \mathbf{P}\left(\max_{0\le l\le n} (-S_{k,l}) > \varepsilon \cdot a(n)\right).
\end{aligned} \tag{30}$$

From conditions (2) we know that $-\alpha_k < 0$. So similarly as in (25) or (28), using (2) and (30), we can prove (29).

Finally, using the triangle inequality, (3), (17), (26) and (29) we obtain for each fixed $\varepsilon > 0$ that

$$\begin{aligned}
&\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{Y}^n, S_k^n) > \varepsilon\right) \\
&\le \mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{Y}^n, \tilde{X}^n) > \varepsilon\right) + \mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{X}^n, \tilde{S}^n) > \varepsilon\right) \\
&+\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{S}^n, \tilde{S}_k^n) > \varepsilon\right) + \mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{S}_k^n, S_k^n) > \varepsilon\right) = 0.
\end{aligned} \tag{31}$$

So, (see (31))

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\tilde{Y}^n, S_k^n) > \varepsilon\right) = 0, \tag{32}$$

if conditions (1) and (2) are fulfilled.

Using the Strassen LIL for partial sums of independent identically distributed random variables, we obtain that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, S_k^n(t) = 1)\right) = 1 \text{ and } \mathbf{P}\left(\underline{\lim_{n\to\infty}}\, S_k^n(t) = -1)\right) = 1.$$

Applying this and (32), we get the proof of Theorem 3.1.

The proof of Theorem 3.1 is complete. ■

The next theorem, establishing the LIL for the global maxima of the waiting time of a customer, is proved similarly as Theorem 3.1.

Theorem 3.2 *If conditions (1) and (2) are fulfilled, then*

$$\boldsymbol{P}\left(\overline{\lim_{n\to\infty}} \frac{\max\limits_{1\le j\le k}\max\limits_{0\le l\le n} W_l^{(j)} - (\alpha_k - \alpha_{k-1})\cdot n}{\hat{\sigma}_k \cdot a(n)} = 1\right) = 1 \quad \textit{and}$$

$$\boldsymbol{P}\left(\underline{\lim_{n\to\infty}} \frac{\max\limits_{1\le j\le k}\max\limits_{0\le l\le n} W_l^{(j)} - (\alpha_k - \alpha_{k-1})\cdot n}{\hat{\sigma}_k \cdot a(n)} = -1\right) = 1.$$

Proof. Denote random functions in **D** as follows

$$\hat{W}^n(t) = \frac{\max\limits_{1\le j\le k}\max\limits_{0\le l\le [nt]} W_l^{(j)} - (\alpha_k - \alpha_{k-1})\cdot [nt]}{\hat{\sigma}_k \cdot a(n)},$$

$$\hat{X}^n(t) = \frac{\max\limits_{1\le j\le k}\max\limits_{0\le l\le [nt]} (\hat{x}_{j,l} - \hat{x}_{j-1,l}) - (\alpha_k - \alpha_{k-1})\cdot [nt]}{\hat{\sigma}_k \cdot a(n)},$$

$$\hat{S}^n(t) = \frac{\max\limits_{1\le j\le k}\max\limits_{0\le l\le [nt]} (-\hat{S}_{j,l}) - (\alpha_k - \alpha_{k-1})\cdot [nt]}{\hat{\sigma}_k \cdot a(n)},$$

$$\hat{S}_k^n(t) = \frac{\max\limits_{0\le l\le [nt]} (-\hat{S}_{k,l}) - (\alpha_k - \alpha_{k-1})\cdot [nt]}{\hat{\sigma}_k \cdot a(n)},$$

$$\tilde{S}_k(t) = \frac{(-\hat{S}_{k,[nt]}) - (\alpha_k - \alpha_{k-1})\cdot [nt]}{\hat{\sigma}_k \cdot a(n)}, \quad t \in [0,1].$$

First we prove that, if conditions (2) are satisfied, then for each fixed $\varepsilon > 0$

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\hat{W}^n, \hat{X}^n) > \varepsilon\right) = 0. \tag{33}$$

Similarly as in (14), we can prove that

$$\left|\max_{1\le j\le k}\max_{0\le l\le n} W_l^{(j)} - \max_{1\le j\le k}\max_{0\le l\le n} (\hat{x}_{j,l} - \hat{x}_{j-1,l})\right| \le c\cdot \hat{\delta}_n, \tag{34}$$

where $c = 8k^3$, $\hat{\delta}_n = \max\limits_{1\le j\le k}\max\limits_{0\le l\le 2n} |\delta_{j,l}|$. The further proof of the theorem is just the same as in (14)-(16). Consequently, (33) is proved.

Now we prove that, if conditions (2) are fulfilled, then for each fixed $\varepsilon > 0$

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\hat{X}^n, \hat{S}^n) > \varepsilon\right) = 0. \tag{35}$$

Thus, (see (18), (21))

$$\hat{x}_{j,n} - \hat{x}_{j-1,n} \le \sum_{i=1}^{n}\left\{\max_{0\le l\le n} \hat{S}_{i,l}\right\} - \hat{S}_{j,n} \le k\cdot c_n - \hat{S}_{j,n},\ j = 1,2,\ldots,k.$$

Hence it follows that

$$\max_{1\le j\le k}\max_{0\le l\le n}(\hat{x}_{j,l}-\hat{x}_{j-1,l})\le\max_{1\le j\le k}\max_{0\le l\le n}(k\cdot c_n-\hat{S}_{j,l})$$
$$=\max_{1\le j\le k}\max_{0\le l\le n}(-\hat{S}_{j,l})+k\cdot c_n. \quad (36)$$

Analogously (see again (18), (21)) for $j=1,2,\dots,k$

$$\hat{x}_{j,n}-\hat{x}_{j-1,n}\ge-\sum_{i=1}^{j-1}\left\{\max_{0\le l\le n}\hat{S}_{i,l}\right\}+S_{j,n}-S_{j-1,n}\ge-k\cdot c_n-\hat{S}_{j,n}. \quad (37)$$

Therefore, (see (37))

$$\max_{1\le j\le k}\max_{0\le l\le n}(\hat{x}_{j,l}-\hat{x}_{j-1,l})\ge\max_{1\le j\le k}\max_{0\le l\le n}(-k\cdot c_n-\hat{S}_{j,l})$$
$$=-k\cdot c_n+\max_{1\le j\le k}\max_{0\le l\le n}(-\hat{S}_{j,l}). \quad (38)$$

(36) and (38) imply that

$$|\max_{1\le j\le k}\max_{0\le l\le n}(\hat{x}_{j,l}-\hat{x}_{j-1,l})-\max_{1\le j\le k}\max_{0\le l\le n}(-\hat{S}_{j,l})|\le k\cdot c_n. \quad (39)$$

Then (see (24) and (39)), for each fixed $\varepsilon>0$

$$\mathbf{P}\left(d(\hat{X}^n,\hat{S}^n)>\varepsilon\right)\le\mathbf{P}\left(\max_{0\le m\le n}|c_m|>\frac{\varepsilon\cdot a(n)}{k}\right)$$
$$\le\mathbf{P}\left(\max_{1\le j\le k}\max_{0\le m\le n}\max_{0\le l\le m}\hat{S}_{j,l}>\frac{\varepsilon\cdot a(n)}{k}\right) \quad (40)$$
$$\le\mathbf{P}\left(\max_{1\le j\le k}\max_{0\le l\le n}\hat{S}_{j,l}>\frac{\varepsilon\cdot a(n)}{k}\right)\le\sum_{j=1}^{k}\mathbf{P}\left(\frac{\max_{0\le l\le n}\hat{S}_{j,l}}{a(n)}>\frac{\varepsilon}{k}\right).$$

Note that $\alpha_{j-1}-\alpha_j<0$, $j=1,2,\dots,k$ (see (2)). Making use of this and analogously as in (25) we obtain, for each fixed $\varepsilon>0$, that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\frac{\max_{0\le l\le n}\hat{S}_{j,l}}{a(n)}>\varepsilon\right)=0,\quad j=1,2,\dots,k. \quad (41)$$

So, (see (3) and (41))

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\hat{X}^n,\hat{S}^n)>\varepsilon\right)=0. \quad (42)$$

The proof that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\hat{S}^n,\hat{S}^n_k)>\varepsilon\right)=0\ \text{ and }\ \mathbf{P}\left(\overline{\lim_{n\to\infty}}\,d(\hat{S}^n_k,\tilde{S}_k)>\varepsilon\right)=0$$

is similar to that of (27) and (30).

From this, the triangle inequality, (33), (35) and (42), we obtain, for each fixed $\varepsilon > 0$, that

$$\mathbf{P}\left(\overline{\lim_{n\to\infty}}\, d(\hat{W}^n, \tilde{S}_k) > \varepsilon\right) = 0. \tag{43}$$

Applying this and the Strassen LIL to partial sums of independent identically distributed random variables (see (43) and the end of the proof of Theorem 3.1), we get the proof of Theorem 3.2.

The proof of Theorem 3.2 is complete. ■

References

Billingsley, P. (1977) *Convergence of probability measures.* Nauka, Moscow.

Bingham, N. (1986) Variants on the law of the iterated logarithm. *Bulletin London Mathematical Society* **18**, 433–467.

Borovkov, A. (1972) *Probability processes in theory of queues.* Nauka, Moscow, (in Russian).

Borovkov, A. (1980) *Asymptotic methods in theory of queues.* Nauka, Moscow, (in Russian).

Kuo-Hwa Chang (1997) Extreme and high-level sojourns of the single server-queue in heavy traffic. *Queueing Systems* **27**, 17–35.

Grigelionis, Br. and Mikulevičius, R. (1987) Functional limit theorems for a queueing system in heavy traffic. *Lietuvos Matematikos Rinkinys* **27**, 441–454, 660–673.

Harrison, J. (1978) The diffusion approximation for tandem queues in heavy traffic. *Advances in Applied Probability* **10**, 886–905.

Harrison, J. (1985) *Brownian Motion and Stochastic Flow Processes.* Wiley, New York.

Harrison, J. and Nguyen, V. (1993) Brownian models of multiclass queueing networks: current status and open problems. *Queueing Systems* **13**, 5–40.

Iglehart, D. (1971) Multiple channel queues in heavy traffic. IV. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **17**(1), 168–180.

Iglehart, D. (1972) Extreme values in the $GI/G/1$ queue. *Annals of Mathematical Statistics* **43** (1), 627–635.

Iglehart, D. (1973) Weak convergence in queueing theory. *Advances in Applied Probability* **5**, 570–594.

Karpelevich, F. I. and Kreinin, A. I. (1994) Heavy traffic limits for multiphase queues. *American Mathematical Society*, Providence, Rhode Island.

Kendall, D. (1961) Some problems in the theory of queues. *J. R. Statist. Soc.*, Ser. B. **13**, 151–185.

Kingman, J. (1962**a)** On queues in heavy traffic. *J. R. Statist. Soc.* **24**, 383–392.

Kingman, J. (1962**b)** The single server queue in heavy traffic. *Proc. Camb. Phil. Soc.* **57**, 902–904.

Kobyashi, H. (1974) Application of the diffusion approximation to queueing networks. *Journal of ACM,* **21**, 316–328.

Minkevičius, S. (1986) Weak convergence in multiphase queues. *Lietuvos Matematikos Rinkinys* **26**, 717–722.

Minkevičius, S. (1991) Transient phenomena in multiphase queues. *Lietuvos Matematikos Rinkinys* **31**, 136–145.

Minkevičius, S. (1997) On the law of the iterated logarithm in multiphase queueing systems. II. *Informatica* **8**, 367–376.

Prohorov, Y. (1963) Transient phenomena in queues. *Lietuvos Matematikos Rinkinys* **3**, 199–206.

Reiman, M. (1984) Open queueing networks in heavy traffic. *Mathematics of operations research*, **9**, 441–459.

Sakalauskas, L. and Minkevičius, S. (2000) On the law of the iterated logarithm in open queueing networks. *European Journal of Operational Research* **120**, 632–640.

Strassen, V. (1964) An invariance principle for the law of the iterated logarithm. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **3**, 211–226.

Whitt, W. (1974) *Heavy Traffic Limit Theorems for Queues: A Survey. Lecture Notes in Economic and Mathematical Systems* **98**, Springer-Verlag, Berlin-Heidelberg-New York, 307–350.