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A model for passive damping of a membrane

by

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Abstract: We introduce a model in which a wire made of shape memory alloy is used to passively damp the vibrations of a membrane. The mechanical energy of the membrane is transformed into heat via the thermo-elastic properties of the shape memory alloy. We describe the model and prove existence and uniqueness of solutions. Finally we will show that for suitable initial and boundary conditions, the energy of the entire system is a decreasing function.

Keywords: thermo-elastic materials, non-linear partial differential equations.

1. Description of the model

Thermo-elastic materials, such as shape memory alloys, have the property that they can transform mechanical (i.e. kinetic and potential) energy into thermal energy and vice versa. This property can be used to damp the vibrations of an elastic structure such as a membrane, plate or beam, by attaching shape memory materials at suitable locations. The mechanical deformations induce phase transitions in the shape memory alloy and a transformation of the mechanical energy into thermal energy via the latent heat of these phase transitions. The thermal energy can be removed from the structure by cooling the shape memory alloy. In this process, the shape memory material does not act as an actuator, but merely as a sink for mechanical energy. No active controls are necessary, except for a cooling mechanism for the shape memory alloy.

In this paper we will investigate a model of passive damping of a membrane to which a shape memory wire is attached. The model can easily be extended to a plate or a beam.

To our knowledge there is no mathematical result obtained for such a model due to two reasons:

1. the wave equation is defined in a nonsmooth domain, and so the general theory of such initial-boundary value problems is not available;
2. the shape memory alloys are highly nonlinear, and the useful theory exists only for one dimensional models.

Therefore, our paper is the first attempt to provide the mathematical model for such elastic structure. It seems that such devices could also be used in applications (Müller and Seelecke, 1998).

Furthermore, the modelling can be used for further optimization problems including optimal location of passing damping devices. Optimal location of shape memory alloys can be formulated as optimal *nonlinear cracks* in solids.

We start by a heuristic introduction to this model. Let $\Omega \subset \mathbb{R}^2$ be a convex domain with a Lipschitz boundary Γ and let $\Omega^\pm \subset \Omega$ be two non-empty subdomains with the following properties:

$$\overline{(\Omega^+ \cup \Omega^-)} = \overline{\Omega}, \tag{1}$$
$$Q = \overline{\Omega^+} \cap \overline{\Omega^-}, \tag{2}$$
$$\Gamma^\pm = \Gamma \cap \overline{\Omega^\pm}, \tag{3}$$

where Q is a line segment. From the convexity of Ω, it follows that Q intersects Γ transversally, i.e at angles in $(0, 2\pi)$. For simplicity, we will assume that $Q = [0,1]$. This geometry is indicated in Fig. 1, below. Furthermore, we let Ω_T, Q_T etc. denote the cylinders $\Omega \times (0,T)$, $Q \times (0,T)$ etc. The membrane occupies the domain Ω and the line segment Q is occupied by a one-dimensional shape memory rod. To formulate the problem let u be the vertical displacement of the membrane, v the vertical displacement of the shape memory rod and θ be the absolute temperature of the rod. We will also assume that the membrane itself is held at a constant temperature θ_0. This assumption can be removed by modeling the heat transfer on the membrane as well. However, this would not introduce any additional insight into the model.

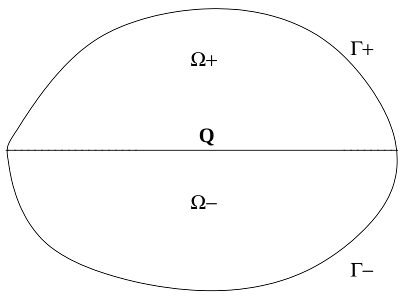

Figure 1. The domain Ω

The displacement u will satisfy a homogeneous wave equation with Dirichlet boundary conditions on $\Omega_T^{\pm}$:

$$u_{tt} - \Delta u = 0 \quad \text{on} \quad \Omega_T^{\pm}, \tag{4}$$

$$u = 0 \quad \text{on} \quad \Gamma_T^{\pm}. \tag{5}$$

Let us observe that such a wave equation in the domain with a crack is highly nonstandard. We use the estimates obtained recently by Kokotov and Plamenevsky for linear wave equation in the domain with cracks. On the common boundary Q_T, we assume continuity with the displacement v of the shape memory rod:

$$u = v \quad \text{on} \quad Q_T. \tag{6}$$

For the shape memory alloy on $Q = [0,1]$, we use the same model as in Brokate and Sprekels (1996), Bubner (1995), Bubner and Sprekels (1998), Sprekels and Zheng (1989) and Zheng (1995). In this model the functions v and θ satisfy the equation

$$v_{tt} - (\sigma(\theta, v_x))_x + R v_{xxxx} = f, \tag{7}$$

$$\theta_t - \kappa\theta_{xx} - \theta\,(\sigma(\theta, v_x))_\theta\, v_{xt} = g. \tag{8}$$

Here the function σ is given by

$$\sigma(\theta, v_x) = \gamma\,(\theta_1 - \theta)\, v_x - \beta v_x^3 + \alpha v_x^5. \tag{9}$$

θ_1, α, β, γ, κ and R being positive constants. We refer the reader to Sprekels and Zheng (1989) for a detailed investigation of this model.

The boundary conditions for these equations are given by

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{10}$$

$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0\ . \tag{11}$$

It remains to describe the source terms f and g in these equations. The deformed membrane will act on the shape memory rod via the elastic force. As in previous papers (see Horn and Sokołowski, 2002), this force is proportional to the jump of the normal derivatives of u across Q. To define this let ν be the outward normal on Q with respect to Ω^+, then we define

$$f = -\left[\frac{\partial u}{\partial \nu}\right] = \lim_{y\to 0^+} \frac{\partial u}{\partial y} + \lim_{y \to 0^-} \frac{\partial u}{\partial y}\ . \tag{12}$$

The source term g provides a mechanism for cooling the rod; g should be a function of $\theta - \theta_0$, which has the opposite sign to $\theta - \theta_0$, i.e. g acts as a sink when $\theta > \theta_0$, and g acts as a source if $\theta < \theta_0$. For simplicity we will use

$$g(\theta) = k\,(\theta_0 - \theta)\,.$$

Initial conditions on u are given as follows:

$$u(x,y,0) = u_0(x,y) \quad \text{on} \quad \Omega, \tag{13}$$
$$u_t(x,y,0) = u_1(x,y) \quad \text{on} \quad \Omega. \tag{14}$$

We will assume that the initial conditions for v and θ are compatible with those for u, i.e.

$$v(x,0) = v_0(x) = u_0(x,y)|_Q\,, \tag{15}$$
$$v_t(x,0) = v_1(x) = u_1(x,y)|_Q\,, \tag{16}$$
$$\theta(x,0) = \theta_0. \tag{17}$$

We conclude this section by formally showing that the energy of this system decreases. Formally, one can compute the energy of this structure as follows. After multiplying (4) by u_t and integrating the result over Ω one receives:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right)\, dxdy = \int_Q \left[\frac{\partial u}{\partial \nu}\right] u_t\, dx. \tag{18}$$

Multiplying (7) by v_t and integrating over Q and adding the result to the integral of (8) over Q yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_Q &\left(|v_t|^2 + R\,|v_{xx}|^2 + \gamma\theta_1\,|v_x|^2 - \beta\,|v_x|^4 + \alpha\,|v_x|^6 + |\theta|\right)\, dx \\ &= \int_Q f v_t\, dx + k\int_Q (\theta_0 - \theta)\, dx. \end{aligned}\tag{19}$$

This computation can be found in Sprekels and Zheng (1989). Next we add (18) and (19) and see that the terms $\int_Q \left[\frac{\partial u}{\partial \nu}\right] u_t\, dx$, and $\int_Q f v_t\, dx$ cancel. We get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right)\, dxdy \\ &\quad+\frac{1}{2}\frac{d}{dt}\int_Q \left(|v_t|^2 + R\,|v_{xx}|^2 + \gamma\theta_1\,|v_x|^2 - \beta\,|v_x|^4 + \alpha\,|v_x|^6 + |\theta|\right)\, dx \\ &\quad= k\int_Q (\theta_0 - \theta)\, dx\ . \end{aligned}\tag{20}$$

The left hand side of this equation is the time derivative of the total energy of the structure, consisting of the kinetic energies of the membrane and the rod, the potential energy of the membrane and the rod and the thermal energy of the rod. The right hand side is the heat flux into or out of the rod. We see that if θ_0 is sufficiently small, the right hand side will be negative, and the total energy will decrease, i.e. the motion of the membrane will be damped.

In the next section we will give a weak formulation of this system. In Section 3 we will prove uniform a priori estimates for the solution, which will justify the heuristic energy balance above. This will also allow us to prove an existence and uniqueness result.

Remark 1.1 *In view of the equality (19) we could consider a shape optimization problem of minimizing the functional*

$$J(Q) = \int_0^T \int_Q f v_t \, dx + k \int_Q (\theta_0 - \theta) \; dxdt$$

over a family of admissible curves Q. However, the value of the functional depends on the initial conditions. The other possibility would be to determine the rate of decreasing of the energy and maximize the rate with respect to the admissible curve. It seems that the appropriate choice of the shape functional is an important issue and should be adressed, taking into account that in general some numerical methods should be applied and the solution obtained for the shape optimization problem under considerations should be stable with respect to imperfections.

2. Weak formulation of the problem

To obtain a weak form of this problem we again start with the linear wave equation (4)

$$u_{tt} - \Delta u = 0 \quad \text{on} \quad \Omega^{\pm}_T, \tag{21}$$

$$u = 0 \quad \text{on} \quad \Gamma^{\pm}_T. \tag{22}$$

We multiply this equation by a smooth test function $\phi \in Y_1$ where

$$Y_1 = \left\{ \phi \in H^{2,2}(\Omega_T) : \phi|_{t=T} = \phi_t|_{t=T} = 0, \, \phi|_\Gamma = 0 \right\}$$

and integrate over $\Omega = \Omega^+ \cup \Omega^- \cup Q$ and over $[0, T]$ to get

$$0 = \int_0^T \int_\Omega (u_{tt} - \Delta u) \, \phi \, dxdt = \int_0^T \int_{\Omega^\pm} (u_{tt} - \Delta u) \, \phi \, dxdt,$$

where

$$\int_{\Omega^\pm} f \, dx = \int_{\Omega^+} f \, dx + \int_{\Omega^-} f \, dx.$$

Integrating by parts twice gives

$$\begin{aligned} \int_0^T \int_{\Omega^\pm} u_{tt}\phi \, dxdt &= - \int_0^T \int_{\Omega^\pm} u_t \phi_t \, dxdt + \int_{\Omega^\pm} u_t \phi \, dx \Big|_{t=0}^T \\ &= \int_0^T \int_{\Omega^\pm} u\phi_{tt} \, dxdt - \int_{\Omega^\pm} u_1 \phi(0,x) \, dx + \int_{\Omega^\pm} u_0 \phi_t(0,x) \, dx. \end{aligned}$$

Next we integrate the term with the Laplacian by parts over Ω^+ to get

$$\begin{aligned} - \int_0^T \int_{\Omega^+} \Delta u \phi \, dxdt &= \int_0^T \int_{\Omega^+} \nabla u \nabla \phi \, dxdt - \int_0^T \int_{\partial\Omega^+} \frac{\partial u}{\partial n} \phi \, d\sigma dt \\ &= - \int_0^T \int_{\Omega^+} u \Delta \phi \, dxdt + \int_0^T \int_{\partial\Omega^+} u \frac{\partial \phi}{\partial n} \, d\sigma dt - \int_0^T \int_{\partial\Omega^+} \frac{\partial u}{\partial n} \phi \, d\sigma dt. \end{aligned}$$

Similarly we integrate by parts over Ω^- to get:

$$
\begin{aligned}
-\int_0^T \int_{\Omega^-} \Delta u\phi \, dxdt &= \int_0^T \int_{\Omega^-} \nabla u \nabla \phi \, dxdt - \int_0^T \int_{\partial\Omega^-} \frac{\partial u}{\partial n}\phi \, d\sigma dt \\
&= -\int_0^T \int_{\Omega^-} u\Delta\phi \, dxdt + \int_0^T \int_{\partial\Omega^-} u\frac{\partial \phi}{\partial n} \, d\sigma dt - \int_0^T \int_{\partial\Omega^-} \frac{\partial u}{\partial n}\phi \, d\sigma dt.
\end{aligned}
$$

As a result we have

$$
\begin{aligned}
-\int_0^T \int_{\Omega} \Delta u\phi \, dxdt &= -\int_0^T \int_{\Omega} u\Delta\phi \, dxdt - \int_0^T \int_Q \left[\frac{\partial u}{\partial n}\right]_Q \phi \, d\sigma dt \\
&\quad - \int_0^T \int_\Gamma \frac{\partial u}{\partial n}\phi \, ds dt + \int_0^T \int_\Gamma \frac{\partial \phi}{\partial n} u \, ds dt.
\end{aligned}
$$

Because of the boundary conditions on u and ϕ on Γ the last two terms vanish. This allows us now to establish the weak form of the wave equation;

$$
\begin{aligned}
&\int_0^T \int_\Omega u\left(\phi_{tt} - \Delta\phi\right) \, dxdt - \int_\Omega u_1\phi(0) \, dx \\
&\quad + \int_\Omega u_0\phi_t(0) \, dx - \int_0^T \int_Q \left[\frac{\partial u}{\partial n}\right]_Q \phi \, d\sigma dt = 0
\end{aligned}
$$

for all $\phi \in Y_1$. The term $\left[\frac{\partial u}{\partial n}\right]_Q$ can now be replaced by the balance of momentum for shape memory alloys as follows

$$
v_{tt} - \left(\sigma(\theta, v_x)\right)_x + Rv_{xxxx} = -\left[\frac{\partial u}{\partial n}\right]_Q,
$$

where the function $\sigma(\theta, v_x)$ is given in (9). After this (23) becomes

$$
\begin{aligned}
&\int_0^T \int_\Omega u\left(\phi_{tt} - \Delta\phi\right) \, dxdt - \int_\Omega u_1\phi(0) \, dx + \int_\Omega u_0\phi_t(0) \, dx \\
&\quad + \int_0^T \int_Q \left(v_{tt} - \left(\sigma(\theta, v_x)\right)_x + Rv_{xxxx}\right) \phi \, d\sigma dt = 0 \,.
\end{aligned} \tag{23}
$$

In the last term we can again integrate by parts to get

$$
\int_0^T \int_Q v_{tt}\phi \, d\sigma dt = \int_0^T \int_Q v\phi_{tt} \, d\sigma dt + \int_Q v_0\phi_t(0) \, d\sigma - \int_Q v_1\phi(0) \, d\sigma,
$$

and

$$
-\int_0^T \int_Q \left(\sigma(\theta, v_x)\right)_x \phi \, d\sigma = \int_0^T \int_Q \sigma(\theta, v_x)\phi_x \, d\sigma,
$$

where the boundary terms vanish because of the choice of Y_1. Finally, we have

$$
\int_0^T \int_Q Rv_{xxxx}\phi \, d\sigma dt = \int_0^T \int_Q Rv_{xx}\phi_{xx} \, d\sigma dt.
$$

Again the boundary terms vanish by the choice of Y_1 and the boundary conditions on v. We can combine this with the weak wave equation (23) to get

$$\int_0^T \int_\Omega u\left(\phi_{tt} - \Delta\phi\right)\, dxdt - \int_\Omega u_1\phi(0)\, dx + \int_\Omega u_0\phi_t(0)\, dx \\ + \int_0^T \int_Q \left(v\phi_{tt} + \sigma(\theta, v_x)\phi_x + Rv_{xx}\phi_{xx}\right)\, d\sigma dt = 0\ . \tag{24}$$

This equation has to be combined with the non-linear heat equation (8). To obtain a weak formulation of (8) one multiplies this equation by a smooth function ψ and integrates it over $Q_T = Q \times (0, T)$ to get

$$\int_0^T \int_Q \left(-\theta\psi_t + \kappa\theta_x\psi_x - \theta\left(\sigma(\theta, v_x)\right)_\theta v_{xt}\psi - g\psi\right)\, d\sigma dt = 0, \tag{25}$$

for all

$$\psi \in Y_2 = \left\{\Psi \in H^{1,1}(Q_T) : \Psi(T) = 0\right\}.$$

The complete problem can now be formulated as:

Find

$$(u, v, \theta) \in L^2(\Omega_T) \times L^2(0, T; H^2(Q)) \cup H^1(0, T; H^1(Q)) \times L^2(0, T; H^1(Q))$$

such that (24) and (25) are satisfied for all $(\phi, \psi) \in Y_1 \times Y_2$.

Remark 2.1 *Observe that for our choice of the function σ given in (9) we have*

$$\theta\left(\sigma(\theta, v_x)\right)_\theta v_{xt} = -\theta\gamma v_x v_{xt},$$

i.e. this expression is an $L^2(Q_T)$ function for $v \in H^1(0, T; H^1(Q))$. This requirement can be eased, but we will see that it is automatically satisfied.

3. Uniform a priori estimates

Before going into details of the entire system we will investigate the wave equation on domains with corners by itself. Whereas there is a rich bibliography on elliptic boundary value problems in domains with corners (see, for example, Kozlov, Maz'ya and Rossmann, 1997; Kozlov, Maz'ya, 1999; Nazarov, Plamenevsky, 1994), there are only few papers on hyperbolic problems with non-homogeneous boundary conditions. The wave equation with homogeneous Dirichlet boundary conditions has been investigated extensively (see, for example Grisvard, 1989). These results all depend on the results for elliptic problems. In our case we do not have homogeneous Dirichlet conditions, which complicates the situation significantly. This case was studied in great detail in a recent paper by Kokotov and Plamenevsky (2000). To start recall (4) together with (6):

$$\begin{aligned} u_{tt} - \Delta u &= 0 \quad \text{on} \quad \Omega_T^{\pm}, \\ u &= 0 \quad \text{on} \quad \Gamma_T^{\pm}, \\ u &= v \quad \text{on} \quad Q_T. \end{aligned} \tag{26}$$

In each of the sub-domains $\Omega^{\pm}$ this system satisfies the requirements for the systems studied in Kokotov and Plamenevsky (2000). Hence we have the following inequality on Ω^+:

$$\lambda^2 \int_0^T \int_{\Omega^+} e^{-2\lambda t} |\nabla u|^2 \, dxdt + \lambda \int_0^T \int_Q e^{-2\lambda t} \left|\frac{\partial u}{\partial n}\right|^2 dsdt \tag{27}$$
$$\leq c\lambda \int_0^T \int_Q e^{-2\lambda t} \left|\frac{\partial u}{\partial \tau}\right|^2 dsdt$$

where $\frac{\partial u}{\partial \tau}$ denotes the tangential derivative along Q, $\lambda > 0$ is a parameter and c a positive constant that does not depend on λ. In the specific case at hand we have of course

$$\frac{\partial u}{\partial \tau} = v_x,$$

and furthermore we have that $0 \leq t \leq T$ and we therefore get

$$\lambda^2 e^{-2\lambda T} \int_0^T \int_{\Omega^+} |\nabla u|^2 \, dxdt + \lambda e^{-2\lambda T} \int_0^T \int_Q \left|\frac{\partial u}{\partial n}\right|^2 dsdt \tag{28}$$
$$\leq c\lambda \int_0^T \int_Q |v_x|^2 \, dsdt,$$

which immediately implies

$$\|u\|_{L^2(0,T;H^1(\Omega^+))} \leq C \|v\|_{L^2(0,T;H^1(Q))} . \tag{29}$$

The same inequality holds also for Ω^- and we can combine them to get

$$\int_0^T \int_{\Omega^\pm} |\nabla u|^2 \, dxdt + \int_0^T \int_Q \left| \left[\frac{\partial u}{\partial n}\right]_Q \right|^2 dsdt \leq C \int_0 \int_Q |v_x|^2 \, dsdt \ . \tag{30}$$

Hence, we have shown the following lemma:

LEMMA 3.1 *Let u satisfy (26) and $v \in L^2(0,T;H^1(Q))$, then u satisfies the following estimate:*

$$\|u\|_{L^2(0,T;H^1(\Omega^\pm))} \leq C \|v\|_{L^2(0,T;H^1(Q))} .$$

Furthermore, we have that

$$\left\| \left[\frac{\partial u}{\partial n}\right]_Q \right\|_{L^2(Q_T)} \leq C \|v\|_{L^2(0,T;H^1(Q))} .$$

This lemma now allows us to prove the following apriori estimate:

LEMMA 3.2 *Let* (v, θ) *satisfy (7) and (8) together with the boundary conditions (10) and (11). Furthermore, let* $u_0|_Q \in H^2(Q)$, $u_1|_Q \in L^2(Q)$ *and* $\theta_0|_Q \in L^2(Q)$. *Then we have* $v \in C(0,T;H^2(Q)), v_t \in C(0,T;L^2(Q))$, *and* $\theta \in C(0,T;L^1(Q))$.

Proof. As in the introduction, we multiply (7) by v_t and integrate over Q_T. Furthermore, we integrate (8) over Q and add the two resulting equations to get

$$\frac{1}{2}\frac{d}{dt}\int_Q \Big(|v_t|^2 + R\,|v_{xx}|^2 + \gamma\theta_1\,|v_x|^2 - \beta\,|v_x|^4 + \alpha\,|v_x|^6 + |\theta|\Big)\,dx$$

$$= \int_Q \left[\frac{\partial u}{\partial n}\right]_Q v_t\,dx + k\int_Q (\theta_0 - \theta)\,dx.$$

The first term on the right can be estimated using the previous Lemma as follows:

$$\left|\int_Q \left[\frac{\partial u}{\partial n}\right]_Q v_t\,dx\right|$$

$$\leq \left(\int_Q \left[\frac{\partial u}{\partial n}\right]_Q^2 dx\right)^{\frac{1}{2}} \|v_t\| \leq C\,\|v_x\|\,\|v_t\| \leq \frac{C}{2}\|v_x\|^2 + \frac{C}{2}\|v_t\|^2.$$

An application of Gronwall's inequality will yield the desired result. ∎

In the next step consider, we start by formally taking the time derivative of the wave equation to get

$$U_{tt} - \Delta U = 0 \quad \text{on} \quad \Omega_T^{\pm} \tag{31}$$

$$U = 0 \quad \text{on} \quad \Gamma_T^{\pm} \tag{32}$$

$$U = v_t \quad \text{on} \quad Q_T \tag{33}$$

together with the initial conditions

$$U(0,x) = u_1 \quad \text{on} \quad \Omega^{\pm} \tag{34}$$

$$U_t(0,x) = \Delta u_0 \quad \text{on} \quad \Omega^{\pm}. \tag{35}$$

This can always be done provided that u_0 satisfies the compatibility condition that $u_0 \in H^2(\Omega^{\pm})$. We can again apply the result of Kokotov and Plamenevsky (2000) to get:

$$\left\|\left[\frac{\partial U}{\partial n}\right]_Q\right\|_{L^2(Q_T)} \leq C\,\|v_t\|_{L^2(0,T;H^1(Q))}. \tag{36}$$

Furthermore, since we have $U = u_t$ a.e. it follows that

$$\left\| \left[\frac{\partial u_t}{\partial n} \right]_Q \right\|_{L^2(Q_T)} \leq C \left\| v_t \right\|_{L^2(0,T;H^1(Q))} . \tag{37}$$

To get higher apriori estimates for v, we multiply (7) by $-v_{xxt}$ an (8) by θ_t and follow exactly the proof of Lemma 4.2.4 of Zheng (1995) to obtain:

$$\begin{aligned}
&\frac{1}{2}\left(\|v_{xt}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|\theta_x(t)\|^2 \right) + \int_0^t \|\theta_t\|^2 \, ds \\
&\quad + \int_0^t \int_Q \left((\sigma(\theta, v_x))_x v_{xxt} - \theta(\sigma(\theta, v_x))_\theta v_{xt}\theta_t \right) \, dxds \\
&\quad = \frac{1}{2}\left(\|v_{xt}(0)\|^2 + \|v_{xxx}(0)\|^2 + \|\theta_x(0)\|^2 \right) \\
&\quad + \int_0^t \int_Q g\theta_t \, dxds - \int_0^t \int_Q \left[\frac{\partial u}{\partial n} \right] v_{xxt} \, dxdt \, .
\end{aligned}$$

Only the last term on the right is new and has to be treated differently. Integration by parts in t yields:

$$\begin{aligned}
\int_0^t \int_Q \left[\frac{\partial u}{\partial n} \right] v_{xxt} \, dxds &= \int_Q \left[\frac{\partial u(t)}{\partial n} \right] v_{xx}(t) \, dx \\
&\quad - \int_Q \left[\frac{\partial u(0)}{\partial n} \right] v_{xx(0)} \, dx - \int_0^t \int_Q \left[\frac{\partial u_t}{\partial n} \right] v_{xx} \, dxds \, .
\end{aligned}$$

The last term can be estimated as follows:

$$\begin{aligned}
\left| \int_0^t \int_Q \left[\frac{\partial u_t}{\partial n} \right] v_{xx} \, dxds \right| &\leq \left\| \left[\frac{\partial u_t}{\partial n} \right]_Q \right\|_{L^2(Q_T)} \|v\|_{L^2(0,T;H^2(Q))} \\
&\leq C \left\| v_t \right\|_{L^2(0,T;H^1(Q))} \|v\|_{L^2(0,T;H^2(Q))} .
\end{aligned}$$

Furthermore, observe that

$$\left| \int_Q \left[\frac{\partial u(t)}{\partial n} \right] v_{xx}(t) \, dx \right| \leq \left(\int_Q \left[\frac{\partial u(t)}{\partial n} \right]^2 dx \right)^{\frac{1}{2}} \|v(t)\|_{H^2(\Omega)} ,$$

and that

$$\begin{aligned}
\int_Q \left[\frac{\partial u(t)}{\partial n} \right]^2 dx &= \int_Q \left[\frac{\partial u(0)}{\partial n} \right]^2 dx + 2\int_0^t \int_Q \left[\frac{\partial u}{\partial n} \right] \left[\frac{\partial u_t}{\partial n} \right] dxds \\
&\leq 2C \|v\|_{L^2(0,T;H^1(Q))} \|v_t\|_{L^2(0,T;H^1(Q))} \, .
\end{aligned}$$

The terms involving $\|v_t\|_{L^2(0,T;H^1(Q))}$ can be treated using Gronwall's inequality. We arrive at:

Lemma 3.3 *Let (v,θ) satisfy (7) and (8) together with the boundary conditions (10) and (11). Furthermore, let $u_0|_Q \in H^2(Q)$, $u_1|_Q \in L^2(Q)$ and $\theta_0|_Q \in L^2(Q)$. Then there exists a constant C which depends only on the data such that*

$$\sup_{0\le t\le T} \frac{1}{2}\left(\|v_{xt}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|\theta_x(t)\|^2\right) + \int_0^T \|\theta_t\|^2 \, ds \le C.$$

Moreover, we have

$$\|\theta\|^2_{L^2(0,T;H^2(Q))} \le C$$

This Lemma has an immediate consequence:

Corollary 3.1 *Let u satisfy the system (26) together with the compatibility condition $u_0 \in H^2(\Omega^\pm)$ then we have $u_t \in L^2(0,T;H^1(\Omega^\pm)$, and $u \in C(0,T;H^1(\Omega^\pm))$.*

The previous lemma gives $v_t \in C(0,T;H^1(Q)$. This in turn can be used in (36) to get the estimate. This Corollary allows us to multiply (4) by u_t to get the energy inequality.

We can now combine these estimates to get the following existence and uniqueness result:

Theorem 3.1 *Let u_0 and u_1 satisfy the hypotheses of the previous lemmas and the corollary. Then there exists a unique triple*

$$(u,v,\theta) \in L^2(\Omega_T) \times L^2(0,T;H^2(Q)) \cup H^1(0,T;H^1(Q)) \times L^2(0,T;H^1(Q))$$

that satisfies (24) and (25) for all $(\phi,\psi) \in Y_1 \times Y_2$. Moreover, we have

$$\begin{aligned} u &\in C(0,T;H^1(\Omega^\pm)), && (38)\\ u_t &\in L^2(0,T;H^1(\Omega^\pm)), && (39)\\ v &\in C(0,T;H^3(Q)), && (40)\\ v_t &\in C(0,T;H^1(Q)), && (41)\\ \theta &\in C(0,T;H^1(Q)) \cap L^2(0,T;H^2(Q)). && (42)\end{aligned}$$

Finally, the triple satisfies the energy relation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|u_t|^2 + |\nabla u|^2\right) dxdy \\ &\quad + \frac{1}{2}\frac{d}{dt}\int_Q \left(|v_t|^2 + R\,|v_{xx}|^2 + \gamma\theta_1\,|v_x|^2 - \beta\,|v_x|^4 + \alpha\,|v_x|^6 + |\theta|\right) dx \quad (43)\\ &\quad = k\int_Q (\theta_0 - \theta)\; dx. \end{aligned}$$

Sketch of Proof. We consider in $Q \times (0,T)$ the system

$$v_{tt} - (\sigma(\theta, v_x))_x + Rv_{xxxx} = f(v),$$
$$\theta_t - \kappa\theta_{xx} - \theta\,(\sigma(\theta, v_x))_\theta\, v_{xt} = g,$$

along with the initial and boundary conditions, where

$$f(v) = \left[\frac{\partial u}{\partial n}\right]_Q,$$

for u a solution of (26). Then f is a linear pseudo-differential operator defined by Dirichlet-to-Neumann map on Q for linear wave equation. The operator is bounded as a map

$$f : L^2(0,T;H^1(Q)) \to L^2(0,T;H^1(Q))$$

and further we have that

$$f : v_t \mapsto f(v_t)$$

is bounded in the same space. The a priori estimates can be used to follow the treatment of Horn and Sokołowski (2002) to obtain a solution to this problem. From there we get a solution to (26). Finally, the a priori estimates allow us to obtain the energy inequality. ■

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