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Fuzzy system reliability analysis using time dependent fuzzy set

by

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Abstract: In this paper the use of the concept of γ-cut (interval of confidence) and time dependent fuzzy set theory is proposed in a general procedure to construct the membership function of the fuzzy reliability, when the failure rate is fuzzy. The failure rate of the system is represented by a triangular fuzzy number. A simple numerical example is also presented.

Keywords: fuzzy reliability, fuzzy failure rate, time dependent fuzzy set.

1. Introduction

The reliable engineering is one of the important engineering tasks in design and development of technical system. The conventional reliability of a system is defined as the probability that the system performs its assigned function properly during a predefined period under the condition that the system behavior can be fully characterized in the context of probability measures. The reliability of a system can be determined on the basis of tests or the acquisition of operational data. However, due to the uncertainity and inaccuracy of this data, the estimation of precise values of probabilities is very difficult in many systems (e.g. power system, electrical machine, hardware etc., Hammer (2001), El-Hawary (2000)). For this reason the fuzzy reliability concept has been introduced and formulated in the context of fuzzy measures. The basis for this approach is constituted by the fundamental works on fuzzy set theory of Zadeh (1978), Dubois and Prade (1980), Zimmerman (1986) and other.

The theory of fuzzy reliability was proposed and development by several authors, Cai, Wen and Zhang (1991, 1993); Cai (1996); Chen, Mon (1993); Hammer (2001); El-Hawary (2000), Onisawa, Kacprzyk (1995); Utkin, Gurov (1995). The recent collection of papers by Onisawa and Kacprzyk (1995), gave many different approach for fuzzy reliability. According to Cai, Wen and Zhang (1991, 1993); Cai (1996) various form of fuzzy reliability theories, including profust reliability theory Dobois, Prade (1980); Cai, Wen and Zhang (1993); Cai (1996); Chen, Mon (1993); Hammer (2001); El-Hawary (2000); Utkin, Gurov (1995), posbist reliability theory, Cai, Wen and Zhang (1991, 1993) and posfust reliability theory, can be considered by taking new assumptions, such as the possibility assumption, or the fuzzy state assumption, in place of the probability assumption or the binary state assumption.

Profust reliability theory is based on the fuzzy-state assumption. Profust reliability function, profust failure rate function and mathematically rigorous relationships among them, lay a solid foundation for profust reliability theory.

In many cases, the notion of representing component reliability indices (such as failure rate or mean repair time) by crisp numbers must be challenged. For instance, a lot of reliability data are obtained by analogy from data bases associated with equipment that is not exactly the one under analysis, either because it was not installed under the same conditions or just because some new types of equipment are being foreseen. Also, repair times depend not only on the components themselves but also on other systematic factors that include company efficiency. Consequently, it is natural that some uncertainty be associated with component indices, and this uncertainty is not of the probabilistic type. In fact, we are dealing with a twofold dimension of uncertainty, getting at a hybrid model that connects stochastic and fuzzy uncertainties: stochastic because we are still dealing with a failure-repair cycle and fuzzy because we cannot accurately describe all the conditions of the "experiments" that would lead to a pure probabilistic model.

Cai, Wen and Zhang (1993) presented a fuzzy set based approach to failure rate and reliability analysis, where profust failure rate is defined in the context of statistics. El-Nawary (2000) presented a models for fuzzy power system reliability analysis, where the failure rate of a system is represented by a triangular fuzzy number. However, in this approach, the membership function of the fuzzy reliability function is not completely described. Our procedure is based on the profust reliability theory. In this paper we propose a general procedure to construct the membership function of the reliability function, when the failure rate is fuzzy. The failure rate of the system is represented by a triangular fuzzy number.

2. Problem formulation

Let X be a universal set. Then a fuzzy subset $\tilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,1]$. We can also write the fuzzy set $\tilde{A}$ as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}\,. \tag{1}$$

The time dependent fuzzy set $\tilde{A}(t)$ of the universe of discourse X is defined as, Virant and Zimic (1996),

$$\tilde{A}(t) = \left\{ \left(x, \mu_{\tilde{A}(t)}(x) \right) \right\} \tag{2}$$

where $\mu_{\tilde{A}(t)}(x)$ is the dynamic membership function for $x \in X$. A convex, normalized fuzzy set defined on the real line $\Re$, whose membership function is piecewise continuous or, equivalently, each γ-cut is a closed interval, is called a fuzzy number. The γ-cut of a time dependent fuzzy set $\tilde{A}(t)$ is defined by

$$\tilde{A}_\gamma(t) = \left\{ x \left| \mu_{\tilde{A}(t)}(x) \geq \gamma, x \in X \right. \right\}, \quad \gamma \in [0, 1]. \tag{3}$$

For example, Fig. 1 shows a fuzzy number $\tilde{A}(t)$ which is both convex and normal. At time t_1 we have fuzzy number $\tilde{A}(t_1)$, at time t_2, fuzzy number $\tilde{A}(t_2)$, etc. Fig. 2 shows a fuzzy number $\tilde{A}(t)$ with γ-cut.

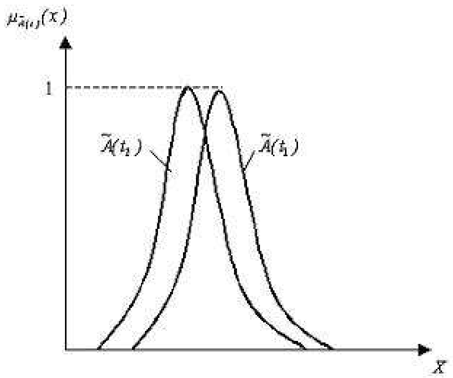

Figure 1. A fuzzy number $\tilde{A}(t)$.

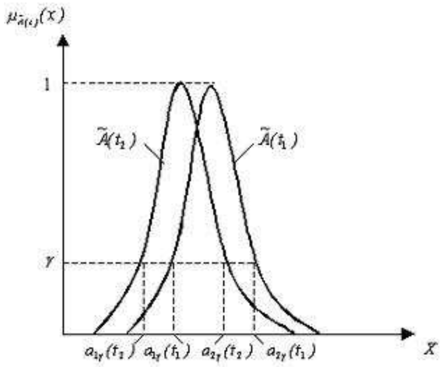

Figure 2. A fuzzy number $\tilde{A}(t)$ with γ-cut.

From Fig. 2 we can see that

$$\tilde{A}_\gamma(t_1) = [a_{1\gamma}(t_1), a_{2\gamma}(t_1)], \quad \tilde{A}_\gamma(t_2) = [a_{1\gamma}(t_2), a_{2\gamma}(t_2)]. \tag{4}$$

Assume that X and U are two crisp sets. Let failure rate function be fuzzy and represented by a fuzzy set $\tilde{H}(t)$, $\tilde{H}(t) = \left\{h,\ \mu_{\tilde{H}(t)}\ h \in X\right\}$. The γ-cut fuzzy set of $\tilde{H}(t)$ is $\tilde{H}_\gamma(t) = \left\{\gamma \in X \left|\mu_{\tilde{H}(t)} \geq \gamma\right.\right\}$. Note that $\tilde{H}_\gamma(t)$ is a crisp set. Suppose that $\tilde{H}(t)$ is a fuzzy number. Then for each choice of γ-cut we have interval $\tilde{H}_\gamma(t) = [h_{1\gamma}(t),\ h_{2\gamma}(t)]$. By the convexity of the fuzzy number, the bounds of the interval are functions of γ and can be obtained as $\tilde{h}_{1\gamma} = \min \mu_{\tilde{H}(t)}(\gamma)$ and $\tilde{h}_{2\gamma} = \max \mu_{\tilde{H}(t)}(\gamma)$, respectively. Let $\psi : X \to U$ be a bounded continuously differentiable function from X to the universe U. We wish to calculate the fuzzy set (fuzzy reliability function) induced on U by applying ψ to the set $\tilde{H}(t)$. If we write $u = \psi(h)$, where $h \in X$ and $\tilde{R}(t) = \left\{u, \mu_{\tilde{R}(t)}(u) \left| u = \psi(h), u \in U\right.\right\}$, then the membership function of $\tilde{R}(t)$ is defined by the extension principle

$$\mu_{\tilde{R}(t)}(u) = \sup_{h \in X} \left\{\mu_{\tilde{H}(t)}(h) \,|\, u = \psi(h)\right\}. \tag{5}$$

We know that if $\tilde{H}(t)$ is normal and convex and ψ is bounded, then $\tilde{R}(t)$is also normal and convex, Mon and Cheng (1995). Therefore we can calculate the corresponding interval $[r_{1\gamma}(t),\ r_{2\gamma}(t)] = \psi\left(\tilde{H}_\gamma(t)\right)$, where $r_{1\gamma}(t)$ and $r_{2\gamma}(t)$ correspond, respectively, to the global minimum and maximum of ψ over the space $\tilde{H}_\gamma(t)$ at γ-level:

$$r_{1\gamma}(t) = \min \psi(h), \quad \text{s.t.} \quad h_{1\gamma}(t) \leq h \leq h_{2\gamma}(t) \tag{6a}$$

$$r_{2\gamma}(t) = \max \psi(h), \quad \text{s.t.} \quad h_{1\gamma}(t) \leq h \leq h_{2\gamma}(t). \tag{6b}$$

This pair of mathematical programs involve the systematic study of how the optimal solutions change as the bounds $h_{1\gamma}(t)$, and $h_{2\gamma}(t)$, vary over the interval $\gamma \in [0, 1]$, and so they fall into the category of parametric programming.

If both $r_{1\gamma}(t)$ and $r_{2\gamma}(t)$ are invertible with respect to γ, then a left shape function $f_{\tilde{R}(t)}(u) = [r_{1\gamma}(t)]^{-1} = \left[\min\limits_{u_1 \leq u \leq u_2} u\right]_\gamma^{-1}$ and a right shape function $g_{\tilde{R}(t)}(u) = [r_{2\gamma}(t)]^{-1} = \left[\max\limits_{u_2 \leq u \leq u_3} u\right]_\gamma^{-1}$ can be obtained, from which the membership function $\mu_{\tilde{R}(t)}(u)$ is constructed:

$$\mu_{\tilde{R}(t)}(u) = \begin{cases} f_{\tilde{R}(t)}(u) & , \quad u_1 \leq u \leq u_2 \\ g_{\tilde{R}(t)}(u) & , \quad u_2 \leq u \leq u_3 \\ 0 & , \quad \text{otherwise,} \end{cases} \tag{7}$$

where $u_1 \leq u_2 \leq u_3$, $f_{\tilde{R}(t)}(u_1) = g_{\tilde{R}(t)}(u_3) = 0$ and $f_{\tilde{R}(t)}(u_2) = g_{\tilde{R}(t)}(u_2) = 1$. It is obvious that $f_{\tilde{R}(t)}(u)$ is a continuous and strictly increasing function on $[u_1, u_2]$, and $g_{\tilde{R}(t)}(u)$ is continuous and strictly decreasing on $[u_2, u_3]$.

3. Main results

The system reliability function is

$$R(t) = \exp\left[-\int_{o}^{t} h(t)dt\right], \quad t \geq 0 \tag{8}$$

where $h(t)$ is failure rate function.

Let failure rate function be represented by a triangular fuzzy number $\tilde{H}(t)$, $\tilde{H}(t) = [m(t) - \alpha(t), m(t), m(t) + \beta(t)]$, where $\alpha(t) > 0$ and $\beta(t) > 0$ are called left and right spreads at time t, respectively. The γ-cut of the fuzzy set $\tilde{H}(t)$ is

$$\tilde{H}_\gamma(t) = [(m(t) - \alpha(t)) + \gamma\alpha(t), (m(t) + \beta(t)) - \gamma\beta(t)], \quad \gamma \in [0, 1].$$

Then, formulation (6) can be written as

$$r_{1\gamma}(t) = \min\left(\exp\left[-\int_{0}^{t} h(\tau)d\tau\right]\right), \quad t \geq 0 \tag{9a}$$

$$\text{s.t. } (m(t) - \alpha(t)) + \gamma\alpha(t) \leq h(t) \leq (m(t) + \beta(t)) - \gamma\beta(t)$$

$$r_{2\gamma}(t) = \max\left(\exp\left[-\int_{0}^{t} h(\tau)d\tau\right]\right), \quad t \geq 0 \tag{9b}$$

$$\text{s.t. } (m(t) - \alpha(t)) + \gamma\alpha(t) \leq h(t) \leq (m(t) + \beta(t)) - \gamma\beta(t).$$

Since the reliability function is a monotonically decreasing function, $\tilde{R}(t)$ attains its extreme at the bound. That is

$$r_{1\gamma}(t) = \exp\left[-\int_{0}^{t} ((m(t) + \beta(t)) - \gamma\beta(t))\, d\tau\right], \quad t \geq 0 \tag{10a}$$

$$r_{2\gamma}(t) = \exp\left[-\int_{0}^{t} ((m(t) - \alpha(t)) + \gamma\alpha(t))\, d\tau\right], \quad t \geq 0. \tag{10b}$$

Taking the inverse of (10) allows for obtaining of the left and right shape function of $\mu_{\tilde{R}(t)}$. Accordingly

$$\mu_{\tilde{R}(t)}(u)=\begin{cases}\dfrac{\ln(u)+\int_0^t(m(t)+\beta(t))dt}{\int_0^t\beta(t)dt}, \\ \qquad \exp\left[-\int_0^t(m(t)+\beta(t))d\tau\right]\le u\le\exp\left[-\int_0^t m(t)d\tau\right] \\ -\dfrac{}{\ln(u)+\int_0^t(m(t)-\alpha(t))d\tau\int_0^t\alpha(t)d\tau}, \\ \qquad \exp\left[-\int_0^t m(t)d\tau\right]\le u\le\exp\left[-\int_0^t(m(t)-\alpha(t))d\tau\right].\end{cases} \tag{11}$$

Now we will show that the fuzzy set $\tilde{R}(t)$ is a fuzzy number. A convex and normalized fuzzy set defined on the line $\Re$ whose membership function is piecewise continuous or, equivalently, each γ - cut is a closed interval, is called a fuzzy number, Chen and Mon (1993), Mon and Cheng (1995), El-Hawary (2000).

Normality implies that $\exists u\in\Re \quad \bigcup_u \mu_{\tilde{R}(t)}=1$, that is, the maximum value of the fuzzy set $\tilde{R}(t)$ in $\Re$ is 1

$$\mu_{\tilde{R}(t)}\left(\exp\left[-\int_0^t m(t)+\beta(t)]d\tau)\right]\right)=1. \tag{12}$$

Convexity means that a γ-cut which is parallel to the horizontal axis $\tilde{R}_\gamma(t)=[r_{1\gamma}(t),r_{2\gamma}(t)]$ yields the property of nesting, that is $(\gamma<\gamma')\Rightarrow(r_{1\gamma'}\le r_{1\gamma},r_{2\gamma'}\ge r_{2\gamma})$. Alternatively, if we represent the γ-cut by $\tilde{R}_\gamma(t)$ as $\tilde{R}_\gamma(t)=[r_{1\gamma}(t),\, r_{2\gamma}(t)]$ and $\tilde{R}_{\gamma'}(t)=[r_{1\gamma'}(t),\, r_{2\gamma'}(t)]$, then the condition of convexity implies that $\gamma'<\gamma\Rightarrow\tilde{R}_\gamma(t)\subset\tilde{R}_{\gamma'}(t)$.

According to the property of the function $\ln(u)$, the left shape function $f_{\tilde{R}(t)}$ of membership function $\mu_{\tilde{R}(t)}(u)$ is strictly continuously increasing on $\left[\exp\left\{-\int_0^t(m(t)+\beta(t))\,d\tau\right\},\ \exp\left\{-\int_0^t m(t)d\tau\right\}\right]$ and the right shape function $g_{\tilde{R}(t)}$ of $\mu_{\tilde{R}(t)}(u)$ is strictly continuously decreasing on $\left[\exp\left\{-\int_0^t m(t)d\tau\right\},\right.$ $\left.\exp\left\{\int_0^t(m(t)-\alpha(t))d\tau\right\}\right]$. One possible realization of the $\mu_{\tilde{R}(t)}(u)$ shown in Fig. 3.

Complex systems usually show a decreasing failure rate at the beginning, followed by a period with a constant failure rate and in the later part of the life cycle, the failure rate increases. This type of failure curve is known as the bathtub curve. There are some models that can be used to analyze this type of

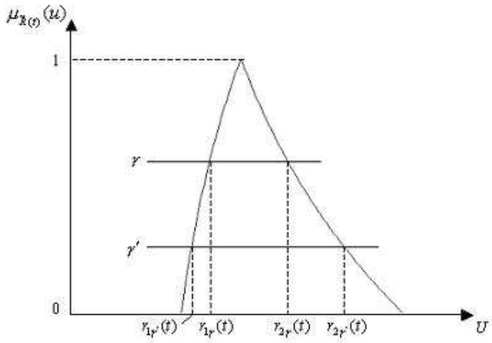

Figure 3. A normal and convex fuzzy number $\tilde{R}_\gamma(t)$ with γ-cut.

failure rate data. Consider the following two models:

Model 1. Let the failure rate function be constant , i.e. $\tilde{H}(t) = \tilde{H}$. Then *m(t)= m=const,* $\alpha(t) = \alpha = const$ and $\beta(t) = \beta = const$. The γ-cut of the fuzzy set $\tilde{H}$ is

$$\tilde{H}_\lambda = \left[(m-\alpha)+\gamma\alpha,\ (m+\beta)-\gamma\beta\right], \quad \gamma \in [0,1]\,. \tag{13}$$

Since $\tilde{R}(0) = 1$ and $\tilde{R}(\infty) = 0$ from (11), we obtain

$$\mu_{\tilde{R}(t)}(u) = \begin{cases} \frac{\ln(u)+(m+\beta)t}{\beta t}, & \beta > 0, 0 < t < \infty \\ & \exp[-(m+\beta)t] \le u \le \exp[-mt], \\ -\frac{\ln(u)+(m-\alpha)t}{\alpha t}, & \alpha > 0, 0 < t < \infty \\ & \exp[-mt] \le u \le \exp[-(m-\alpha)t]. \end{cases} \tag{14}$$

Model 2. Let the failure rate function not be constant. The triangular fuzzy number $\tilde{H}(t)$ and its membership function $\mu_{\tilde{H}(t)}(h)$ depends only on the three parameters $m(t)$, $\alpha(t)$ and $\beta(t)$. Let $\alpha(t) = \alpha = const$, $\beta(t) = \beta = const$ and

$$m(t) = ce^{\omega t} \tag{15}$$

where c is positive constant. Since $\tilde{R}(0) = 1$ and $\tilde{R}(\infty) = 0$ from (11), we obtain

$$\mu_{\tilde{R}(t)}(u) = \begin{cases} \frac{\ln(u)+\frac{c}{k}[\exp(\omega t)-1]+\beta t}{\beta t}, \quad \beta > 0,\ 0 < t < \infty \\ \quad \exp\left[-\frac{c}{k}[\exp(\omega t)-1]-\beta t\right] \le u \le \exp\left[-\frac{c}{k}[\exp(\omega t)-1]\right] \\ -\frac{\ln(u)+\frac{c}{k}[\exp(\omega t)-1]-\alpha t}{\alpha t}, \quad \alpha > 0, 0 < t < \infty \\ \quad \exp\left[-\frac{c}{k}[\exp(\omega t)-1]\right] \le u \le \exp\left[-\frac{c}{k}[\exp(\omega t)-1]+\alpha t\right]. \end{cases} \tag{16}$$

4. Numerical example

In many system studies the raw data are usually constituted by the failure rate and mean repair time. These data may be found in several databases. However, it is typical for technical system components to have small failure rate values and small repair times (compared to functioning times). Therefore, it is classical to approximate the frequency of failures with failure rate, and therefore, we will assume that raw data extracted from database are failure rates.

But a database is a collection of information about pieces of system possibility installed in very diverse conditions, or some different technological generations, or operated by utilities with different views about maintenance or quality. Besides, when planning a system, one may only apply failure rates, by analogy, to a new system that will be built in the future. In fact, failure rate will be changing over time because of the aging of system or because of rehabilitation actions.

It is not surprising that we may accept, for a failure rate relative to some type of system, instead of crisp number such as 0.0015 failures/year, an interval of confidence such as [0.001, 0.002] failures/year or even a fuzzy number. Recall that designation "interval of confidence", as used here, does not relate to any classical statistical concepts but to the discourse of "fuzzy set community". In this sense, an interval of confidence γ corresponds to the cut set at level γ defined to the membership function of fuzzy set. Assume that the failure rate is uncertain and represented by the triangular fuzzy number such as $h \to$ "more or less 0.015" failures/year $\to \tilde{H} = [0.001, 0.0015, 0.002]$ (Fig. 4).

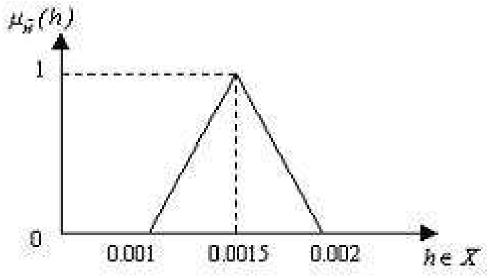

Figure 4. A membership function of the triangular fuzzy number $\tilde{H}$.

A direct substitution into Eq.(14) yields

$$\mu_{\tilde{R}(t)}(u) = \begin{cases} \frac{\ln(u)+0.002t}{0.0005t}, & 0 < t < \infty \\ & \exp(-0.002t) \le u \le \exp(-0.0015t) \\ -\frac{\ln(u)+0.001t}{0.0005t}, & 0 < t < \infty \\ & \exp(-0.0015t) \le u \le \exp(-0.001t). \end{cases}$$

The membership functions of $\tilde{R}(t)$ shown in Fig. 5.

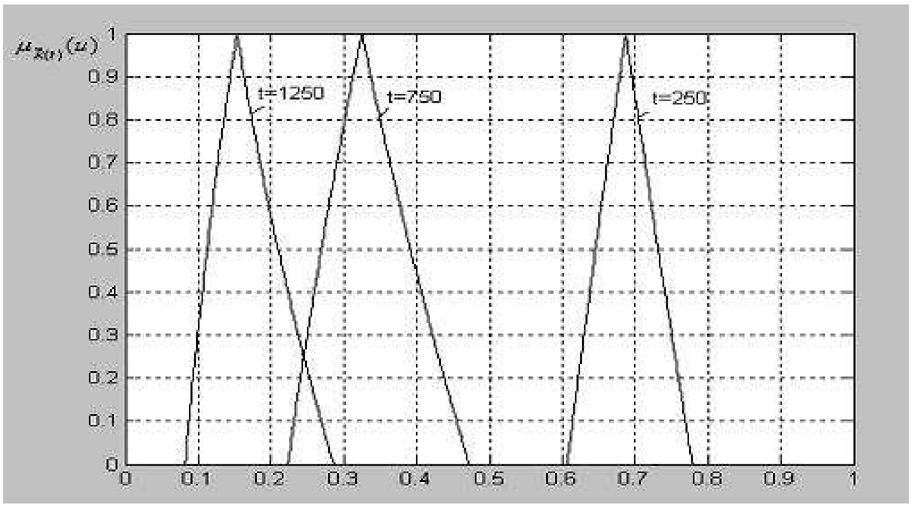

Figure 5. A membership function of $\tilde{R}(t)$.

5. Conclusion

Fuzzy reliability is based on the concept of fuzzy set. When the failure rate is fuzzy, according to Zadeh's extension principle, the reliability measure will be fuzzy as well. In this paper the use of the concept of γ-cut (interval of confidence) and time dependent fuzzy set theory leads to the proposal of a general procedure to construct the membership function of the reliability function, when the failure rate is fuzzy. The failure rate of the system is represented by a triangular fuzzy number. The obtained result is acceptable also for the membership functions which are invertible with respect to γ (e.g. the trapezoidal, gamma etc.).

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