Control and Cybernetics

vol. **33** (2004) No. 4

Hard and soft sub-time-optimal controllers for a mechanical system with uncertain mass

by

Piotr Kulczycki[1], Rafael Wisniewski[2], Piotr Kowalski[3] and Karol Krawiec[3]

[1] Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01-447 Warsaw, Poland
e-mail: kulczycki@ibspan.waw.pl
[2] Department of Control Engineering, Aalborg University
Fredrik Bajers Vej 7, 9220 Aalborg Ø, Denmark
[3] Department of Automatic Control, Cracow University of Technology
ul. Warszawska 24, 31-155 Cracow, Poland

Abstract: An essential limitation in using the classical optimal control has been its limited robustness to modeling inadequacies and perturbations. This paper presents the concepts of two practical control structures based on the time-optimal approach, a hard and soft one. The hard structure is defined by the parameters selected in accordance with the rules of the statistical decision theory; however, the soft structure allows additionally for elimination of rapid changes in control values. The object is a basic mechanical system, with uncertain (also non-stationary) mass treated as a stochastic process. The methodology proposed here is of a universal nature and may easily be applied with respect to other elements of uncertainty of time-optimal controlled mechanical systems.

Keywords: optimal control, mechanical system, uncertain mass, stochastic process, suboptimal structure, robustness.

1. Introduction

The main constraint on the application possibilities of systems based on the principles of the classical optimal control theory (Athans and Falb, 1966) has been their excessive sensitivity to the modeling inaccuracy of object dynamics, the identification of object parameters, as well as perturbations and noise naturally accompanying real processes. In the extreme cases, even a small error in parameter identification, which is unavoidable in practice, completely disqualifies an optimal control system. However, the very idea of optimal control often turns out to be a proper basis to design a suboptimal structure in which excessive sensitivity would be eliminated; for details, see Friedland (1996), Isidori (1995), Khalil (1996), Lyshevski (2001), Weinmann (1991), Zhou et al. (1996).

The two basic types of optimal control are related with quadratic and time-optimal (minimum-time) performance indexes. The time-optimal approach is very significant from the viewpoint of many technological processes, because it allows to maximally reduce considerable technological interruptions, which are economically ineffective. On the other hand, time-optimal structures, such as controls with extreme values, are exceptionally sensitive to the above-mentioned identification inaccuracies and disturbances.

In this paper, the time-optimal control of an object described using the second principle of Newton's dynamics, i.e. from the physical point of view, representing mass subjected to force, will be considered. Such a mechanical system is a basic element accompanying all considerations in robotics (Sciavicco and Siciliano, 1996). The uncertainty problem will be considered in the example of the main parameter of such an object, i.e. the value of mass (or the moment of inertia). In practice, that value can only be given with the precision that results from accurate measurement. Moreover, in many applications (e.g. shifting or transport tasks) this value is not subject to measurement at all, but rather grossly estimated on the basis of the assumed value. Furthermore, in other situations, a mass may be variable, along with the consumption of fuel or other substances used in the technological process.

In this paper, the above problem has been solved by the introduction of a random factor; namely, load will be treated as the realization of a stochastic process with almost all realizations being piecewise continuous and jointly bounded. The introduction of a random factor makes it possible to take into account errors in the identification of mass, whereas the fluctuations of the particular realizations describe its changes, including also those of discontinuous nature.

The paper is organized as follows. Section 2 specifies mathematical grounds regulating strict theoretical justification for practical controlling structures presented in Section 3: a hard one, where parameters are selected in accordance with the rules of the statistical decision theory, and a soft one, which allows additionally for elimination of rapid changes of control values by making the function of a feedback controller continuous. The concept presented is universal and may be supplemented by and generalized with a number of various aspects occurring in such tasks. These tasks, together with the results of numerical verification, constitute the subject of the last Section 5.

The material presented provides a summary of the previous research on the hard structure (Kulczycki, 1996a, b, 2000; Kulczycki and Wisniewski, 2002), which forms here a basis for new investigations concerning the soft approach. This material was presented in its preliminary version as Kulczycki et al. (2004).

2. Theoretical results

The random approach to the control problem, worked out in this paper is based on the concept of an almost certain time-optimal control. This is defined as a stochastic process such that almost all of its realizations are controls which, for proper deterministic systems obtained by fixing the random factor, bring the state of the system to the target set in a minimal and finite time. The almost certain time-optimal control is unique if every time-optimal control is a process stochastically equivalent to it. This notion was introduced in Kulczycki (1996b). Similarly, an almost certain solution of a random differential equation means such a stochastic process that almost all its realizations are solutions of proper deterministic equations obtained for a fixed random factor. The almost certain solution is unique if every almost certain solution is a process stochastically equivalent to it. The solution of a deterministic differential equation will be considered below in Caratheodory sense, i.e. as a function which is absolutely continuous at every compact subinterval of its time domain and fulfils the differential equation almost everywhere; for details see Kulczycki (1996c).

Consider a mechanical system with a single degree of freedom, whose dynamics is described by the second law of Newtonian mechanics

$$m\ddot{s}(t) = u(t) \ , \tag{1}$$

where m, s, u mean the load (mass or moment of inertia), position (linear or angular), and control (force or moment), respectively. If the parameter m is treated as a realization of a stochastic process M, then denoting by $\omega \in \Omega$ a random factor, and by X_1, X_2, U real stochastic processes which represent the position, velocity and control respectively, the dynamics of the system under consideration can now be described by the following random differential equation:

$$\dot{X}_1(\omega, t) = X_2(\omega, t) \tag{2}$$

$$\dot{X}_2(\omega, t) = \frac{1}{M(\omega, t)} U(\omega, t) \ , \tag{3}$$

with the initial condition

$$\begin{bmatrix} X_1(\omega, t_0) \\ X_2(\omega, t_0) \end{bmatrix} = x_0 \qquad \text{for almost all } \ \omega = \Omega, \tag{4}$$

given the following assumptions

(A1) $t_0 \in \mathbf{R}$, $T = [t_0, \infty)$;

(A2) $x_0 = [x_{01}, x_{02}]^{\mathrm{T}} \in \mathbf{R}^2$ and $x_f = [x_{f1}, x_{f2}]^{\mathrm{T}} \in \mathbf{R}^2$ constitute the initial and the target states, respectively;

(A3) the values of admissible controls are limited to the interval $[-1, 1]$;

(A4) (Ω, Σ, P) denotes a complete probability space;

(A5) M is a real stochastic process with almost all realizations being piecewise continuous and satisfying the boundary condition $M(\omega, t) \in [m_-, m_+]$ for $t \in T$, where $0 < m_- \leq m_+$.

Let us introduce also the following subdivision of the state space $\mathbf{R}^2$ into the disjoint sets $R_+, R_-, Q_+, Q_-, \{x_f\}$; see Fig. 1. Specifically, let K_{+-}, K_{++}, denote sets of all states which can be brought to the target by the control $U \equiv +1$, if $M \equiv m_-$ or $M \equiv m_+$, respectively; analogously K_{--} and K_{-+} for $U \equiv -1$ if $M \equiv m_-$ or $M \equiv m_+$. Moreover, let:

$$Q_+ = \{[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2 \text{ such that there exist } [x_1', x_2]^{\mathrm{T}} \in K_{+-} \text{ and } [x_1'', x_2]^{\mathrm{T}} \in K_{++} \text{ with } x_1' \le x_1 \le x_1'' \text{ or } x_1'' \le x_1 \le x_1'\} \tag{5}$$

$$Q_- = \{[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2 \text{ such that there exist } [x_1', x_2]^{\mathrm{T}} \in K_{-+} \text{ and } [x_1'', x_2]^{\mathrm{T}} \in K_{--} \text{ with } x_1' \le x_1 \le x_1'' \text{ or } x_1'' \le x_1 \le x_1'\} \tag{6}$$

$$R_+ = \{[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2 \backslash Q \text{ such that there exist } [x_1', x_2]^{\mathrm{T}} \in Q \text{ with } x_1 < x_1'\} \tag{7}$$

$$R_- = \{[x_1, x_2]^{\mathrm{T}} \in \mathbf{R}^2 \backslash Q \text{ such that there exist } [x_1', x_2]^{\mathrm{T}} \in Q \text{ with } x_1' < x_1\}, \tag{8}$$

where $Q = Q_+ \cup \{x_f\} \cup Q_-$. Therefore, the sets K_{+-}, K_{++}, represent all those states which can be brought to the target by the control $+1$, at the minimum and maximum possible values of a mass. The set Q_+ contains intermediate points. The sets K_{-+}, K_{--}, and Q_- may be interpreted analogously for the control -1. Note also that K_{+-} and K_{++} belong to Q_+ as K_{-+} and K_{--} belong to Q_-. For illustration, see Fig. 1.

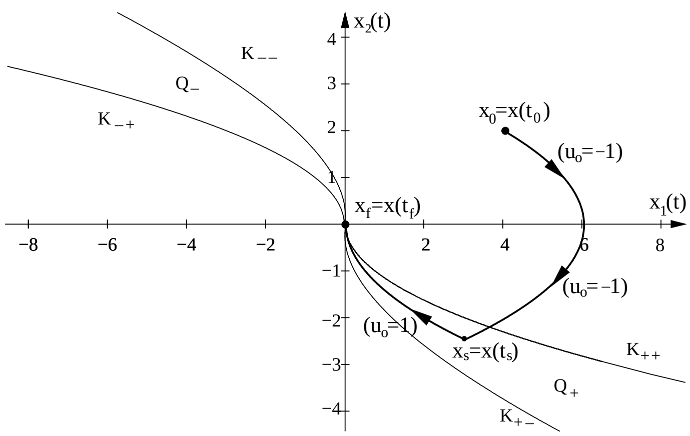

Figure 1. Illustration of the theorem

Theorem 2.1 *For a dynamic system described by random differential equation (2)-(4), under assumptions (A1)-(A5), there exists a unique almost certain time-optimal control U_o, generating a unique almost certain solution $X = [X_1, X_2]^{\mathrm{T}}$, where with probability* 1*:*

(T1) if $x_0 \in R_-$, the function $U_o(\omega, \cdot)$ takes on the value -1 for $t \in [t_0, t_s(\omega))$ and $+1$ for $t \in [t_s(\omega), t_f(\omega)]$, where $t_0 < t_s(\omega) < t_f(\omega) < \infty$ and $X(\omega, t) \in Q_+$ for $t \in [t_s(\omega), t_f(\omega))$; (for interpretation see Fig. 1);

(T2) if $x_0 \in R_+$, the function $U_o(\omega, \cdot)$ takes on the value $+1$ for $t \in [t_0, t_s(\omega))$ and -1 for $t \in [t_s(\omega), t_f(\omega)]$, where $t_0 < t_s(\omega) < t_f(\omega) < \infty$ and $X(\omega, t) \in Q_-$ for $t \in [t_s(\omega), t_f(\omega))$;

(T3) if $x_0 \in Q_-$, the function $U_o(\omega, \cdot)$ takes on the form described above in points (T1) or (T2) or takes on the value -1 for $t \in [t_0(\omega), t_f(\omega)]$, where $t_0 < t_f(\omega) < \infty$ and $X(t) \in Q_-$ for $t \in [t_0(\omega), t_f(\omega))$;

(T4) if $x_0 \in Q_+$, the function $U_o(\omega, \cdot)$ takes on the form described above in points (T1) or (T2) or takes on the value $+1$ for $t \in [t_0(\omega), t_f(\omega)]$, where $t_0 < t_f(\omega) < \infty$ and $X(t) \in Q_+$ for $t \in [t_0(\omega), t_f(\omega))$.

The functions $t_s : \Omega \to \mathbf{R}$ and $t_f : \Omega \to \mathbf{R}$ introduced above, representing the time of the changes in the value of the function $U_o(\omega, \cdot)$ and the time to reach the target by the solution $X(\omega, \cdot)$, respectively, are random variables.

The proof of the above Theorem is analogous to one for the auxiliary task of motion resistance, presented in Kulczycki (1996a, 1996b). The optimality can be shown based on the theory of differential inequalities (Kulczycki, 1996a), while the measurability of the functions t_s and t_f as well as $U_o(\cdot, t)$ and $X(\cdot, t)$ can be shown by a superposition of the corresponding mappings (Kulczycki, 1996b).

The change of sign in the particular realizations of the control U_o (switching of the control) can occur only when the system state belongs to the set Q. For this reason it will be called a switching region. Finally: the switching curve γ familiar from the classic case of the time-optimal point-to-point transfer of the fixed mass m (Athans and Falb, 1966; Chapter 7.2), has been generalized by the above to the switching region Q ($\gamma = Q$ when $m_- = m_+ = m$).

3. Implications: suboptimal control structures

Except for specific cases, direct implementation of a system generating almost certain time-optimal control encounters difficulties because of its dependence on the random factor, in fact unknown a priori. However, thanks to the results of Theorem given in Section 2, the presented material constitutes a useful basis for creation of suboptimal control laws, from which such a dependence is removed.

3.1. The hard structure

The following concept will be based on the form of differential equation (3). Namely, after its bilateral integration one may observe that the impact of the particular realizations of the stochastic process M can be estimated by using their mean-values over any interval of time in which no special event – for example control switching – occurs. To obtain a suboptimal controller, consider a particular case of the probability measure P connected with the process M (see Assumptions (A4)-(A5)) which is concentrated on constant realizations (interpreted as the average values). If the value of these constant realizations is known and equal to m, then with the notation of Theorem 2.1 presented in the previous section, $m_- = m_+ = m$, therefore, $K_{+-} = K_{++}$, and $K_{-+} = K_{--}$, hence the switching region Q is confined to the switching curve whose shape is dependent on the value of the parameter m. Denote as $\hat{m}$ its estimate used in the feedback control law; therefore, it can be interpreted as an (indefinite) knowledge about the parameter m needed for the purpose of the synthesis of the feedback controller equations.

The analysis of sensitivity to the error of the estimation of the parameter m by the value $\hat{m}$ will be presented below.

The case where the second coordinate of the target state is equal to zero, i.e. with $x_{f2} = 0$, will be considered first. If $\hat{m} = m$, the control is time-optimal; the state of the system is brought to the switching curve, and being permanently included in this curve hereafter, it reaches the target in a minimal and finite time. When $\hat{m} < m$, as a result of its having oscillations around the target, over-regulations occur in the system; the target is reached in a finite time. If $\hat{m} > m$, after the switching curve is crossed, sliding trajectories appear in the system; here, too, the target is reached in a finite time. In both of the last two cases, i.e. with $\hat{m} \neq m$, the time to reach the target state increases from the optimal more or less proportionally to the difference between the values $\hat{m}$ and m.

The remaining case, $x_{f2} \neq 0$, will now be presented. If $\hat{m} = m$, the control is time-optimal, and the phenomena are identical as before for $x_{f2} = 0$. When $\hat{m} < m$, the trajectories occurring in the system generate limit cycles; the target is not reached. Finally if $\hat{m} > m$, even though some of the trajectories temporarily diverge from the switching curve in the part between the axis x_1 and the target state, ultimately the target is reached in a finite time; sliding trajectories exist on the switching curve; the time to reach the target increases along with the growth in the difference $\hat{m} - m$.

Based on the sensitivity analysis presented above, some elements of statistical decision theory will be applied to obtain the optimal value of the estimator $\hat{m}$ needed for the purpose of the synthesis of the feedback controller equations. The basic task of statistical decision theory (Berger, 1980) is the optimal selection of one element from among all possible decisions on the sole basis of probabilistic information about the state of nature (reality), especially when its actual state is unknown. In the problem considered here, the real value of the parameter m is treated as an unknown state of reality, while the fixed value of the estimator $\hat{m}$ constitutes a decision. The loss function l is required, whose value $l(\hat{m}, m)$ is interpreted as losses resulting from making the decision $\hat{m}$ when hypothetically the value m occurs in reality. Two basic procedures are commonly used: the "flexible" Bayes rule minimizes the expected value of losses, whereas the "radical" minimax rule minimizes the greatest possible loss that may occur after a given decision is made. For details, see Berger (1980).

Assume – according to the results of the sensitivity analysis – that the loss function is described in the linear and nonsymmetrical form:

$$l(\hat{m}, m) = \begin{cases} -p(\hat{m} - m) & \text{if} \quad \hat{m} - m < 0 \\ 0 & \text{if} \quad \hat{m} - m = 0, \\ q(\hat{m} - m) & \text{if} \quad \hat{m} - m > 0 \end{cases} \tag{9}$$

where $p, q \in \mathbf{R}_+ \cup \{\infty\}$, but only one of them can be infinite. Suppose – in reference to Assumption (A5) – that the random variable characterizing the distribution of the mass m has a support of the form $[m_-, m_+]$ such that $[m_-, m_+] \subset (0, \infty)$.

It is readily shown (Kulczycki and Wisniewski, 2002) that if $p = \infty$, i.e. with infinite values of loss function (9) for $\hat{m} < m$, the minimax decision is realized by

$$\hat{m} = m_+ \, . \tag{10}$$

In turn, the Bayes decision with the positive numbers p and q, is given as a solution of the following equation with the argument $\hat{m}$:

$$F(\hat{m}) = \frac{p}{p+q} \, , \tag{11}$$

where F denotes the distribution function of the random variable characterizing the mass m. This solution is unique owing to connectivity of its support. The practical algorithm to solve equation (11) is presented in Kulczycki (2001). For this purpose, one can also use artificial neural networks, according to the procedure presented in Schiøler and Kulczycki (1997).

The results given by formulas (10) and (11) will be applied below.

Once again the case $x_{f2} = 0$ is considered first.

If over-regulations can be allowed, it is worthwhile to use the flexible Bayes rule with real values for the loss function, i.e. according to equation (11). Such a choice is possible because the determination of the estimator $\hat{m}$ value that is either less than, equal to, or greater than m allows for the system state to be brought to the target in a finite time. (However, this time increases approximately proportionally to the difference between the values $\hat{m}$ and m.)

If over-regulations are not allowed, this determination needs to be carried out on the basis of the minimax rule, assuming infinite values of the loss function for $\hat{m} < m$, i.e. using formula (11). This enables the over-regulations to be avoided, because they occur only if $\hat{m} < m$.

Assume now $x_{f2} \neq 0$.

The value of the parameter $\hat{m}$ should be determined using the minimax rule with infinite values of the loss function for $\hat{m} < m$, i.e. by dependence (10).

Such a choice guarantees that the generation of the inadmissible limit cycles, which appear when $\hat{m} < m$, is avoided. If, however, this value is greater than m, the state of the system is brought to the target in a finite time. (Note that in the case of $x_{f2} \neq 0$, the over-regulations cannot be avoided at all.) A somewhat improved structure can be obtained by dividing the switching region (curve) Q into two parts at the point of its intersection with the axis x_1. For each of them, the values of the parameter $\hat{m}$ should be determined in a different manner. Namely, in the case of the part which lies on the same side of the axis x_1 as the target state, it should be done – as previously – by using the minimax rule with infinite values of the loss function for $\hat{m} < m$, i.e. using formula (10); in the case of the part located on the opposite side, however, by the Bayes rule with real values of the loss function, i.e. according to equation (11). This change does not pose the risk that a cycle will occur, while the use of the flexible Bayes rule makes it possible to render more efficiently the potential sliding process occurring along the part of the switching curve located on the side of the axis x_1 opposite to the target.

If one has the value $\hat{m}$ obtained according to the above procedure, the feedback controller equations can be calculated. Thus, the equations of the switching curve K take on the form

$$x_1 = -\frac{\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} \quad \text{for} \quad x_2 \in (x_{f2}, \infty) \tag{12}$$

$$x_1 = \frac{\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} \quad \text{for} \quad x_2 \in (-\infty, x_{f2}) \ . \tag{13}$$

Formula (12) defines the set $K_{-+} = K_{--}$, while dependence (13) defines the set $K_{+-} = K_{++}$. In the case when, for $x_{f2} \neq 0$, the switching curve is divided into two parts at the point of its intersection with the axis x_1, the equation for the part lying on the side of this axis opposite to the target should be modified as follows:

$$x_1 = \operatorname{sgn}(x_{f2}) \left(\frac{\hat{m}_b}{2} x_2^2 - \frac{\hat{m}}{2} x_{f2}^2 \right) + x_{f1} \ , \tag{14}$$

where $\hat{m}_b$ denotes the additional estimator defining that part, obtained through Bayes rule with real values of the loss function, i.e. by equation (11). The sets R_- and R_+ constitute adequate areas resulting from the division of the plane $\mathbf{R}^2$ by the curve K, according to formulas (7)–(8). For the sets K_-, K_+, R_-, R_+ obtained in this way, the value of the suboptimal control is defined by the equation

$$u_{\text{hard}}(t) = \begin{cases} -1 & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in (R_- \cup K_-) \\ 0 & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in \{x_f\} \\ +1 & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in (R_+ \cup K_+) \end{cases} , \tag{15}$$

where $[x_1(t), x_2(t)]^{\mathrm{T}}$ means the object state, obtained by a real-time measurement process for any $t \in T$. Fig. 2 provides an illustration of the control structure worked out here with the representative trajectory it generates.

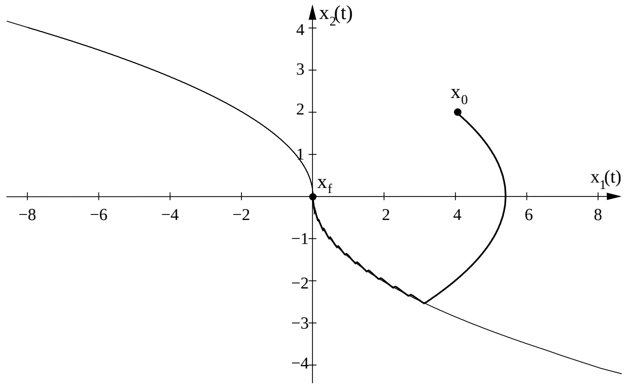

Figure 2. Hard structure (15) and the trajectories it generates for the case of $x_{f2} = 0$

3.2. The soft structure

The control designed in the previous subsection may lead to frequent switchings between the extreme – according to the assumption (A3) – values +1 and −1 along sliding trajectories, which should be avoided in mechanical systems, since it can have a negative impact on the endurance of a device and on user comfort. Based on the results of Theorem presented in Section 2 and under the condition that the control may take any value from the interval $[-1,\ 1]$, this goal can be obtained by substituting a modified control law, rendered "soft" instead of "hard" (15). A general concept of soft structures is described in Lyshevski (2001).

Let the sets K_{--} and K_{+-}, be defined as previously but for the value of the parameter $\hat{m}$ calculated in the previous section for the discontinuous structure. Let also the additional positive constant $\Delta\hat{m}$ be given and the sets K_{-+} and K_{++}, be defined for the value $\hat{m} + \Delta\hat{m}$.

As before, the case $x_{f2} = 0$ will be considered first. Let a feedback controller be as follows

$$u_{\text{soft}}(t) = \begin{cases} -1 & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in R_- \\ z(x_1(t), x_2(t)) & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in Q_- \\ 0 & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in \{x_f\} \\ z(x_1(t), x_2(t)) & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in Q_+ \\ +1 & \text{if} \quad [x_1(t), x_2(t)]^{\mathrm{T}} \in R_+ \end{cases}, \tag{16}$$

with the function $z : \mathbf{R}^2 \to \mathbf{R}$ continuously and strictly increasing from the value -1 on the sets K_{--} and K_{++} to the value $+1$ on the sets K_{-+} and K_{+-} (see also Fig. 1). If solution $X(\omega, \cdot)$ is "too close" – with respect to the real value of the mass – to the set K_{+-}, then control (16) is "too great" and it makes this solution further from the set K_{+-} to the interior of the set Q_+. And inversely, if the solution is "too far" to the set K_{+-}, then control (16) is "too small" and brings the trajectory closer to this set (see Figs. 1 and 3). The result obtained in the above manner is similar to the effect achieved on a bob-sled track thanks to the appropriate modeling of its shape. It is a smooth movement, therefore, allowing such a structure to be named "soft". An analogous situation occurs between the sets K_{-+} and K_{--}. The value of the parameter $\Delta\hat{m}$ influences the speed of the control fluctuations in the set Q: the greater the value, the milder the fluctuations. For initial investigations one can suggest $\Delta\hat{m} = \hat{m}/10$.

Having the value $\hat{m}$ obtained according to the material presented in subsection 3.1, and assuming the constant $\Delta\hat{m}$, one can calculate the equation of the set K_{+-}

$$x_1 = \frac{\hat{m}}{2} x_2^2 + x_{f1} - \varepsilon \quad \text{for} \quad x_2 \in (-\infty, 0) \tag{17}$$

and for the set K_{++}

$$x_1 = \frac{\hat{m} + \Delta\hat{m}}{2} x_2^2 + x_{f1} + \varepsilon \quad \text{for} \quad x_2 \in (-\infty, 0), \tag{18}$$

where the additional parameter $\varepsilon \geq 0$ is closer to (but is not greater than) precise positioning (i.e. assumed in practice precision of reaching the target state) and has been introduced to avoid the over-increasing of the function z near the axis x_1. The function z can be proposed in the following manner:

$$z(x_1, x_2) = a(x_2)[x_1 - c(x_2)]^d - 1 \quad \text{for} \quad x_2 \in (-\infty, 0)\ , \tag{19}$$

with

$$a(x_2) = \frac{4}{\Delta\hat{m} x_2^2 + 4\varepsilon} \tag{20}$$

$$c(x_2) = \frac{\hat{m} + \Delta\hat{m}}{2} x_2^2 + x_{f1} + \varepsilon, \tag{21}$$

while the value of the positive parameter d presents a compromise between the speed of action of the sub-time-optimal control system and its robustness. Namely, $d = 1$ can be treated as neutral; the value $d < 1$ result in making the solutions nearer to the curves K_{-+} or K_{++}, which slows down the process but increases robustness; and the inverse when $d > 1$. For the initial experimental research $d = 0.25$ is proposed.

Analogous dependencies are given for in the sets K_{--} and K_{-+}, respectively

$$x_1 = -\frac{\hat{m}}{2} x_2^2 + x_{f1} + \varepsilon \quad \text{for} \quad x_2 \in (0, \infty) \tag{22}$$

$$x_1 = -\frac{\hat{m} + \Delta\hat{m}}{2} x_2^2 + x_{f1} - \varepsilon \quad \text{for} \quad x_2 \in (0, \infty)\ . \tag{23}$$

The function z can be proposed here as

$$z(x_1, x_2) = a(x_2)[x_1 - c(x_2)]^{1/d} - 1 \quad \text{for} \quad x_2 \in (0, \infty) \ , \tag{24}$$

with

$$a(x_2) = \frac{-4}{\Delta\hat{m}x_2^2 + 4\varepsilon} \tag{25}$$

$$c(x_2) = -\frac{\hat{m}}{2}x_2^2 + x_{f1} + \varepsilon. \tag{26}$$

Let now $x_{f2} \neq 0$. The concept introduced in the preceding paragraph should be transferred here in a natural way. For simplicity of notation, the case $x_{f2} > 0$ will be investigated below; if $x_{f2} < 0$, the respective considerations are symmetrical. A feedback controller is also defined here by formula (16).

The sets K_{+-} and K_{++}, in the part between the target and the axis x_1, should be given like for the hard structure, both defined by the equation

$$x_1 = \frac{\hat{m} + \Delta\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} \quad \text{for} \quad x_2 \in [0, x_{f2}), \tag{27}$$

with

$$z(x_1, x_2) = 1 \quad \text{for} \quad x_2 \in [0, x_{f2}). \tag{28}$$

For the part lying in lower half-plane, the set K_{++} is defined by

$$x_1 = \frac{\hat{m} + \Delta\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} \quad \text{for} \quad x_2 \in (-\infty, 0) \tag{29}$$

and the set K_{+-} by

$$x_1 = \frac{\hat{m}}{2}x_2^2 - \frac{\hat{m} + \Delta\hat{m}}{2}x_{f2}^2 + x_{f1} - \varepsilon \quad \text{for} \quad x_2 \in (-\infty, 0) \ . \tag{30}$$

The function z is given as

$$z(x_1, x_2) = a(x_2)[x_1 - c(x_2)]^{d} - 1 \quad \text{for} \quad x_2 \in (-\infty, 0) \ , \tag{31}$$

with

$$a(x_2) = \frac{-4}{\Delta\hat{m}x_2^2 + 4\varepsilon} \tag{32}$$

$$c(x_2) = \frac{\hat{m} + \Delta\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} + \varepsilon. \tag{33}$$

Finally, the sets K_{--} and K_{-+} are defined by

$$x_1 = -\frac{\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} + \varepsilon \quad \text{for} \quad x_2 \in (x_{f2}, \infty) \tag{34}$$

$$x_1 = -\frac{\hat{m} + \Delta\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} - \varepsilon \quad \text{for} \quad x_2 \in (x_{f2}, \infty), \tag{35}$$

respectively, and the function z is given as

$$z(x_1, x_2) = a(x_2)[x_1 - c(x_2)]^{1/d} - 1 \quad \text{for} \quad x_2 \in (x_{f2}, \infty), \tag{36}$$

with

$$a(x_2) = \frac{-4}{\Delta\hat{m}(x_2^2 - x_{f2}^2) + 4\varepsilon} \tag{37}$$

$$c(x_2) = -\frac{\hat{m}}{2}(x_2^2 - x_{f2}^2) + x_{f1} + \varepsilon. \tag{38}$$

An illustration of the control structure thus obtained, along with the trajectories it generates, is provided in Fig. 3. Frequent switchings of the control

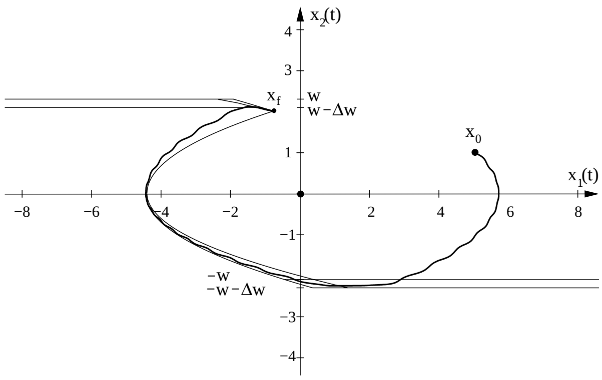

Figure 3. Soft structure (16) and the trajectories it generates for the case of $x_{f2} \neq 0$

along the sliding trajectories have been eliminated, according to the assumed goal of the soft structure. The control changes its value smoothly in full range of the interval $[-1, 1]$.

4. Final suggestions and remarks

The material presented in this paper is of a universal nature, and owing to its clear interpretation it may be easily supplemented by a number of auxiliary aspects frequently occurring in robust control tasks. As a representative example, the problem of velocity limitation, described by the condition $|X_2(\omega, t)| \leq w$ for almost every $\omega \in \Omega$ and every $t \in [t_0(\omega), t_f(\omega)]$, while $w > 0$ and $-w < x_{f2} < w$, will be investigated. Let also the auxiliary parameter Δw, such that $0 < \Delta w \leq w$ and $\Delta w - w \leq x_{f2} \leq w - \Delta w$, be introduced. By defining the function $v : \mathbf{R}^2 \to \mathbf{R}$ (similar to the function z) continuously and strictly increasing from the value −1 on the set $\mathbf{R} \times \{w\}$ to the value +1 on the set $\mathbf{R} \times \{w - \Delta w\}$, with the formula

$$v(x_1, x_2) = 2\left(\frac{w - x_2}{\Delta w}\right)^D - 1 \quad \text{for} \quad x_2 \in [w - \Delta w, w], \tag{39}$$

where the parameter $D > 0$ plays the same role as d introduced in dependence (19), one can obtain soft structure (16) supplemented with the problem of velocity limitation:

$$u_{\text{soft}}(t) = \begin{cases} -1 & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in R_- \cup \{\mathbf{R} \times (w, \infty)\} \\ v(x_1(t), x_2(t)) & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in R_+ \cap \{\mathbf{R} \times [w, -\Delta w, w]\} \\ \min\{v(x_1(t), x_2(t), z_1(t), x_2(t))\} & \\ & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in Q_- \cap \{\mathbf{R} \times [w, -\Delta w, w]\} \\ z(x_1(t), x_2(t)) & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in Q_- \cap \{\mathbf{R} \times [\Delta w - w, w - \Delta w]\} \\ 0 & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in \{x_f\} \\ z(x_1(t), x_2(t)) & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in Q_+ \cap \{\mathbf{R} \times (\Delta w - w, w - \Delta w)\} \\ \max\{-v(x_1(t), -x_2(t), z_1(t), x_2(t))\} & \\ & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in Q_+ \cap \{\mathbf{R} \times [-w, \Delta w - w]\} \\ -v(-x_1(t), -x_2(t)) & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in R_- \cap \{\mathbf{R} \times [-w, \Delta w - w]\} \\ +1 & \text{if } [x_1(t), x_2(t)]^{\text{T}} \in R_+ \cup \{\mathbf{R} \times (-\infty, -w)\}. \end{cases} \tag{40}$$

For interpretation, see Fig. 3.

The concept presented can also be applied for many other similar, auxiliary issues appearing in optimal control, e.g. modeling of motion resistance (Kulczycki, 1996a, b). As an example, consider the initial system (1) supplemented with the discontinuous model of motion resistance $\mathscr{b}\,\text{sgn}(\dot{s}(t))$, i.e.

$$m\ddot{s}(t) = u(t) - \mathscr{b}\,\text{sgn}(\dot{s}(t)), \tag{41}$$

where $\mathscr{b} \in [0, 1)$; then under- or overestimation of the value of the parameter $\mathscr{b}$ will entail similar raising or lowering of the parameter m, and further considerations are analogous to those presented above for the concepts of hard and soft controlling structures.

The correct functioning of the suboptimal structures investigated in this paper has been verified by numerical simulation. The object is a mechanical system (1) with unknown (random) and/or varying load. In the case $x_{f2} = 0$, if it is assumed that over-regulations are undesirable, then they did not occur in the controlled object. For $x_{f2} \neq 0$, limit cycles did not appear. If the Bayes rule was applied for determining the hard structure parameters, the sliding trajectories occurring there did not have frequent switches. In the case of the soft structure, sliding trajectories were eliminated.

Typical trajectories generated by control structures (15) and (16) are shown in Figs. 2 and 3. Tables 1 and 2 show times to reach the target set when $x_{f2} = 0$ and $x_{f2} \neq 0$, respectively. The results are shown for the optimal control (under the practically unrealistic assumption that the true value of the mass m is known exactly) and the suboptimal structures: hard and soft ones. It is not surprising that the shortest times to reach the target were obtained for optimal control (owing to the hypothetical assumption of an exactly known mass), followed by the hard structure (although at the cost of frequent and arduous switches on sliding trajectories), while the longest times for the soft structure are inversely proportional to the value of the parameter d. If, however, each value of m was supplemented by perturbation, with the value of $0.5m\sin(25t)$, the results favored the soft structure at small values of the parameter d, as the most robust. Note that in the case of the soft structure, the results were satisfying even when temporarily $m \notin [m_-, m_+]$.

Table 1. Times to reach the target set for $x_0 = [5,0]^T$, $x_f = [0,0]^T$, $\hat{m} = 1.5$, $\Delta\hat{m} = 0.3$

Control structures	**Optimal**	**Hard**	**Soft**		
			$d = 0.2$	$d = 1$	$d = 5$
$m = 0.6$	3.446	4.537	4.805	4.638	4.560
$m = 1.0$	4.442	4.955	5.154	5.010	4.966
$m = 1.4$	5.250	5.340	5.430	5.356	5.343

Table 2. Times to reach the target set for $x_0 = [5,0]^T$, $x_f = [2,2]^T$, $\hat{m} = 1.5$, $\Delta\hat{m} = 0.3$

Control structures	**Optimal**	**Hard**	**Soft**		
			$d = 0.2$	$d = 1$	$d = 5$
$m = 0.6$	4.3635	7.0253	7.8483	7.6609	7.5716
$m = 1.0$	6.4588	7.9402	8.7259	8.5706	8.5218
$m = 1.4$	8.4851	8.7839	9.4997	9.4164	9.4009

References

Athans, M. and Falb, P.L. (1966) *Optimal Control*. McGraw-Hill, New York.

Berger, J.O. (1980) *Statistical Decision Theory*. Springer-Verlag, New York.

Friedland, B. (1996) *Advanced Control System Design*. Prentice-Hall, Englewood Cliffs.

Isidori, A. (1995) *Nonlinear Control Systems*. Springer-Verlag, London.

Khalil, H.K. (1996) *Nonlinear Systems*. Prentice-Hall, Englewood Cliffs.

KULCZYCKI, P. (1996a) Time-Optimal Stabilization of a Discontinuous and Non-Autonomous Dynamic Object. *Control and Cybernetics* **25** (4), 707–720.

KULCZYCKI, P. (1996b) Almost Certain Time-Optimal Positional Control. *IMA Journal of Mathematical Control & Information* **13** (1), 63–77.

KULCZYCKI, P. (1996c) Some Remarks on Solutions of Discontinuous Differential Equations Applied in Automatic Control. *Industrial Mathematics* **46** (2), 119–128.

KULCZYCKI, P. (2000) Fuzzy Controller for Mechanical Systems. *IEEE Transactions on Fuzzy Systems* **8** (5), 645–652.

KULCZYCKI, P. (2001) An Algorithm for Bayes Parameter Identification. *Journal of Dynamic Systems, Measurement, and Control* **123** (4), 611–614.

KULCZYCKI, P. and WISNIEWSKI, R. (2002) Fuzzy Controller for a System with Uncertain Load. *Fuzzy Sets and Systems* **131** (2), 185–195.

KULCZYCKI, P., WISNIEWSKI, R., KOWALSKI, P. and KRAWIEC, K. (2004) Soft time-suboptimal controlling structure for mechanical systems. *4rd International Workshop on Robot Motion and Control*, Puszczykowo (Poland), 17-20 June 2004, 237–242.

LYSHEVSKI, S.E. (2001) *Control Systems Theory with Engineering Applications*. Birkhäuser, New York.

SCIAVICCO, L. and SICILIANO, B. (1996) *Modeling and Control of Robot Manipulators*. McGraw-Hill, New York.

SCHIØLER, H. and KULCZYCKI, P. (1997) Neural Network for Estimating Conditional Distributions. *IEEE Transactions on Neural Networks* **8** (5), 1015–1025.

WEINMANN, A. (1991) *Uncertain Models and Robust Control*. Springer-Verlag, Vienna.

ZHOU, K., DOYLE, J.C. and GLOVER, K. (1996) *Robust and Optimal Control*. Prentice Hall, Englewood Cliffs.