

Hemivariational inequalities governed by the p -Laplacian -
Dirichlet problem

by

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Abstract: A hemivariational inequality involving p -Laplacian is studied under the hypothesis that the nonlinear part fulfills the unilateral growth condition (Naniewicz, 1994). The existence of solutions for problems with Dirichlet boundary conditions is established by making use of Chang's version of the critical point theory for non-smooth locally Lipschitz functionals (Chang, 1981), combined with the Galerkin method. A class of problems with nonlinear potentials fulfilling the classical growth hypothesis without Ambrosetti-Rabinowitz type assumption is discussed. The approach is based on the recession technique introduced in Naniewicz (2003).

Keywords: Dirichlet problem, hemivariational inequality, unilateral growth condition, critical point theory, locally Lipschitz functional.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. The problem under consideration is as follows: Find $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} -\Delta_p u(x) & \in -\partial j(x, u(x)) \text{ a.e. on } \Omega \\ u|_{\partial\Omega} & = 0, \quad 2 \leq p < \infty, \end{cases} \quad (1)$$

where $-\Delta_p u := -\operatorname{div}(|Du|^{p-2}Du)$ stands for the p -Laplacian operator. By $\partial j(x, u)$ we denote the generalized gradient of Clarke (Clarke, 1983) of a locally Lipschitz $\mathbb{R} \ni \xi \mapsto j(x, \xi)$ (for a.e. $x \in \Omega$). For the right hand side of (1) we suppose that it satisfies the unilateral growth condition of the form (Naniewicz, 1994):

$$j^0(x; \xi, -\xi) \leq \kappa(1 + |\xi|^q), \quad \forall \xi \in \mathbb{R}, \text{ for a.e. } x \in \Omega, \quad q < p^*.$$

Thus the problem to be studied involves nonlinear, nonconvex function $j(\cdot, u)$ which is not summable for every $u \in W_0^{1,p}(\Omega)$ and consequently, the corresponding energy functional $\mathcal{R}(u) = \frac{1}{p} \|Du\|_{L^p(\Omega; \mathbb{R}^N)}^p + \int_{\Omega} j(x, u(x)) dx$ is not locally Lipschitz and has no longer the whole space $W_0^{1,p}(\Omega)$ as its effective domain. The direct use of the critical point theory developed for locally Lipschitz functionals (Chang, 1981) is therefore not available. We use the Galerkin method and solve the discretized problems in finite dimensional subspaces of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by making use of the recession technique for semicoercive problems introduced in Naniewicz (2003) and then pass to the limit to get a solution.

The class of hemivariational inequalities considered in the paper can be referred to as variational problems with discontinuities, widely studied recently. For elliptic problems with the classical growth conditions we refer to Motreanu and Panagiotopoulos (1995, 1996, 1999), Goeleven, Motreanu and Panagiotopoulos (1997), Radulescu (1993), Gasiński and Papageorgiou (2001b). Nonsmooth problems within the framework of the unilateral growth conditions can be found in Motreanu and Naniewicz (2001, 2002, 2003), Halidias and Naniewicz (2004), Naniewicz (2003) and the references quoted there. For elliptic problems involving p -Laplacian we refer to Arcoya and Orsina (1997), Bouchala and Drabek (200), Anane and Gossez (1990) (smooth potentials) and to Gasiński and Papageorgiou (2001a, c), Papalini (2002), Halidias and Naniewicz (2004) in the case of nonsmooth potentials.

The notion of hemivariational inequalities has been first introduced in the early eighties with the works of P. D. Panagiotopoulos (Panagiotopoulos, 1981, 1983). The main reason for its birth was the need for description of important problems in physics and engineering, where nonmonotone, multivalued boundary or interface conditions occur, or where some nonmonotone, multivalued relations between stress and strain, or reaction and displacement have to be taken into account. The theory of hemivariational inequalities (as the generalization of variational inequalities, see Duvaut and Lions (1972) has been proved to be very useful in the understanding of many problems of mechanics and engineering involving nonconvex, nonsmooth energy functionals. For the general study of hemivariational inequalities in both scalar and vector-valued function spaces the reader is referred to Panagiotopoulos (1985, 1993), Motreanu and Panagiotopoulos (1999), Motreanu and Naniewicz (1996), Naniewicz and Panagiotopoulos (1995) and the references quoted there.

2. Mathematical background

Let us recall some facts and definitions from the critical point theory for locally Lipschitz functionals and the generalized gradient of Clarke (Clarke, 1983).

Let Y be a subset of a Banach space X . A function $f : Y \rightarrow \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative

scalar K , one has

$$|f(y) - f(x)| \leq K\|y - x\|_X$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x , and let v be any vector in X . The generalized directional derivative of f at x in the direction v , denoted by $f^0(x; v)$, is defined as follows:

$$f^0(x; v) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \rightarrow f^0(x; v)$ is finite, positively homogeneous, subadditive and satisfies the conditions $|f^0(x; v)| \leq K\|v\|_X$ and $f^0(x; -v) = (-f)^0(x; v)$. Now we are ready to introduce the generalized gradient $\partial f(x)$ defined in Clarke (1983):

$$\partial f(x) = \{w \in X^* : f^0(x; v) \geq \langle w, v \rangle_X \text{ for all } v \in X\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly-star compact subset of X^* and $\|w\|_{X^*} \leq K$ for every w in $\partial f(x)$;

(b) For every v in X , one has

$$f^0(x; v) = \max\{\langle w, v \rangle : w \in \partial f(x)\};$$

(c) If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Let us recall the (P.S.)-condition introduced by Chang (Chang, 1981).

DEFINITION A locally Lipschitz function f is said to satisfy the Palais - Smale condition if any sequence $\{x_n\}$ along which $|f(x_n)|$ is bounded and

$$\lambda(x_n) = \min_{w \in \partial f(x_n)} \|w\|_{X^*} \rightarrow 0,$$

possesses a convergent subsequence.

Let us mention some facts about the first eigenvalue of the p -Laplacian.

Consider the first nonzero eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(\Omega))$. It is well known (see Lindqvist, 1990) that $\lambda_1 > 0$ and it is characterized by the Rayleigh quotient:

$$\lambda_1 := \inf \left\{ \frac{\|Dw\|_{L^p(\Omega; \mathbb{R}^N)}^p}{\|w\|_{L^p(\Omega)}^p} : w \in W_0^{1,p}(\Omega), w \neq 0 \right\}.$$

Each eigenfunction $w \in W_0^{1,p}(\Omega)$ corresponding to λ_1 has the properties that $\|Dw\|_{L^p(\Omega;\mathbb{R}^N)}^p = \lambda_1 \|w\|_{L^p(\Omega)}^p$ and it is a solution of the problem

$$\begin{cases} -\Delta_p w &= \lambda_1 |w|^{p-2} w \text{ a.e. on } \Omega \\ w|_{\partial\Omega} &= 0, \quad 2 \leq p < \infty. \end{cases} \quad (2)$$

Moreover, the generalized Poincaré inequality due to Fleckinger-Pellé-Tkáč (2002) holds: There exists a positive constant $c > 0$ such that:

$$\int_{\Omega} |Du|^p dx - \lambda_1 \int_{\Omega} |u|^p dx \geq c \left(|e|^{p-2} \int_{\Omega} |D\theta|^{p-2} |D\hat{u}|^2 dx + \int_{\Omega} |D\hat{u}|^p dx \right), \quad \forall u \in W_0^{1,p}(\Omega), \quad (3)$$

where θ is the λ_1 -eigenfunction and $u = e\theta + \hat{u}$ is an orthogonal decomposition of u in $L^2(\Omega)$, $e = \|\theta\|_{L^2(\Omega)}^{-2} \langle u, \theta \rangle_{L^2(\Omega)}$ and $\langle \hat{u}, \theta \rangle_{L^2(\Omega)} = 0$.

Let $f : X \rightarrow \mathbb{R}$ be a locally Lipschitz function on a Banach space. A point $x \in X$ is said to be a critical point of f if $0 \in \partial f(x)$ and $c = f(x)$ is then its critical value.

The theorems below characterize conditions under which the existence of critical points follows. They are due to Chang (Chang, 1981) and extend to a nonsmooth setting the well known classical results of the critical point theory.

THEOREM 2.1 *If a locally Lipschitz function $f : X \rightarrow \mathbb{R}$ on the reflexive Banach space X satisfies the (PS)-condition and there exist a positive constant $\rho > 0$ and $e \in X$ with $\|e\| > \rho$ such that*

$$\max\{f(0), f(e)\} < \inf_{\|x\|=\rho} \{f(x)\},$$

then f has a critical point $u \in X$ with its critical value $c = f(u)$ characterized by

$$c = \inf_{g \in G} \max_{t \in [0,1]} f(g(t))$$

where

$$G = \{g \in C([0,1], X) : g(0) = 0, g(1) = e\}.$$

THEOREM 2.2 *Suppose that a locally Lipschitz function $f : X \rightarrow \mathbb{R}$ satisfies the (PS)-condition and is bounded from below. Then $c = \inf_X \{f(x)\}$ is a critical value of f .*

3. Auxiliary results

Let us denote by $V_0 = \{s\theta\}_{s \in \mathbb{R}}$ the one-dimensional eigenspace spanned by the eigenfunction θ corresponding to the first eigenvalue λ_1 of $(-\Delta_p, W_0^{1,p}(\Omega))$.

It is well known (see Anane, 1988, Lindqvist, 1990) that $\theta \in L^\infty(\Omega)$, θ does not change its sign in Ω . Therefore one can normalize the eigenfunction by assuming that $\theta > 0$ a.e. in Ω and $\|\theta\|_{W_0^{1,p}(\Omega)} = 1$. By V^\perp we denote the orthogonal complement of V_0 in $L^2(\Omega)$. Accordingly, for any $u \in W_0^{1,p}(\Omega)$ the decomposition follows

$$u = e\bar{\theta} + \hat{u} \quad \text{with} \quad e \geq 0, \bar{\theta} \in \{\pm\theta\} \subset V_0, \hat{u} \in \hat{V}, \tag{4}$$

where $\hat{V} := V^\perp \cap W_0^{1,p}(\Omega)$, and by (3) we have the inequality

$$\int_\Omega |Du|^p dx - \lambda_1 \int_\Omega |u|^p dx \geq c \int_\Omega |D\hat{u}|^p dx, \quad \forall u \in W_0^{1,p}(\Omega). \tag{5}$$

LEMMA 3.1 *Assume that*

(H1) $j(\cdot, 0) \in L^1(\Omega)$ and $j(x, \cdot)$ is Lipschitz continuous on the bounded subsets of \mathbb{R} uniformly with respect to $x \in \Omega$, i.e., $\forall r > 0 \exists K_r > 0$ such that $\forall |y_1|, |y_2| \leq r$,

$$|j(x, y_1) - j(x, y_2)| \leq K_r |y_1 - y_2|, \quad \text{for a.e. } x \in \Omega;$$

(H2) *One of the two conditions below holds (the Ambrosetti-Rabinowitz type conditions):*

(i) *There exist $\mu > p$, $1 \leq \sigma < p$, $a \in L^1(\Omega)$ and a constant $k \geq 0$ such that*

$$\mu j(x, \xi) - j^0(x, \xi; \xi) + \lambda_1 \frac{\mu - p}{p} |\xi|^p \geq -a(x) - k|\xi|^\sigma, \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(ii) *There exist $0 < \nu < p$, $1 \leq \sigma < p$, $a \in L^1(\Omega)$ and a constant $k \geq 0$ such that*

$$-\nu j(x, \xi) - j^0(x, \xi; -\xi) + \lambda_1 \frac{p - \nu}{p} |\xi|^p \geq -a(x) - k|\xi|^\sigma, \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(H3) *Suppose that*

$$J^\infty(\bar{\theta}) + \lambda_1 \int_\Omega |\bar{\theta}|^p dx > 0 \quad \text{for each } \bar{\theta} \in \{\pm\theta\},$$

where

$$J^\infty(\bar{\theta}) := \liminf_{\substack{t \rightarrow +\infty \\ \eta \xrightarrow{L^p(\Omega)} \bar{\theta}}} \frac{1}{t^{p-1}} \int_\Omega -j^0(x, t\eta(x); -\bar{\theta}(x)) dx, \quad \bar{\theta} \in \{\pm\theta\},$$

is the recession function of nonconvex, nonsmooth $J(\cdot) = \int_\Omega j(x, \cdot) dx$ (see Naniewicz, 2003, and also Goeleven and Théra, 1995, Baiocchi, Buttazzo, Gastaldi and Tomarelli, 1988).

Moreover, suppose that for a sequence $\{u_n\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ there exists $\varepsilon_n \searrow 0$ such that the conditions below are fulfilled:

$$\begin{aligned} & \int_{\Omega} |Du_n(x)|^{p-2} \langle Du_n(x), Dv(x) - Du_n(x) \rangle_{\mathbb{R}^N} dx \\ & + \int_{\Omega} j^0(x, u_n(x); v(x) - u_n(x)) dx \geq -\varepsilon_n \|v - u_n\|_{W_0^{1,p}(\Omega)}, \\ & \forall v \in \text{Lin}(\{u_n, \theta\}), \end{aligned} \quad (6)$$

and

$$\left| \frac{1}{p} \int_{\Omega} |Du_n(x)|^p dx + \int_{\Omega} j(x, u_n(x)) dx \right| \leq C, \quad C > 0. \quad (7)$$

where $\text{Lin}(\{u_n, \theta\})$ is the linear subspace of $W_0^{1,p}(\Omega)$ spanned by $\{\theta, u_n\}$. Then the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, i.e. there exists $M > 0$ such that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq M. \quad (8)$$

Proof. Suppose on the contrary that the claim is not true, i.e. there exists a sequence $\{u_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\|u_n\|_{W_0^{1,p}(\Omega)} \rightarrow \infty$ for which (6) and (7) hold. Under (H2)_(i) combining (7) multiplied by $\mu > p$ with (6) (with $v = 2u_n$ substituted) yields

$$\begin{aligned} \mu C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} & \geq \frac{\mu-p}{p} \left(\|Du_n\|_{L^p(\Omega; \mathbb{R}^N)}^p - \lambda_1 \|u_n\|_{L^p(\Omega)}^p \right) \\ & + \int_{\Omega} \left(\mu j(u_n) - j^0(u_n; u_n) + \lambda_1 \frac{\mu-p}{p} |u_n|^p \right) dx. \end{aligned} \quad (9)$$

Taking into account the decomposition $u_n = e_n \theta_n + \hat{u}_n$, where $\hat{u}_n \in \hat{V}$, $e_n \geq 0$, $\theta_n \in \{\pm\theta\}$, $\|\theta\|_{W_0^{1,p}(\Omega)} = 1$, by (5) and (H2)_(i) we have

$$\mu C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} \geq c \frac{\mu-p}{p} \|D(\hat{u}_n)\|_{L^p(\Omega; \mathbb{R}^N)}^p - c_1 \|\hat{u}_n + e_n \theta_n\|_{L^p(\Omega)}^\sigma - \|a\|_{L^1(\Omega)}. \quad (10)$$

Hence

$$\begin{aligned} \mu C + \varepsilon_n (\|\hat{u}_n\|_{W_0^{1,p}(\Omega)} + e_n) & \geq c \frac{\mu-p}{p} \|D\hat{u}_n\|_{L^p(\Omega; \mathbb{R}^N)}^p \\ & - c_2 \|\hat{u}_n\|_{W_0^{1,p}(\Omega)}^\sigma - c_3 e_n^\sigma - \|a\|_{L^1(\Omega)}. \end{aligned} \quad (11)$$

Thus, it follows that $e_n \rightarrow \infty$ because otherwise we would get the boundedness of $\{\hat{u}_n\}$ and consequently the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$, contrary to our supposition. Dividing this inequality by e_n yields

$$\begin{aligned} \frac{\mu C}{e_n} + \varepsilon_n \left(\frac{\|\hat{u}_n\|_{W_0^{1,p}(\Omega)}}{e_n} + 1 \right) & \geq e_n^{p-1} c \frac{\mu-p}{p} \|D(\frac{\hat{u}_n}{e_n})\|_{L^p(\Omega; \mathbb{R}^N)}^p \\ & - e_n^{\sigma-1} c_2 \frac{\|\hat{u}_n\|_{W_0^{1,p}(\Omega)}^\sigma}{e_n} - e_n^{\sigma-1} c_3 - \frac{\|a\|_{L^1(\Omega)}}{e_n}. \end{aligned} \quad (12)$$

which means that $\{\frac{\widehat{u}_n}{e_n}\}$ is bounded in $W_0^{1,p}(\Omega)$. Further, in view of $\sigma < p$ and $e_n \rightarrow \infty$, this leads to the conclusion that

$$\|\frac{\widehat{u}_n}{e_n}\|_{W_0^{1,p}(\Omega)} \rightarrow 0. \quad (13)$$

Now let us turn back to (6). By passing to a subsequence one can suppose also that $\theta_n = \theta$ (or $\theta_n = -\theta$). Thus, substituting $v = \widehat{u}_n$ into (6) yields

$$\begin{aligned} e_n^p \int_{\Omega} |D(\frac{\widehat{u}_n}{e_n}) + D\theta|^{p-2} \langle D(\frac{\widehat{u}_n}{e_n}) + D\theta, -D\theta \rangle_{\mathbb{R}^N} dx \\ + e_n \int_{\Omega} j^0(e_n(\frac{\widehat{u}_n}{e_n} + \theta); -\theta) dx \geq -\varepsilon_n e_n. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\varepsilon_n}{e_n^{p-1}} \geq \int_{\Omega} \left[-\frac{1}{e_n^{p-1}} j^0(e_n(\frac{\widehat{u}_n}{e_n} + \theta); -\theta) - \lambda_1 |\frac{\widehat{u}_n}{e_n} + \theta|^{p-2} (\frac{\widehat{u}_n}{e_n} + \theta)(-\theta) \right] dx \\ + \int_{\Omega} |D(\frac{\widehat{u}_n}{e_n}) + D\theta|^{p-2} \langle D(\frac{\widehat{u}_n}{e_n}) + D\theta, D\theta \rangle_{\mathbb{R}^N} dx \\ - \lambda_1 \int_{\Omega} |\frac{\widehat{u}_n}{e_n} + \theta|^{p-2} (\frac{\widehat{u}_n}{e_n} + \theta)\theta dx. \end{aligned}$$

Now we are ready to pass to the limit with $n \rightarrow \infty$. For this purpose notice that in view of (13) there is

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\{ \int_{\Omega} |D(\frac{\widehat{u}_n}{e_n}) + D\theta|^{p-2} \langle D(\frac{\widehat{u}_n}{e_n}) + D\theta, D\theta \rangle_{\mathbb{R}^N} dx \right. \\ \left. - \lambda_1 \int_{\Omega} |\frac{\widehat{u}_n}{e_n} + \theta|^{p-2} (\frac{\widehat{u}_n}{e_n} + \theta)\theta dx \right\} = \|D\theta\|_{L^p(\Omega; \mathbb{R}^N)}^p - \lambda_1 \|\theta\|_{L^p(\Omega)}^p = 0. \end{aligned}$$

Accordingly, we arrive at

$$0 \geq \limsup_{n \rightarrow \infty} \frac{1}{e_n^{p-1}} \int_{\Omega} -j^0(e_n(\frac{\widehat{u}_n}{e_n} + \theta); -\theta) dx + \lambda_1 \int_{\Omega} |\theta|^p dx \geq J^\infty(\theta) + \lambda_1 \int_{\Omega} |\theta|^p dx.$$

But this contradicts (H3).

Under (H2)_(ii), combining (7) multiplied by $\nu < p$ with (6) (with $v = 0$ substituted) yields

$$\begin{aligned} \nu C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} \geq \frac{p-\nu}{p} \left(\|Du_n\|_{L^p(\Omega; \mathbb{R}^N)}^p - \lambda_1 \|u_n\|_{L^p(\Omega)}^p \right) \\ + \int_{\Omega} \left(-\nu j(u_n) - j^0(u_n; -u_n) + \lambda_1 \frac{p-\nu}{p} |u_n|^p \right) dx. \end{aligned} \quad (14)$$

Now we can proceed as previously to establish the result. The proof of Lemma 3.1 is complete. \blacksquare

LEMMA 3.2 Assume that (H1) and the hypotheses below hold:

(H4) The unilateral growth condition (Naniewicz, 1994): There exist $p < q < p^* = \frac{Np}{N-p}$, and a constant $\kappa \geq 0$ such that

$$j^0(x, \xi; -\xi) \leq \kappa(1 + |\xi|^q), \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(H5) Uniformly for a.e. $x \in \Omega$,

$$\liminf_{\xi \rightarrow 0} \frac{pj(x, \xi)}{|\xi|^p} \geq \phi(x) \geq -\lambda_1,$$

with $\phi(x) \in L^\infty(\Omega)$ and $\phi(x) > -\lambda_1$ on a set of positive measure.

Then there exists $\rho > 0$ such that

$$\mathcal{R}(u) := \frac{1}{p} \|Du\|_{L^p(\Omega; \mathbb{R}^N)}^p + \int_{\Omega} j(u) dx \geq \eta, \quad \eta = \text{const} > 0, \quad (15)$$

is valid for any $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega)} = \rho$.

Proof. Suppose the assertion is not true. Thus there exist sequences $\{u_n\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $\rho_n \searrow 0$ such that $\|u_n\|_{W_0^{1,p}(\Omega)} = \rho_n$ and $\mathcal{R}(u_n) \leq \rho_n^{p+1}$. So we have

$$\|Du_n\|_{L^p(\Omega; \mathbb{R}^N)}^p + \int_{\Omega} pj(u_n) dx \leq p\rho_n^{p+1}. \quad (16)$$

Further, from (H5) it follows that for any $\varepsilon > 0$, uniformly for all $x \in \Omega$ one can find $\delta > 0$ such that

$$pj(x, \xi) \geq \phi(x)|\xi|^p - \varepsilon|\xi|^p, \quad |\xi| \leq \delta.$$

Moreover, (H4) allows to conclude that (see Lemma 2.1, pp. 119-120, Naniewicz, 1997):

$$j(x, \xi) \geq -\kappa_0(1 + |\xi|^q), \quad \forall \xi \in \mathbb{R}, \text{ for a.e. } x \in \Omega, \quad \kappa_0 = \text{const} > 0. \quad (17)$$

Thus it is easy to see that

$$pj(x, \xi) \geq (\phi(x) - \varepsilon)|\xi|^p - \gamma|\xi|^q, \quad \forall \xi \in \mathbb{R}, \quad (18)$$

for some positive $\gamma = \gamma(\delta) > 0$. Then by (16) it follows

$$\begin{aligned} & \|Du_n\|_{L^p(\Omega; \mathbb{R}^N)}^p - \lambda_1 \|u_n\|_{L^p(\Omega)}^p + \int_{\Omega} [(\phi(x) - \varepsilon)|u_n(x)|^p + \lambda_1 |u_n(x)|^p] dx \\ & \leq p\rho_n^{p+1} + \gamma \int_{\Omega} |u_n(x)|^q dx. \end{aligned} \quad (19)$$

Since $W_0^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ we have

$$\begin{aligned} \|Du_n\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|u_n\|_{L^p(\Omega)}^p + \int_{\Omega} (\phi(x) + \lambda_1 - \varepsilon) |u_n(x)|^p dx \\ \leq p\rho_n^{p+1} + \gamma_1 \|u_n\|_{W_0^{1,p}(\Omega)}^q, \quad \gamma_1 = \text{const} > 0. \end{aligned} \tag{20}$$

Dividing inequality (20) by ρ_n^p yields

$$\begin{aligned} \|Dy_n\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|y_n\|_{L^p(\Omega)}^p + \int_{\Omega} (\phi(x) + \lambda_1 - \varepsilon) |y_n(x)|^p dx \\ \leq p\rho_n + \gamma_1 \rho_n^{q-p}. \end{aligned} \tag{21}$$

The norms $\|D(\cdot)\|_{L^p(\Omega;\mathbb{R}^N)}$ and $\|\cdot\|_{W_0^{1,p}(\Omega)}$ are equivalent on $W_0^{1,p}(\Omega)$ and $\|y_n\|_{W_0^{1,p}(\Omega)} = 1$. Therefore we can suppose that for a subsequence (again denoted by the same symbol) $y_n \rightharpoonup y$ weakly in $W_0^{1,p}(\Omega)$ and $y_n \rightarrow y$ strongly in $L^p(\Omega)$ (the Rellich theorem) for some $y \in W_0^{1,p}(\Omega)$. Passing to the limit and the weak lower semicontinuity of the norm allow the conclusion

$$\|Dy\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|y\|_{L^p(\Omega)}^p + \int_{\Omega} (\phi(x) + \lambda_1 - \varepsilon) |y(x)|^p dx \leq 0, \tag{22}$$

which is valid for an arbitrary $\varepsilon > 0$. Therefore we get

$$\|Dy\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|y\|_{L^p(\Omega)}^p + \int_{\Omega} (\phi(x) + \lambda_1) |y(x)|^p dx \leq 0. \tag{23}$$

Application of the Rayleigh quotient characterization of λ_1 and (H5) leads to the equalities

$$\|Dy\|_{L^p(\Omega;\mathbb{R}^N)}^p = \lambda_1 \|y\|_{L^p(\Omega)}^p, \tag{24}$$

$$\int_{\Omega} (\phi(x) + \lambda_1) |y(x)|^p dx = 0. \tag{25}$$

Now we show that $y \neq 0$. Indeed, from the results obtained it follows that

$$\|Dy_n\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|y_n\|_{L^p(\Omega)}^p \rightarrow 0$$

and by the compactness of the imbedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$ we get

$$\|y_n\|_{L^p(\Omega)} \rightarrow \|y\|_{L^p(\Omega)}.$$

Since $\|Dy_n\|_{L^p(\Omega;\mathbb{R}^N)} \geq c \|y_n\|_{W_0^{1,p}(\Omega)} = c$, $c > 0$ (the equivalence of the norms), we arrive at $\lambda_1 \|y\|_{L^p(\Omega)}^p \geq c^p$ which establishes the assertion. Therefore, taking into account (24) we conclude that $y \neq 0$ is an λ_1 -eigenfunction. Since $\phi(x) > -\lambda_1$ on a set of positive measure (by (H5)) and $|y(x)| > 0$ for a.e. $x \in \Omega$

(see Lindqvist, 1990), we are led to the contradiction with (25). The proof of Lemma 3.2 is complete. \blacksquare

If we strengthen the hypotheses (H4) and (H5) as shown below then the statements of Lemma 3.1 and Lemma 3.2 still hold true. This is the case when the Ambrosetti-Rabinowitz type conditions $(H2)_{(i)}$ and $(H2)_{(ii)}$ are redundant.

LEMMA 3.3 *Assume the hypotheses (H1), (H3). Moreover, assume the following:*

(H4)₁ *The classical growth condition: There exists a constant $\kappa > 0$ such that*

$$|\partial_\xi j(x, \xi)| \leq \kappa(1 + |\xi|^{p-1}), \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(H5)₁ *The inequality holds:*

$$pj(x, \xi) \geq \phi(x)|\xi|^p, \quad \forall \xi \in \mathbb{R} \text{ and a.e. } x \in \Omega,$$

with $\phi(x) \in L^\infty(\Omega)$ and $\phi(x) > -\lambda_1$ a.e. in Ω .

Moreover, suppose that for a sequence $\{u_n\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ there exists $\varepsilon_n \searrow 0$ such that the conditions below are fulfilled:

$$\begin{aligned} & \int_{\Omega} |Du_n(x)|^{p-2} \langle Du_n(x), Dv(x) - Du_n(x) \rangle_{\mathbb{R}^N} dx \\ & + \int_{\Omega} j^0(x, u_n(x); v(x) - u_n(x)) dx \geq -\varepsilon_n \|v - u_n\|_{W_0^{1,p}(\Omega)}, \\ & \forall v \in \text{Lin}(\{u_n, \theta\}), \end{aligned} \quad (26)$$

and

$$\left| \frac{1}{p} \int_{\Omega} |Du_n(x)|^p dx + \int_{\Omega} j(x, u_n(x)) dx \right| \leq C, \quad C > 0. \quad (27)$$

where $\text{Lin}(\{u_n, \theta\})$ is the linear subspace of $W_0^{1,p}(\Omega)$ spanned by $\{\theta, u_n\}$. Then the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, i.e. there exists $M > 0$ such that

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq M. \quad (28)$$

Moreover, there exists $\rho > 0$ such that

$$\mathcal{R}(u) := \frac{1}{p} \|Du\|_{L^p(\Omega; \mathbb{R}^N)}^p + \int_{\Omega} j(u) dx \geq \eta, \quad \eta = \text{const} > 0, \quad (29)$$

is valid for any $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{W_0^{1,p}(\Omega)} = \rho$.

Proof. Let us begin with (28). Suppose on the contrary that the claim is not true, i.e. there exists a sequence $\{u_n\}_{n=1}^\infty \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with

$\|u_n\|_{W_0^{1,p}(\Omega)} \rightarrow \infty$ for which (26) and (27) hold. Combining (27) multiplied by any $\mu > p$ with (26) (with $v = 2u_n$ substituted) yields

$$\begin{aligned} \mu C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} &\geq \frac{\mu-p}{p} \left(\|Du_n\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|u_n\|_{L^p(\Omega)}^p \right) \\ &+ \int_{\Omega} \left(\mu j(u_n) - j^0(u_n; u_n) + \lambda_1 \frac{\mu-p}{p} |u_n|^p \right) dx. \end{aligned} \quad (30)$$

In view of $(H4)_1$, $-j^0(u_n; u_n) \geq -k|u_n|^p - k$ for some $k > 0$, so by (5) we obtain

$$\begin{aligned} \mu C + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} &\geq c \frac{\mu-p}{p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &+ \int_{\Omega} \left(\mu j(u_n) - j^0(u_n; u_n) + \lambda_1 \frac{\mu-p}{p} |u_n|^p \right) dx \\ &\geq c \frac{\mu-p}{p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p + \int_{\Omega} \left[\frac{\mu}{p} (\phi + \lambda_1) - \lambda_1 - k \right] |u_n|^p dx - k|\Omega|. \\ &\geq c \frac{\mu-p}{2p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p + c \frac{\mu-p}{2p} \left(\|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p - \lambda_1 \|\widehat{u}_n\|_{L^p(\Omega)}^p \right) \\ &+ \int_{\Omega} \left(\left[\frac{\mu}{p} (\phi + \lambda_1) - \lambda_1 - k \right] |u_n|^p + c \frac{\mu-p}{2p} \lambda_1 |\widehat{u}_n|^p \right) dx - k|\Omega|, \end{aligned}$$

$|\Omega|$ being the Lebesgue measure of Ω . Now we state the estimate that will be useful for our further investigations:

$$\begin{aligned} \mu C + k|\Omega| + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} &\geq c \frac{\mu-p}{2p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &+ \int_{\Omega} \left(\left[\frac{\mu}{p} (\phi + \lambda_1) - \lambda_1 - k \right] |u_n|^p + c \frac{\mu-p}{2p} \lambda_1 |\widehat{u}_n|^p \right) dx \\ &\geq c \frac{\mu-p}{2p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &- \int_{\Omega_{\mu}^-} \left(\left[\lambda_1 + k - \frac{\mu}{p} (\phi + \lambda_1) \right] |u_n|^p dx + c \frac{\mu-p}{2p} \lambda_1 \int_{\Omega_{\mu}^-} |\widehat{u}_n|^p \right) dx, \end{aligned} \quad (31)$$

where $\Omega_{\mu}^- := \{x \in \Omega : \frac{\mu}{p}(\phi + \lambda_1) < \lambda_1 + k\}$. Notice that due to $\phi + \lambda_1 > 0$ a.e. in Ω it follows that $|\Omega_{\mu}^-| \rightarrow 0$ as $\mu \rightarrow +\infty$. This (by $|a \pm b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, $a, b \in \mathbb{R}$) implies

$$\begin{aligned} \mu C + k|\Omega| + \varepsilon_n \|u_n\|_{W_0^{1,p}(\Omega)} &\geq c \frac{\mu-p}{2p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &+ \int_{\Omega_{\mu}^-} \left[c \frac{\mu-p}{2p} \lambda_1 + 2^{p-1} \left(\frac{\mu}{p} (\phi + \lambda_1) - \lambda_1 - k \right) \right] |\widehat{u}_n|^p dx \\ &- e_n^p 2^{p-1} \int_{\Omega_{\mu}^-} \left(\lambda_1 + k - \frac{\mu}{p} (\phi + \lambda_1) \right) |\theta|^p dx. \end{aligned}$$

Since $\phi + \lambda_1 > 0$, one can choose μ large enough to get

$$c \frac{\mu-p}{2p} \lambda_1 + 2^{p-1} \left(\frac{\mu}{p} (\phi + \lambda_1) - \lambda_1 - k \right) \geq 0. \quad (32)$$

This yields the estimate

$$\begin{aligned} \mu C + k|\Omega| + \varepsilon_n (\|\widehat{u}_n\|_{W_0^{1,p}(\Omega)} + e_n) &\geq c \frac{\mu-p}{2p} \|D(\widehat{u}_n)\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &\quad - e_n^p \int_{\Omega_\mu^-} (\lambda_1 + k - \frac{\mu}{p}(\phi + \lambda_1)) |\theta|^p dx, \end{aligned}$$

which allows the conclusion that $e_n \rightarrow \infty$. Otherwise, we would have the boundedness of $\{\widehat{u}_n\}$ and consequently the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$, contrary to our supposition. Dividing (31) by e_n^p yields

$$\begin{aligned} \frac{\mu C + k|\Omega|}{e_n^p} + \frac{\varepsilon_n}{e_n^{p-1}} (\|\frac{\widehat{u}_n}{e_n}\|_{W_0^{1,p}(\Omega)} + 1) &\geq c \frac{\mu-p}{2p} \|D(\frac{\widehat{u}_n}{e_n})\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &\quad - \int_{\Omega_\mu^-} \left([\lambda_1 + k - \frac{\mu}{p}(\phi + \lambda_1)] |\frac{\widehat{u}_n}{e_n} + \theta_n|^p \right) dx + c \frac{\mu-p}{2p} \lambda_1 \int_{\Omega_\mu^-} |\frac{\widehat{u}_n}{e_n}|^p dx. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\mu C + k|\Omega|}{e_n^p} + \frac{\varepsilon_n}{e_n^{p-1}} (\|\frac{\widehat{u}_n}{e_n}\|_{W_0^{1,p}(\Omega)} + 1) &\geq c \frac{\mu-p}{2p} \|D(\frac{\widehat{u}_n}{e_n})\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &\quad + \int_{\Omega_\mu^-} [2^{p-1}(-\lambda_1 - k + \frac{\mu}{p}(\phi + \lambda_1)) + c \frac{\mu-p}{2p} \lambda_1] |\frac{\widehat{u}_n}{e_n}|^p dx \\ &\quad - 2^{p-1} \int_{\Omega_\mu^-} \left([\lambda_1 + k - \frac{\mu}{p}(\phi + \lambda_1)] |\theta|^p \right) dx. \end{aligned}$$

Choosing μ like in (32) gives rise to

$$\begin{aligned} \frac{\mu C + k|\Omega|}{e_n^p} + \frac{\varepsilon_n}{e_n^{p-1}} (\|\frac{\widehat{u}_n}{e_n}\|_{W_0^{1,p}(\Omega)} + 1) &\geq c \frac{\mu-p}{2p} \|D(\frac{\widehat{u}_n}{e_n})\|_{L^p(\Omega;\mathbb{R}^N)}^p \\ &\quad - 2^{p-1} \int_{\Omega_\mu^-} \left([\lambda_1 + k - \frac{\mu}{p}(\phi + \lambda_1)] |\theta|^p \right) dx, \end{aligned}$$

which means that $\{\frac{\widehat{u}_n}{e_n}\}$ is bounded in $W_0^{1,p}(\Omega)$. We claim that, in fact, $\frac{\widehat{u}_n}{e_n} \rightarrow 0$ strongly in $W_0^{1,p}(\Omega)$. Indeed, because $|\Omega_\mu^-| \rightarrow 0$ as $\mu \rightarrow +\infty$, for an arbitrary $\varepsilon > 0$ one can choose μ sufficiently large, say $\mu \geq \mu_0$, so that

$$2^{p-1} \int_{\Omega_\mu^-} [\lambda_1 + k - \frac{\mu}{p}(\phi + \lambda_1)] |\theta|^p dx \leq \frac{\varepsilon}{2}.$$

For such a μ one can find n_0 large enough to fulfill

$$\frac{\mu C + k|\Omega|}{e_n^p} + \frac{\varepsilon_n}{e_n^{p-1}} (\|\frac{\widehat{u}_n}{e_n}\|_{W_0^{1,p}(\Omega)} + 1) \leq \frac{\varepsilon}{2}, \quad n \geq n_0.$$

Therefore we are led to the estimate

$$\varepsilon \geq c \frac{\mu-p}{2p} \|D(\frac{\widehat{u}_n}{e_n})\|_{L^p(\Omega;\mathbb{R}^N)}^p \geq c \|D(\frac{\widehat{u}_n}{e_n})\|_{L^p(\Omega;\mathbb{R}^N)}^p, \quad \mu \geq \max\{3p, \mu_0\}, \quad n \geq n_0,$$

which establishes the strong convergence $\frac{\widehat{u}_n}{e_n} \rightarrow 0$ in $W_0^{1,p}(\Omega)$. Proceeding like in the proof of Lemma 3.1 we get (28). For (29) it is sufficient to invoke Lemma 3.2. \blacksquare

From now on we shall assume the hypothesis:

(H6) $\int_{\Omega} j(x, 0) dx \leq 0$ and there exists $e \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $e \neq 0$, such that

$$\mathcal{R}(se) \leq 0, \quad \forall s \geq 1.$$

LEMMA 3.4 *Assume that (H1)-(H2) are satisfied, $\int_{\Omega} j(x, 0) dx \leq 0$ and for some $\bar{\theta} \in V_0$, $\bar{\theta} \neq 0$,*

$$\liminf_{s \rightarrow +\infty} \int_{\Omega} j(x, s\bar{\theta}(x)) + \frac{\lambda_1}{p} s^p |\bar{\theta}(x)|^p dx < 0. \tag{33}$$

Then (H6) holds.

Proof. The assertion easily holds for $e = s_0 \bar{\theta}$ with sufficiently large $s_0 > 0$. ■

LEMMA 3.5 *Assume that (H1) is fulfilled and instead of (H2) the stronger hypothesis is satisfied:*

(H2)' *One of the two conditions below holds (the Ambrosetti-Rabinowitz conditions):*

(i) *There exist $\mu > p$, $1 \leq \sigma < p$, $a \in L^1(\Omega)$ and a constant $k \geq 0$ such that*

$$\mu j(x, \xi) - j^0(x, \xi; \xi) \geq -a(x) - k|\xi|^\sigma, \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

(ii) *There exist $0 < \nu < p$, $1 \leq \sigma < p$, $a \in L^1(\Omega)$ and a constant $k \geq 0$ such that*

$$-\nu j(x, \xi) - j^0(x, \xi; -\xi) \geq -a(x) - k|\xi|^\sigma, \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega,$$

Moreover, assume that $\int_{\Omega} j(x, 0) dx \leq 0$ and for some $v_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (Motreanu and Panagiotopoulos, 1999),

$$\liminf_{s \rightarrow +\infty} s^{-\sigma} \int_{\Omega} j(x, sv_0(x)) dx < \frac{k}{\sigma - \mu} \|v_0\|_{L^\sigma(\Omega)}^\sigma, \tag{34}$$

with the positive constants k, μ, σ entering (H2). Then (H6) holds.

Proof. We follow the lines of Motreanu and Panagiotopoulos (1999). For all $\tau \neq 0$, $x \in \Omega$ and $\xi \in \mathbb{R}$, the formula below of generalized gradient (with respect to τ) holds

$$\partial_\tau(\tau^{-\mu} j(x, \tau\xi)) = \tau^{-\mu-1} [-\mu j(x, \tau\xi) + \partial_\xi j(x, \tau\xi)(\tau\xi)],$$

for the constant $\mu > p$ fulfilling (H2). Since the function $\tau \mapsto \tau^{-\mu} j(x, \tau\xi)$ is differentiable a.e. on \mathbb{R} , the equality above and a classical property of Clarke's generalized directional derivative imply that

$$\begin{aligned} t^{-\mu} j(x, t\xi) - j(x, \xi) &= \int_1^t \frac{d}{d\tau} (\tau^{-\mu} j(x, \tau\xi)) d\tau \\ &\leq \int_1^t \tau^{-\mu-1} [-\mu j(x, \tau\xi) + j^0(x, \tau\xi; \tau\xi)] d\tau, \quad \forall t > 1, \text{ a.e. } x \in \Omega, \xi \in \mathbb{R}. \end{aligned}$$

In view of assumption (H2) we infer that

$$\begin{aligned}
t^{-\mu}j(x, t\xi) - j(x, \xi) &\leq \int_1^t \tau^{-\mu-1} [a(x) + k\tau^\sigma |\xi|^\sigma] d\tau \\
&= \left[a(x) \left(-\frac{1}{\mu} t^{-\mu} + \frac{1}{\mu} \right) + k|\xi|^\sigma \left(\frac{1}{\sigma-\mu} t^{\sigma-\mu} - \frac{1}{\sigma-\mu} \right) \right] \\
&\leq \mu^{-1}a(x) + (\mu-\sigma)^{-1}k|\xi|^\sigma, \quad \forall t > 1, \text{ a.e. } x \in \Omega, \xi \in \mathbb{R}.
\end{aligned} \tag{35}$$

Set $\xi = sv_0(x)$ with $x \in \Omega$ and $s > 0$. We find from (35) the estimate

$$\begin{aligned}
j(x, tsv_0(x)) &\leq t^\mu [j(x, sv_0(x)) + \mu^{-1}a(x) \\
&\quad + (\mu-\sigma)^{-1}ks^\sigma |v_0(x)|^\sigma], \quad \forall t > 1, s > 0, \text{ a.e. } x \in \Omega.
\end{aligned} \tag{36}$$

Combining (36) with (34) yields

$$\begin{aligned}
\mathcal{R}(tsv_0) &\leq \frac{1}{p}t^p s^p \|Dv_0\|_{L^p(\Omega; \mathbb{R}^N)}^p \\
&\quad + t^\mu s^\sigma \left[s^{-\sigma} \int_\Omega j(x, sv_0(x)) dx + k(\mu-\sigma)^{-1} \|v_0\|_{L^\sigma(\Omega)}^\sigma + s^{-\sigma} \mu^{-1} \|a\|_{L^1(\Omega)} \right], \\
\forall t > 1, s > 0.
\end{aligned} \tag{37}$$

Assumption (34) allows to fix some number $s_0 > 0$ such that

$$s_0^{-\sigma} \int_\Omega j(x, s_0 v_0(x)) dx + k(\mu-\sigma)^{-1} \|v_0\|_{L^\sigma(\Omega)}^\sigma + s_0^{-\sigma} \mu^{-1} \|a\|_{L^1(\Omega)} < 0. \tag{38}$$

With such an $s_0 > 0$ we can pass to the limit as $t \rightarrow +\infty$ in (37) and obtain (in view of $\mu > p$) that $\mathcal{R}(ts_0 v_0) \rightarrow -\infty$ as $t \rightarrow +\infty$. Consequently, setting $e = t_0 s_0 v_0$ with sufficiently large $t_0 > 0$ we establish the assertion. This completes the proof of Lemma 3.5. \blacksquare

3.1. Finite dimensional approximation

Let us denote by Λ the family of all finite dimensional subspaces F of $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying the conditions:

$$\begin{aligned}
F \in \Lambda &\Leftrightarrow F = V_0 + \widehat{F} \text{ for some finite dimensional subspace } \widehat{F} \subset \widehat{V} \cap L^\infty(\Omega) \\
&\text{and } e \in F, \tag{39}
\end{aligned}$$

with $e \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ from (H6).

For every subspace $F \in \Lambda$ we introduce the functional $\mathcal{R}_F : F \rightarrow \mathbb{R}$ which is the restriction of \mathcal{R} to F , i.e.

$$\mathcal{R}_F(v) = \frac{1}{p} \|Dv\|_{L^p(\Omega; \mathbb{R}^N)}^p + \int_\Omega j(x, v(x)) dx, \quad \forall v \in F. \tag{40}$$

It is obvious (see (H1)) that the functional \mathcal{R}_F is locally Lipschitz and its generalized gradient fulfills the condition

$$\partial\mathcal{R}_F(v) \subset i_F^* A i_F v + \bar{i}_F^* \partial J(v), \quad \forall v \in F, \tag{41}$$

where $i_F : F \rightarrow W_0^{1,p}(\Omega)$ and $\bar{i}_F : F \rightarrow L^\infty(\Omega)$ are the inclusion maps with their dual projections $i_F^* : W^{-1,p'}(\Omega) \rightarrow F^*$ and $\bar{i}_F^* : L^1(\Omega) \rightarrow F^*$, respectively, while $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is defined by

$$\langle Au, v \rangle_{W_0^{1,p}(\Omega)} = \int_{\Omega} |Du|^{p-2} \langle Du, Dv \rangle_{\mathbb{R}^N} dx. \tag{42}$$

By $\partial J(\cdot)$ the generalized Clarke gradient of $J : L^\infty(\Omega) \rightarrow \mathbb{R}$ given by

$$J(v) = \int_{\Omega} j(x, v(x)) dx, \quad \forall v \in L^\infty(\Omega)$$

has been denoted. Notice that in view of (H1), J is locally Lipschitz on $L^\infty(\Omega)$, so the generalized gradient $\partial J(\cdot)$ is well defined. The pairing over $F^* \times F$ will be denoted by $\langle \cdot, \cdot \rangle_F$.

PROPOSITION 3.1 *Assume that the hypotheses $\{(H1)-(H6)\}$ or $\{(H1),(H3), (H4)_1, (H5)_1, (H6)\}$ are fulfilled. Then for each $F \in \Lambda$*

Problem (P_F): Find $u_F \in F$ such as to satisfy the hemivariational inequality:

$$\int_{\Omega} |Du_F|^{p-2} \langle Du_F, Dv - Du_F \rangle_{\mathbb{R}^N} dx + \int_{\Omega} j^0(u_F; v - u_F) dx \geq 0, \quad \forall v \in F, \tag{43}$$

has at least one solution $u_F \neq 0$. Moreover, there exist constants $M > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$ not depending on $F \in \Lambda$ such that

$$\|u_F\|_{W_0^{1,p}(\Omega)} \leq M, \quad \forall F \in \Lambda \tag{44}$$

$$\gamma_1 \leq \mathcal{R}(u_F) \leq \gamma_2, \quad \forall F \in \Lambda. \tag{45}$$

Proof. By the results obtained it follows that it is sufficient to consider the hypotheses (H1)-(H6).

First we show that the functional $\mathcal{R}_F : F \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition in the sense of Chang (Chang, 1981). Let $\{u_n\} \subset F$ and $\{w_n\} \subset F^*$ be sequences such that $|\mathcal{R}_F(u_n)| \leq C$, for all $n \geq 1$, with a constant $C > 0$, and $w_n \in \partial\mathcal{R}_F(u_n)$, $\|w_n\|_{F^*} = \varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Since F is finite dimensional, it remains to show that $\{u_n\}$ is bounded in F . According to (41) we see that w_n can be expressed as follows

$$w_n = i_F^* A u_n + \bar{i}_F^* \chi_n, \quad \text{with } \chi_n \in \partial J(u_n). \tag{46}$$

Let us notice that the hypothesis of Theorem 2.7.3 of Clarke (1983), p. 80, is verified. Therefore we obtain

$$\partial J(v) \subset \int_{\Omega} \partial j(x, v(x)) dx, \quad \forall v \in L^\infty(\Omega). \tag{47}$$

Thus

$$\begin{aligned} \langle Au_n, v - u_n \rangle_{W_0^{1,p}(\Omega)} + \int_{\Omega} j^0(u_n; v - u_n) dx &\geq \langle w_n, v - u_n \rangle_F \geq -\varepsilon_n \|v - u_n\|_F \\ &\geq -c\varepsilon_n \|v - u_n\|_{W_0^{1,p}(\Omega)}, \quad \forall v \in F, \quad c = \text{const} > 0, \end{aligned}$$

because the norms $\|\cdot\|_F$ and $\|\cdot\|_{W_0^{1,p}(\Omega)}$ are equivalent in F (F is finite dimensional). Since $\text{Lin}(\theta, u_n) \subset F$, the hypotheses of Lemma 3.1 are verified. Consequently $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ which means that

$$\|u_F\|_{W_0^{1,p}(\Omega)} \leq M_F \tag{48}$$

for some $M_F > 0$.

Following the lines of the proof of Lemma 3.2 (with $W_0^{1,p}(\Omega)$ replaced by F) we conclude the existence of positive constants $\rho_F > 0$ and $\eta_F > 0$ such that

$$\mathcal{R}_F(v) \geq \eta_F, \quad \forall v \in \{w \in F : \|w\|_F = \rho_F\}. \tag{49}$$

By (H6) we know that $\mathcal{R}(te) \leq 0$ for any $t \geq 1$, therefore $\rho_F < \|e\|_F$. Thus taking into account that $\mathcal{R}_F(0) \leq 0$ and $\mathcal{R}_F(e) \leq 0$ we are allowed to apply the mountain pass theorem and deduce the existence of a critical point $u_F \in F$ of \mathcal{R}_F . This leads to the finite dimensional hemivariational inequality (43) (see Motreanu and Panagiotopoulos, 1999).

Let us recall that the critical value $\mathcal{R}_F(u_F)$ is characterized by (see Motreanu and Panagiotopoulos, 1999)

$$\mathcal{R}_F(u_F) = \inf_{\gamma \in C_F} \max_{t \in [0,1]} \mathcal{R}_F(\gamma(t)), \tag{50}$$

where

$$C_F = \{\gamma \in C([0,1], F) : \gamma(0) = 0, \gamma(1) = e\},$$

is the the family of all continuous curves in F joining points 0 and e in F i.e. $\gamma(0) = 0$ and $\gamma(1) = e$, $\gamma(t) \subset F$. Further, from Lemma 3.2 it follows that for a certain positive $\rho > 0$ one can find $\eta > 0$ with

$$\mathcal{R}_F(v) \geq \eta, \quad \forall v \in S_{\rho} \cap L^{\infty}(\Omega), \tag{51}$$

where $S_{\rho} := \{v \in W_0^{1,p}(\Omega) : \|v\|_{W_0^{1,p}(\Omega)} = \rho\}$, while (H6) ensures the existence of $e \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, $e \neq 0$, such that

$$\mathcal{R}(te) \leq 0, \quad \forall t \geq 1. \tag{52}$$

Therefore, for any $F \in \Lambda$, if $\gamma \in C_F([0,1]; F)$ then γ meets points of S_{ρ} which means that

$$\max_{t \in [0,1]} \mathcal{R}_F(\gamma(t)) \geq \eta. \tag{53}$$

Hence

$$\eta \leq \mathcal{R}(u_F) = \inf_{\gamma \in C_F} \max_{t \in [0,1]} \mathcal{R}_F(\gamma(t)) \leq \max_{t \in [0,1]} \mathcal{R}(te), \quad \forall F \in \Lambda \tag{54}$$

and (45) results.

Now we are ready to show that $M_F > 0$ in (48) is independent of $F \in \Lambda$. For this purpose suppose that a sequence $\{u_{F_n}\}_{F_n \in \Lambda}$ of solutions of (P_{F_n}) has the property that $\|u_{F_n}\|_{W_0^{1,p}(\Omega)} \rightarrow \infty$. Taking into account (43) and (54) it is easy to check that the hypotheses (7) and (6) of Lemma 3.1 hold (with F replaced by F_n and $\varepsilon_n = 0$). Following the lines of the proof of Lemma 3.1 we arrive at the contradiction which establishes the assertion. The proof of Proposition 3.1 is complete. ■

For the case when the energy functional is bounded from below we can formulate the following result.

PROPOSITION 3.2 *Assume that (H1) holds. Moreover, let (H4)₂ There exist $1 < s < p$, and a constant $\kappa \geq 0$ such that*

$$j^0(x, \xi; -\xi) \leq \kappa(1 + |\xi|^s), \quad \forall \xi \in \mathbb{R} \text{ and for a.e. } x \in \Omega;$$

Then for each $F \in \Lambda$ there exists $u_F \in F$ such as to satisfy the hemivariational inequality (43). Moreover, (44) is fulfilled.

Proof. As it is known, under (H4)₂, (see Lemma 2.1, pp. 119-120, Naniewicz, 1997):

$$j(x, \xi) \geq -c(1 + |\xi|^s), \quad \forall \xi \in \mathbb{R}, \text{ for a.e. } x \in \Omega,$$

so we easily deduce that

$$\inf\{\mathcal{R}_F(v) : v \in \mathcal{W}\} = \alpha > -\infty,$$

with α independent of the choice of $F \in \Lambda$ and that \mathcal{R}_F satisfies the (P.S.) condition. Thus by Theorem 2.2 we conclude the existence of a critical point $u_F \in F$ of \mathcal{R}_F . Accordingly, (43) is fulfilled (see Motreanu and Panagiotopoulos, 1999). It is an easy task to show the uniform boundedness (44). The proof of Proposition 3.2 is complete. ■

For the restriction of J to F , $J_F := J|_F : F \rightarrow \mathbb{R}$, we have $\partial J_F(u_F) \subset \bar{i}_F^* \partial J(u_F)$. Therefore Proposition 3.1 and Proposition 3.2 can be reformulated as follows.

COROLLARY 3.1 *Assume the hypotheses $\{(H1)-(H6)\}$ or $\{(H1),(H3),(H4)_1,(H5)_1,(H6)\}$ or $\{(H1),(H4)_2\}$. Then, for each $F \in \Lambda$ there exist $u_F \in F$ and $\chi_F \in L^1(\Omega)$ such that*

$$\int_{\Omega} |Du_F|^{p-2} \langle Du_F, Dv - Du_F \rangle_{\mathbb{R}^N} dx + \int_{\Omega} \chi_F (v - u_F) dx = 0, \quad \forall v \in F, \\ \chi_F \in \partial j(u_F) \text{ a.e. in } \Omega. \tag{55}$$

According to the results obtained we know that to any $F \in \Lambda$ a pair $(u_F, \chi_F) \in F \times L^1(\Omega)$ can be assigned for which (55) holds. Moreover, the family $\{u_F\}_{F \in \Lambda}$ is uniformly bounded in $W_0^{1,p}(\Omega)$ ((44) holds). The question arises concerning the behavior of $\{\chi_F\}_{F \in \Lambda}$.

PROPOSITION 3.3 *Assume that $(u_F, \chi_F) \in F \times L^1(\Omega)$ satisfies (55). Then the set $\{\chi_F\}_{F \in \Lambda}$ is weakly precompact in $L^1(\Omega)$.*

Proof. Since Ω is bounded, according to the Dunford-Pettis theorem (see, e.g., Ekeland and Temam, 1976, p. 239) it suffices to show that for each $\varepsilon > 0$ a number $\delta > 0$ can be determined such that for any $\omega \subset \Omega$ with $|\omega| < \delta$,

$$\int_{\omega} |\chi_F| dx < \varepsilon, \quad \forall F \in \Lambda. \quad (56)$$

Choose $\bar{q} \in (q, p^*)$. Then the injection $W_0^{1,p}(\Omega) \subset L^{\bar{q}}(\Omega)$ is compact. Further, from (H4) it follows that there exists a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that (see Remark 5.6, p. 156 in Naniewicz and Panagiotopoulos, 1995, and Lemma 1, p. 95, Motreanu and Naniewicz, 2002)

$$j^0(x, \xi; \eta - \xi) \leq \alpha(r)(1 + |\xi|^q), \quad \forall \xi, \eta \in \mathbb{R}, |\eta| \leq r, r \geq 0. \quad (57)$$

Fix $r > 0$ and let $\eta \in \mathbb{R}$ be such that $|\eta| \leq r$. Then, by (55), $\chi_F(\eta - u_F) \leq j^0(x, u_F; \eta - u_F)$, from which we get

$$\chi_F \eta \leq \chi_F u_F + \alpha(r)(1 + |u_F|^q) \quad \text{for a.e. } x \in \Omega. \quad (58)$$

Let us set $\eta \equiv r \operatorname{sgn} \chi_F(x)$ where $\operatorname{sgn} y = 1$ if $y > 0$, $\operatorname{sgn} y = 0$ if $y = 0$, $\operatorname{sgn} y = -1$ if $y < 0$. One obtains that $|\eta| \leq r$ and $\chi_F(x)\eta = r|\chi_F(x)|$ for almost all $x \in \Omega$. Therefore from (58) there is

$$r|\chi_F| \leq \chi_F u_F + \alpha(r)(1 + |u_F|^q).$$

Integrating this inequality over $\omega \subset \Omega$ yields

$$\int_{\omega} |\chi_F| dx \leq \frac{1}{r} \int_{\omega} \chi_F u_F dx + \frac{1}{r} \alpha(r)|\omega| + \frac{1}{r} \alpha(r)|\omega|^{\frac{\bar{q}-q}{\bar{q}}} \|u_F\|_{L^{\bar{q}}(\Omega)}^q. \quad (59)$$

Consequently, from (44) and (59) it follows that

$$\int_{\omega} |\chi_F| dx \leq \frac{1}{r} \int_{\omega} \chi_F u_F dx + \frac{1}{r} \alpha(r)|\omega| + \frac{1}{r} \alpha(r)|\omega|^{\frac{\bar{q}-q}{\bar{q}}} \gamma^q M^q, \quad (60)$$

where $\gamma > 0$ is a constant satisfying $\|\cdot\|_{L^{\bar{q}}(\Omega)} \leq \gamma \|\cdot\|_{H_0^1(\Omega)}$ (which holds since $\bar{q} < p^*$).

We claim

$$\int_{\omega} \chi_F u_F dx \leq C \quad (61)$$

for some positive constant C not depending on $\omega \subset \Omega$ and $F \in \Lambda$. Indeed, from (57) we derive that

$$\chi_F u_F + \alpha(0)(|u_F|^q + 1) \geq 0 \quad \text{for a.e. in } \Omega.$$

Thus it follows that

$$\begin{aligned} \int_{\omega} \chi_F u_F \, dx &\leq \int_{\omega} (\chi_F u_F + \alpha(0)(|u_F|^q + 1)) \, dx \\ &\leq \int_{\Omega} (\chi_F u_F + \alpha(0)(|u_F|^q + 1)) \, dx \\ &\leq \int_{\Omega} \chi_F u_F \, dx + \bar{k}_1 (\|u_F\|_{H_0^1(\Omega)}^q + |\Omega|), \end{aligned}$$

where $\bar{k}_1 > 0$ is a constant. By (44) and (55) (with $v = 0$) it turns out that

$$\int_{\Omega} \chi_F u_F \, dx = - \int_{\Omega} |Du_F|^p \, dx \leq 0.$$

The estimates above imply (61).

Further, (60) and (61) entail

$$\int_{\omega} |\chi_F| \, dx \leq \frac{1}{r} C + \frac{1}{r} \alpha(r) |\omega| + \frac{1}{r} \alpha(r) |\omega|^{\frac{q-q}{q}} \gamma^q M^q, \quad \forall r > 0. \tag{62}$$

Corresponding to $\varepsilon > 0$, fix $r > 0$ with

$$\frac{1}{r} C < \frac{\varepsilon}{2} \tag{63}$$

and then take $\delta > 0$ small enough to have

$$\frac{1}{r} \alpha(r) |\omega| + \frac{1}{r} \alpha(r) |\omega|^{\frac{q-q}{q}} \gamma^q M^q < \frac{\varepsilon}{2} \tag{64}$$

provided that $|\omega| < \delta$. Using this together with (62) and (63) we obtain that (56) is justified whenever $|\omega| < \delta$. This completes the proof. ■

4. Main result

THEOREM 4.1 *Assume the hypotheses (H1)-(H6) and let*

(H7) *One of the two conditions below holds:*

(i) *There exists $w_0 \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} j^0(x; 0, w_0(x)) \, dx < 0$;*

or

(ii) *For any sequence $\{v_k\} \subset L^\infty(\Omega)$, $v_k \rightarrow 0$ strongly in $L^p(\Omega)$, the condition*

$$\int_{\Omega} \min\{\psi(x)v_k(x) : \psi(x) \in \partial j(x, v_k(x))\} \, dx \leq 0,$$

implies

$$\limsup_{k \rightarrow \infty} \int_{\Omega} j(x, v_k(x)) \, dx \leq 0.$$

Then there exists $u \in W_0^{1,p}(\Omega)$ with $u \neq 0$ and $j(u) \in L^1(\Omega)$, such as to satisfy the hemivariational inequality

$$\int_{\Omega} |Du|^{p-2} \langle Du, Dv - Du \rangle_{\mathbb{R}^N} \, dx + \int_{\Omega} j^0(u; v-u) \, dx \geq 0, \quad \forall v \in W_0^{1,p}(\Omega). \quad (65)$$

Moreover, there exists $\chi \in L^1(\Omega)$ with the property that

$$\int_{\Omega} |Du|^{p-2} \langle Du, Dv - Du \rangle_{\mathbb{R}^N} \, dx + \int_{\Omega} \chi(v-u) \, dx = 0, \\ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad (66)$$

$$\chi u \in L^1(\Omega) \quad \text{and} \quad \chi \in \partial j(u) \quad \text{a.e. in } \Omega. \quad (67)$$

Proof. The proof is carried out in a sequence of steps.

Step 1. For every $F \in \Lambda$ we introduce

$$U_F = \{u_F \in W_0^{1,p}(\Omega) : \text{for some } \chi_F \in L^1(\Omega), (u_F, \chi_F) \text{ is a solution of } (P_F)\}$$

and

$$W_F = \bigcup_{\substack{F' \in \Lambda \\ F' \supset F}} U_{F'}.$$

By Proposition 3.1, W_F is nonempty (even U_F is nonempty) and contained in the ball $B_M = \{v \in W_0^{1,p}(\Omega) : \|v\|_{W_0^{1,p}(\Omega)} \leq M\}$. We denote by $\text{weakcl}(W_F)$ the closure of W_F in the weak topology of $W_0^{1,p}(\Omega)$. Proposition 3.1 ensures that $\text{weakcl}(W_F)$ is weakly compact in $W_0^{1,p}(\Omega)$. We claim that the family $\{\text{weakcl}(W_F)\}_{F \in \Lambda}$ has the finite intersection property. Indeed, if $F_1, \dots, F_k \in \Lambda$ then $W_{F_1} \cap \dots \cap W_{F_k} \supset W_F$, with $F = F_1 + \dots + F_k$ and the assertion follows. Thus we are allowed to conclude that there exists an element $u \in W_0^{1,p}(\Omega)$ with

$$u \in \bigcap_{F \in \Lambda} \text{weakcl}(W_F).$$

Let us choose $G \in \Lambda$ arbitrarily. Since $W_0^{1,p}(\Omega)$ is reflexive, one can extract an increasing sequence of subspaces $\{G_n\}$, each containing G , and for each n an element $u_n \in U_{G_n}$ such that $u_n \rightarrow u$ weakly in $W_0^{1,p}(\Omega)$ as $n \rightarrow \infty$ (Proposition 11, p. 274, Browder and Hess, 1972). Let us denote by $\{\chi_n\} \subset L^1(\Omega)$ the corresponding sequence with the property that for each n a pair (u_n, χ_n) is

a solution of (P_{G_n}) . By Proposition 3.3 we can suppose without loss of generality that $\chi_n \rightarrow \chi^G$ weakly in $L^1(\Omega)$ for some $\chi^G \in L^1(\Omega)$. Thus we have asserted that

$$u_n \rightarrow u \text{ weakly in } W_0^{1,p}(\Omega) \tag{68}$$

$$\chi_n \rightarrow \chi^G \text{ weakly in } L^1(\Omega) \tag{69}$$

and that (55) with F replaced by G_n reads

$$\langle Au_n, v - u_n \rangle_{W_0^{1,p}(\Omega)} + \int_{\Omega} \chi_n (v - u_n) dx = 0, \quad \forall v \in G_n, \tag{70}$$

where $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is defined by (42).

Step 2. Now we prove that $\chi^G \in \partial j(u)$ a.e. in Ω . Since $W_0^{1,p}(\Omega)$ is compactly imbedded into $L^p(\Omega)$, due to (44) one may suppose that

$$u_n \rightarrow u \text{ strongly in } L^p(\Omega). \tag{71}$$

This implies that for a subsequence of $\{u_n\}$ (again denoted by the same symbol) one gets $u_n \rightarrow u$ a.e. in Ω . Thus Egoroff's theorem can be applied from which it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $|\omega| < \varepsilon$ can be determined such that $u_n \rightarrow u$ uniformly in $\Omega \setminus \omega$ with $u \in L^\infty(\Omega \setminus \omega)$. Let $v \in L^\infty(\Omega \setminus \omega)$ be an arbitrary function. From the estimate

$$\int_{\Omega \setminus \omega} \chi_n v d\Omega \leq \int_{\Omega \setminus \omega} j^0(u_n; v) d\Omega$$

combined with the weak convergence in $L^1(\Omega)$ of χ_n to χ^G , (71) and with the upper semicontinuity of

$$L^\infty(\Omega \setminus \omega) \ni u_n \mapsto \int_{\Omega \setminus \omega} j^0(u_n; v) dx$$

it follows that

$$\int_{\Omega \setminus \omega} \chi^G v d\Omega \leq \int_{\Omega \setminus \omega} j^0(u; v) d\Omega, \quad \forall v \in L^\infty(\Omega \setminus \omega).$$

But the last inequality amounts to saying that $\chi^G \in \partial j(u)$ a.e. in $\Omega \setminus \omega$. Since $|\omega| < \varepsilon$ and ε was chosen arbitrarily,

$$\chi^G \in \partial j(u) \text{ a.e. in } \Omega, \tag{72}$$

as claimed.

Step 3. Now it will be shown that

$$\limsup_{n \rightarrow \infty} \int_{\Omega} j^0(u_n; v - u_n) dx \leq \int_{\Omega} j^0(u; v - u) dx \tag{73}$$

holds for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. It can be supposed that $u_n \rightarrow u$ a.e. in Ω , since $u_n \rightarrow u$ in $L^q(\Omega)$. Fix $v \in L^\infty(\Omega)$ arbitrarily. In view of $\chi_n \in \partial j(u_n)$ and (57) we get

$$j^0(u_n; v - u_n) \leq \alpha(\|v\|_{L^\infty(\Omega)})(1 + |u_n|^q). \quad (74)$$

From Egoroff's theorem it follows that for any $\varepsilon > 0$ a subset $\omega \subset \Omega$ with $|\omega| < \varepsilon$ can be determined such that $u_n \rightarrow u$ uniformly in $\Omega \setminus \omega$. One can also suppose that ω is small enough to fulfill $\int_\omega \alpha(\|v\|_{L^\infty(\Omega)})(1 + |u_n|^q) dx \leq \varepsilon$, $n = 1, 2, \dots$, and $\int_\omega \alpha(\|v\|_{L^\infty(\Omega)})(1 + |u|^q) dx \leq \varepsilon$. Hence

$$\int_\Omega j^0(u_n; v - u_n) dx \leq \int_{\Omega \setminus \omega} j^0(u_n; v - u_n) dx + \varepsilon$$

which by Fatou's lemma and upper semicontinuity of $j^0(\cdot; \cdot)$ yields

$$\limsup_{n \rightarrow \infty} \int_\Omega j^0(u_n; v - u_n) dx \leq \int_\Omega j^0(u; v - u) dx + 2\varepsilon.$$

By arbitrariness of $\varepsilon > 0$ one obtains (73), as required.

Step 4. Now we show that

$$\chi^G u \in L^1(\Omega). \quad (75)$$

$$\liminf_{n \rightarrow \infty} \int_\Omega \chi_n u_n dx \geq \int_\Omega \chi^G u dx \quad (76)$$

For this purpose let $\{\epsilon_k\} \subset L^\infty(\Omega)$ be such that (Hedberg, 1978):

$$\begin{aligned} \{(1 - \epsilon_k)u\} &\subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad 0 \leq \epsilon_k \leq 1 \\ \tilde{u}_k &:= (1 - \epsilon_k)u \rightarrow u \text{ strongly in } W_0^{1,p}(\Omega) \text{ as } k \rightarrow \infty. \end{aligned} \quad (77)$$

Without loss of generality it can be assumed that $\tilde{u}_k \rightarrow u$ a.e. in Ω . Since it is already known that $\chi^G \in \partial j(u)$, one can apply (H4) to obtain $\chi^G(-u) \leq j^0(u; -u) \leq \kappa(1 + |u|^q)$. Hence

$$\chi^G \tilde{u}_k = (1 - \epsilon_k)\chi^G u \geq -\kappa(1 + |u|^q). \quad (78)$$

This implies that the sequence $\{\chi^G \tilde{u}_k\}$ is bounded from below by an integrable function and $\chi^G \tilde{u}_k \rightarrow \chi^G u$ a.e. in Ω . On the other hand, one gets

$$\int_\Omega \chi_n(\tilde{u}_k - u_n) dx \leq \int_\Omega j^0(u_n; \tilde{u}_k - u_n) dx.$$

Thus, passing to the limit with $n \rightarrow \infty$ yields

$$\int_\Omega \chi^G \tilde{u}_k dx - \liminf_{n \rightarrow \infty} \int_\Omega \chi_n u_n dx \leq \limsup_{n \rightarrow \infty} \int_\Omega j^0(u_n; \tilde{u}_k - u_n) dx,$$

and due to (73) we are led to the estimate

$$\begin{aligned} \int_{\Omega} \chi^G \tilde{u}_k \, dx &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \chi_n u_n \, dx + \int_{\Omega} j^0(u; \tilde{u}_k - u) \, dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \chi_n u_n \, dx + \int_{\Omega} j^0(u; -\epsilon_k u) \, dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\Omega} \chi_n u_n \, dx + \int_{\Omega} \epsilon_k \kappa (1 + |u|^q) \, dx \leq C, \quad C = \text{const}. \end{aligned}$$

Thus by Fatou's lemma we are allowed to conclude that $\chi^G u \in L^1(\Omega)$, i.e. (75) holds. Taking into account that $\epsilon_k \rightarrow 0$ a.e. in Ω as $k \rightarrow \infty$ (passing to a subsequence if necessary) we establish (76), as required.

Step 5. It will be shown that

$$\langle Au, v - u \rangle_{W_0^{1,p}(\Omega)} + \int_{\Omega} \chi^G (v - u) \, dx = 0, \quad \forall v \in \bigcup_{n=1}^{\infty} G_n \supset G \quad (Q^G)$$

$$\chi^G \in \partial j(u).$$

Since A is bounded and $\{u_F\}_{F \in \Lambda} \subset \{v \in W_0^{1,p}(\Omega) : \|v\|_{W_0^{1,p}(\Omega)} \leq M\}$, there exists $K > 0$ such that $\{Au_F\}_{F \in \Lambda} \subset \{l \in W^{-1,p'}(\Omega) : \|l\|_{W^{-1,p'}(\Omega)} \leq K\}$. From (70) it follows that for any fixed $G \in \Lambda$ we get

$$\left| \int_{\Omega} \chi^G v \, dx \right| \leq K \|v\|_V, \quad \forall v \in \bigcup_{n=1}^{\infty} G_n, \quad \chi^G \in \partial j(u), \quad (79)$$

because $\{G_n\}$ is an increasing sequence. Further, by making use of (75) and (76) we have $\chi^G u \in L^1(\Omega)$ and

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle_{W_0^{1,p}(\Omega)} \leq \int_{\Omega} \chi^G (v - u) \, dx, \quad \forall v \in \bigcup_{n=1}^{\infty} G_n. \quad (80)$$

Since $u_n \in G_n$ and $u_n \rightarrow u$ weakly in $W_0^{1,p}(\Omega)$, the closure of $\bigcup_{n=1}^{\infty} G_n$ in the strong topology of $W_0^{1,p}(\Omega)$, $\overline{\bigcup_{n=1}^{\infty} G_n}$, must contain u . Thus there exists a sequence $\{w_i\} \subset \bigcup_{n=1}^{\infty} G_n$ converging strongly to u in $W_0^{1,p}(\Omega)$ as $i \rightarrow \infty$. We claim that for such a sequence,

$$\int_{\Omega} \chi^G w_i \, dx \rightarrow \int_{\Omega} \chi^G u \, dx \quad \text{as } i \rightarrow \infty. \quad (81)$$

Indeed, let $\{\tilde{u}_k\}_{k=1}^{\infty}$ be given by (77). From (78) it follows that

$$-\kappa(1 + |u|^q) \leq \chi^G \tilde{u}_k \leq |\chi^G u|, \quad k = 1, 2, \dots, \quad (82)$$

with the bounds $-\kappa(1+|u|^q)$ and $|\chi^G u|$ being integrable in Ω . Thus there exists a constant $C > 0$ such that

$$\left| \int_{\Omega} \chi^G \tilde{u}_k dx \right| \leq C \|\tilde{u}_k\|_{W_0^{1,p}(\Omega)}, \quad k = 1, 2, \dots \quad (83)$$

Denote by \mathcal{A} a linear subspace spanned by $\{\tilde{u}_k\}_{k=1}^{\infty}$ and define a linear functional $\widehat{l}_{\chi^G} : (\bigcup_{n=1}^{\infty} G_n + \mathcal{A}) \rightarrow \mathbb{R}$ by the formula

$$\widehat{l}_{\chi^G}(v) := \int_{\Omega} \chi^G v dx, \quad v \in \bigcup_{n=1}^{\infty} G_n + \mathcal{A}.$$

Taking into account (79) and (83), from the Hahn-Banach theorem it follows that \widehat{l}_{χ^G} admits its linear continuous extension onto $W_0^{1,p}(\Omega)$, $l_{\chi^G} \in W^{-1,p'}(\Omega)$. By the dominated convergence,

$$\int_{\Omega} \chi^G \tilde{u}_k dx \rightarrow \int_{\Omega} \chi^G u dx, \quad \text{as } k \rightarrow \infty,$$

so we get $l_{\chi^G}(u) = \int_{\Omega} \chi^G u dx$ which, in particular, implies (81), as claimed.

Taking into account (80) and (81) we conclude

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle_{W_0^{1,p}(\Omega)} \leq 0, \quad (84)$$

which by the pseudomonotonicity of A implies

$$Au_n \rightarrow Au \text{ weakly in } W_0^{1,p}(\Omega) \quad (85)$$

$$\langle Au_n, u_n \rangle_{W_0^{1,p}(\Omega)} \rightarrow \langle Au, u \rangle_{W_0^{1,p}(\Omega)}. \quad (86)$$

Hence from (70) we are led to (Q^G) , as desired. Notice that (85) and (86) imply the strong convergence $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$.

Step 6. It remains to show that there exists $\chi \in \partial j(u)$ with the associated linear functional defined by

$$\widehat{l}_{\chi}(v) := \int_{\Omega} \chi v dx, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega),$$

admitting a continuous extension $l_{\chi} \in W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$, such that

$$Au + l_{\chi} = 0, \quad \langle l_{\chi}, u \rangle_{W_0^{1,p}(\Omega)} = \int_{\Omega} \chi u dx. \quad (87)$$

For every $G \in \Lambda$ let us introduce

$$V^{(G)} = \{\chi^G \in L^{\infty}(\Omega) : (Q^G) \text{ holds}\}$$

and

$$Z^{(G)} = \bigcup_{\substack{G' \in \Lambda \\ G' \supset G}} V^{(G')}.$$

As in the proof of Proposition 3.3 we show that the family $\{\chi^G\}_{G \in \Lambda}$ is weakly precompact in $L^1(\Omega)$. Denoting by $\text{weakcl}(Z^{(G)})$ the closure of $Z^{(G)}$ in the weak topology of $L^1(\Omega)$ we prove analogously that the family $\{\text{weakcl}(Z^{(G)})\}_{G \in \Lambda}$ has the finite intersection property. Thus, there exists an element $\chi \in \partial j(u)$ such that for any $G \in \Lambda$ there holds

$$\langle Au, v \rangle_{W_0^{1,p}(\Omega)} + \int_{\Omega} \chi v \, dx = 0, \quad \forall v \in G.$$

Since $G \in \Lambda$ has been chosen arbitrarily and Λ is dense in $W_0^{1,p}(\Omega)$, (87) results, as desired.

Step 7. It remains to show (65). From (66) we obtain easily its validity for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Let us consider the case $j^0(u; v - u) \in L^1(\Omega)$ with $v \in W_0^{1,p}(\Omega)$. There exists a sequence $\tilde{v}_k = (1 - \epsilon_k)v$ such that $\{\tilde{v}_k\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\tilde{v}_k \rightarrow v$ strongly in $W_0^{1,p}(\Omega)$. Since, as already has been established,

$$\langle Au, \tilde{v}_k - u \rangle_{W_0^{1,p}(\Omega)} + \int_{\Omega} j^0(u; \tilde{v}_k - u) \, dx \geq 0,$$

so in order to show (65) it remains to deduce that

$$\limsup_{k \rightarrow \infty} \int_{\Omega} j^0(u; \tilde{v}_k - u) \, dx \leq \int_{\Omega} j^0(u; v - u) \, dx.$$

For this purpose let us observe that $\tilde{v}_k - u = (1 - \epsilon_k)(v - u) + \epsilon_k(-u)$, which, combined with the convexity of $j^0(u; \cdot)$ yields the estimate

$$j^0(u; \tilde{v}_k - u) \leq (1 - \epsilon_k)j^0(u; v - u) + \epsilon_k j^0(u; -u) \leq |j^0(u; v - u)| + \kappa(1 + |u|^q).$$

Thus Fatou's lemma implies the assertion.

Consider the case $j^0(u; v - u) \notin L^1(\Omega)$. Recall that if $j^0(u; v - u) \notin L^1(\Omega)$ then according to the convention that $+\infty - \infty = +\infty$ we have

$$\begin{aligned} & \int_{\Omega} j^0(u; v - u) \, dx = \\ & = \begin{cases} +\infty & \text{if } \int_{\Omega} [j^0(u; v - u)]^+ \, dx = +\infty \\ -\infty & \text{if } \int_{\Omega} [j^0(u; v - u)]^+ \, dx < +\infty \text{ and } \int_{\Omega} [j^0(u; v - u)]^- \, dx = +\infty, \end{cases} \end{aligned}$$

where the notation has been used: $r^+ := \max\{r, 0\}$ and $r^- := \max\{-r, 0\}$ for any $r \in \mathbb{R}$. Thus, if $\int_{\Omega} j^0(u; v - u) \, dx = +\infty$ then (65) holds immediately.

Now we show that the case $\int_{\Omega} j^0(u; v - u) dx = -\infty$ is not allowed for any $v \in W_0^{1,p}(\Omega)$. Indeed, if we suppose that for some $v \in W_0^{1,p}(\Omega)$, $\int_{\Omega} j^0(u; v - u) dx = -\infty$, then one can find a sequence $\tilde{v}_k = (1 - \epsilon_k)v$ such that $\{\tilde{v}_k\} \subset W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, $\tilde{v}_k \rightarrow v$ strongly in $W_0^{1,p}(\Omega)$. Since, as already has been established,

$$\langle Au, \tilde{v}_k - u \rangle_{W_0^{1,p}(\Omega)} + \int_{\Omega} j^0(u; \tilde{v}_k - u) dx \geq 0,$$

we get

$$\int_{\Omega} j^0(u; \tilde{v}_k - u) dx \geq \langle Au, -\tilde{v}_k + u \rangle_{W_0^{1,p}(\Omega)} \geq -C, \quad C = \text{const},$$

and consequently

$$\int_{\Omega} [j^0(u; \tilde{v}_k - u)]^+ dx \geq \int_{\Omega} [j^0(u; \tilde{v}_k - u)]^- dx - C. \quad (88)$$

By the hypothesis we have $\int_{\Omega} [j^0(u; v - u)]^- dx = +\infty$ and $\int_{\Omega} [j^0(u; v - u)]^+ dx < +\infty$. Since

$$j^0(u; \tilde{v}_k - u) \leq (1 - \epsilon_k)j^0(u; v - u) + \epsilon_k j^0(u; -u) \leq (1 - \epsilon_k)j^0(u; v - u) + \kappa(1 + |u|^q),$$

so we obtain

$$\int_{\Omega} [j^0(u; v_k - u)]^+ dx \leq \int_{\Omega} [j^0(u; v - u)]^+ dx + \int_{\Omega} \kappa(1 + |u|^q) dx \leq D, \quad D = \text{const},$$

which, combined with (88), yields

$$\int_{\Omega} [j^0(u; \tilde{v}_k - u)]^- dx \leq C + D.$$

The application of Fatou's lemma concludes

$$\int_{\Omega} [j^0(u; v - u)]^- dx \leq C + D,$$

which is a contradiction with the assumption that $\int_{\Omega} j^0(u; v - u) dx = -\infty$. This contradiction completes the proof of (65).

Step 8. In order to show that $j(u) \in L^1(\Omega)$ it is enough to use (17) and (45) to get

$$\int_{\Omega} j(u_n) dx \leq \gamma_2 - \frac{1}{p} \|Du_n\|_{L^p(\Omega; \mathbb{R}^N)}^p \leq \gamma_2$$

and

$$j(u_n) \geq -\kappa_0(1 + |u_n|^q).$$

Since $j(u_n) \rightarrow j(u)$ a.e. in Ω as $n \rightarrow \infty$, we are allowed to apply Fatou's lemma which yields the assertion.

Step 9. The existence of a nontrivial solution $u \neq 0$ follows from (H7). Indeed, (H7)_(i) excludes $u = 0$ as a solution of (66) directly. If (H7)_(ii) holds, the supposition $u = 0$ leads to the contradiction. Indeed, as shown previously, $\{u_n\} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $u_n \rightarrow 0$ strongly in $W_0^{1,p}(\Omega)$. Taking into account (70) with $v = 2u_n$ leads to

$$\int_{\Omega} \min\{\psi u_n : \psi \in \partial j(u_n)\} dx \leq \int_{\Omega} \chi_n u_n dx = -\|Du_n\|_{L^p(\Omega; \mathbb{R}^N)}^p \leq 0.$$

Hence,

$$\limsup_{n \rightarrow \infty} \int_{\Omega} j(u_n) dx \leq 0,$$

and consequently,

$$\limsup_{n \rightarrow \infty} \mathcal{R}(u_n) \leq 0,$$

which contradicts (45). This contradiction yields the assertion. The proof of Theorem 4.1 is complete. ■

Analogously one can show the following result:

THEOREM 4.2 *Assume that the hypotheses (H1), (H4)₂ are valid. Then there exists $u \in W_0^{1,p}(\Omega)$ with $j(u) \in L^1(\Omega)$, such as to satisfy (65). Moreover, there exists $\chi \in L^1(\Omega)$ such that (66) and (67) hold.*

Under the classical growth conditions we have also the following result:

THEOREM 4.3 *Assume that the hypotheses $\{(H1), (H3), (H4)_1, (H5)_1, (H6)\}$ are valid. Then there exists $u \in W_0^{1,p}(\Omega)$, such as to satisfy (65). Moreover, there exists $\chi \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$, such that (66) and (67) hold.*

COROLLARY 4.1 *Under the hypotheses of Theorem 4.1 or Theorem 4.2 the problem: Find $u \in W_0^{1,p}(\Omega)$ and $\chi \in L^1(\Omega)$ such that*

$$(P) \quad \begin{cases} -\Delta_p u = -\chi & \text{in the distributional sense} \\ \chi \in \partial j(u) & \text{a.e. in } \Omega \\ \chi u \in L^1(\Omega) \\ j(u) \in L^1(\Omega) \\ u = 0 & \text{on } \partial\Omega \text{ (in the sense of traces),} \end{cases}$$

has at least one nontrivial solution.

COROLLARY 4.2 Under the hypotheses of Theorem 4.3 : Find $u \in W_0^{1,p}(\Omega)$ and $\chi \in L^p(\Omega)$ such that

$$(P) \quad \begin{cases} -\Delta_p u = -\chi & \text{in the distributional sense} \\ \chi \in \partial j(u) & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \text{ (in the sense of traces),} \end{cases}$$

has at least one nontrivial solution.

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