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Adaptive control for a jump linear system with quadratic cost

by

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Abstract: The adaptive control problem for a jump linear system with quadratic cost functional on infinite time interval is solved in this paper. It is assumed that the coefficients of the state equation are unknown but a compact set that contains the parameters is known. A diminishing excitation accompanies the adaptive control signal to ensure the strong consistency of the weighted least squares algorithm.

Keywords: adaptive control, jump linear systems, linear quadratic problem.

1. Introduction

The problem of finding a control that minimizes an ergodic, quadratic cost functional for a linear system with unknown parameters is probably the most well-known stochastic adaptive control problem. There is a huge literature devoted to this problem. The latest publications dealing with this class of adaptation are Duncan, Guo, Pasik-Duncan (1999), Guo (1996), and Prandini and Campi (2001).

In this paper a similar problem for systems with jump parameters is investigated. These models are characterized by their hybrid state space. To the usual Euclidean space, on which we model the basic dynamics x, we append a finite set S. Let r be a discrete Markov chain with state space S. In applications r, called mode, is a labeling process indicating the context within which x evolves. Considerable research devoted to these models is motivated by significant applications. This class of processes has been used successfully to model air traffic (Blom, 1990), manufacturing systems (Boukas, Haurie, 1990), power systems (Sworder, Rogers, 1983), fault tolerant systems (Świerniak, Simek, Boukas, 1998), and multiplex redundant systems (Siljak, 1980).

For systems with jump parameters adaptive control can be understood in two ways. In the first one we assume that the states of the Markov chain cannot be observed directly but only partially through a certain noisy channel. This approach is presented in Dufour and Elliott (1998) and Pan and Bar-Shalom (1996). In the second way the word "adaptive" refers to the situation presented above for standard systems, i.e., we assume that the coefficients of linear model which describes the dynamics of $x(k)$, $k = 0, 1, ...$ are unknown. In this paper we consider the latter situation.

The system under study is described by the following equation:

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k) + w(k+1), \tag{1}$$

with the state $x(k) \in R^n$, control $u(k) \in R^m$, disturbance $w(k) \in R^n$, and the abrupt changes are incorporated into the model via the ergodic Markov chain $r(k)$ taking values from a finite set $S = \{1, ..., s\}$ according to the stationary probability matrix $P = [p_{ij}]$,

$$P\left(r(k+1) = j\,|r(k) = i\,\right) = p_{ij}, i, j \in S,$$

initial distribution $P\left(r(0) = i_0\right) = 1$ and limit distribution $(\pi_i)_{i \in S}$. Throughout this paper $\left(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0}^{k=\infty}, \mathcal{P}\right)$ is a fixed stochastic basis, with $(\mathcal{F}_k)_{k=0}^{k=\infty}$ denoting a filtration, where $\mathcal{F}_k$ stands for the σ-field generated by $\{r(0), ..., r(k)\}$, and $\mathcal{P}$ a probability measure on $(\Omega, \mathcal{F})$. Moreover we assume that $w(k)$, $k = 0, 1....$ is a second-order independent identically distributed sequence of random variables with $Ew(k) = 0$ and

$$Ew(k)w^T(k) = I, \tag{2}$$

and that $w(k)$, $k = 1, 2....$ and $r(k)$, $k = 0, 1....$ are independent. The initial condition $x(0) = x_0$ in (1) is assumed to be a constant vector. The control $u = (u(0), u(1), ...)$ is such that $u(k)$ is $\mathcal{F}_k$-measurable. Together with (1) we will consider the following cost functional to be minimized

$$J\left(x_0, i_0, u\right) = \lim_{N\to\infty} \frac{1}{N} E \sum_{k=0}^{N-1} \left[\left\langle Q_{r(k)}x(k), x(k)\right\rangle + \left\langle R_{r(k)}u(k), u(k)\right\rangle\right], \tag{3}$$

where the matrices Q_i, $i \in S$ are nonnegative definite and R_i, $i \in S$ are positive definite.

The objective of this paper is to find a control that minimizes J under assumption that the Markov chain is perfectly observed and the coefficients A_i and B_i are unknown. If the coefficients of the system (1) are known the solution to the control problem is given by the following Theorem (see Costa, Fragoso, 1995).

Theorem 1.1 *Suppose that* $\{A_i, B_i, i \in S\}$ *is mean square stabilizable and* $\left\{\sqrt{Q_i}, A_i, i \in S\right\}$ *is mean square detectable. Then the coupled Riccati equation*

$$P_i = Q_i + (A_i - B_i G_i)^T \left(\sum_{j\in S} p_{ij} P_j\right)(A_i - B_i G_i) + G_i' R_i G_i, \quad i \in S \tag{4}$$

where

$$G_i = \left(R_i + B_i^T \left(\sum_{j\in S} p_{ij} P_j\right) B_i\right)^{-1} B_i^T \left(\sum_{j\in S} p_{ij} P_j\right) A_i \tag{5}$$

has a unique positive semidefinite solution and the optimal control for the problem (1), (3) is given by

$$u(k) = -G_{r(k)} x(k), \tag{6}$$

the closed loop system is MSS. Moreover the minimal value of the cost functional is

$$\sum_{i\in S}\sum_{j\in S} \pi(i) p_{ij} tr\left(P_j\right). \tag{7}$$

Definitions of stochastic stabilizability and stochastic detectability are given in the next section.

This paper is organized as follows: In the next section we present two definitions of stability of jump linear system and we study their properties. In the third section we characterize a class of controls that are optimal for control problem (1), (3). The problem of parameter estimation is investigated in the fourth section. The main result of this paper is presented in section five where the adaptive control is constructed. Finally, section six contains concluding remarks.

2. Stability of stochastic system

We begin with definitions of stochastic stability, stabilizability, and detectability, which are taken from Costa, Fragoso (1993, 1995). Suppose that for each $i \in S$ a sequence $\overline{A}_i(k)$, $k = 0, 1, ...$ of $n \times n$ random matrices is given.

Definition 2.1 *The system*

$$x(k+1) = A_{r(k)} x(k) \tag{8}$$

is mean square stable (MSS) if

$$\lim_{N\to\infty} E\left\|x(N)\right\|^2 = 0$$

for any initial conditions (i_0, x_0), *and almost sure stable (ASS) if*

$$\lim_{N\to\infty} \|x(N)\|^2 = 0$$

for any initial conditions (i_0, x_0).

LEMMA 2.1 *(Costa, Fragoso, 1993) MSS implies ASS.*

The next theorem contains three conditions equivalent to MSS. The proof can be found, for example, in Ji et al. (1991).

THEOREM 2.1 *The following conditions are equivalent to MSS of (8):*

1. *For each* $i_0 \in S$ *there exists a positive definite matrix* P_{i_0} *such that for all* $x_0 \in R^n$

$$E\sum_{N=0}^{\infty} \|x(N)\|^2 \leq \langle P_{i_0} x_0, x_0 \rangle . \tag{9}$$

2. *For all positive definite matrices* Q_i, $i \in S$ *there exists a positive definite solution* P_i, $i \in S$, *of the following coupled Lyapunov equation*

$$P_i = A_i' \left(\sum_{j=1}^{s} p_{ij} P_j \right) A_i + Q_i, \quad i \in S. \tag{10}$$

3. *For each* $i_0 \in S$ *there exist* $\beta > 0$ *and* $q \in (0,1)$, *such that*

$$E \left\| \prod_{k=0}^{N} A_{r(k)} \right\|^2 \leq \beta q^N,$$

where

$$\prod_{k=0}^{N} A_{r(k)} = A_{r(N)} ... A_{r(0)}.$$

In our further considerations we will deal with models of the form

$$z(k+1) = \overline{A}_{r(k)}(k) z(k). \tag{11}$$

The next result shows that if the sequence $\overline{A}_i(k)$ converges for each $i \in S$ and the limiting system is MSS then so is (11).

LEMMA 2.2 *Suppose that (8) is MSS. Moreover, let for each* $i \in S$ *a sequence* $\overline{A}_i(N)$ *of* $n \times n$ *random matrices be such that*

$$\lim_{N\to\infty} \overline{A}_i(N) = A_i \quad a.s.$$

and there is a constant c *such that* $\left\|\overline{A}_i(N)\right\| < c$ *for all* $i \in S$ *and* $N = 1, 2, \ldots$. *Then for each* $z(0) \in R^n$, $i_0 \in S$ *we have*

$$\lim_{N\to\infty} E\left\|z(N)\right\|^2 = 0 \tag{12}$$

where $z(k)$ *is given by (11). Moreover, convergence in (12) is exponential, that is-for each* $i_0 \in S$ *there exist* $\beta > 0$ *and* $q \in (0,1)$, *such that*

$$E\left\|\prod_{k=0}^{N} \overline{A}_{r(k)}(k)\right\|^2 \leq \beta q^N. \tag{13}$$

Proof. Let P_i, $i \in S$, be the solution of (10) with $Q_i = I$, $i \in S$. Define a function $V : R^n \times S \to R$, by

$$V(z,i) = \langle P_i z, z\rangle$$

and let

$$\widetilde{V}(z,k,i) = E\left(V\left(z(k+1), r(k+1)\right)\right| z(k) = z, r(k) = i) - V(z,i).$$

Moreover denote

$$\widetilde{A}_i(k) = \overline{A}_i(k) - A_i.$$

Using definitions of conditional expectation and P_i, we get

$$\widetilde{V}(z,k,i) = E\left(\left\langle \overline{A}'_{r(k)}(k) P_{r(k+1)} \overline{A}_{r(k)}(k) z(k), z(k)\right\rangle \middle| z(k) = z, r(k) = i\right)$$
$$-V(z,i) =$$
$$E\left(\left\langle \left(\widetilde{A}_{r(k)}(k) + A_{r(k)}\right)' P_{r(k+1)} \left(\widetilde{A}_{r(k)}(k) + A_{r(k)}\right) z, z\right\rangle \middle| r(k) = i\right)$$
$$-V(z,i) =$$
$$E\left(\left\langle \widetilde{A}_{r(k)}(k)' P_{r(k+1)} \widetilde{A}_{r(k)}(k) z, z\right\rangle \middle| r(k) = i\right) +$$
$$2E\left(\left\langle \widetilde{A}_{r(k)}(k)' P_{r(k+1)} A_{r(k)} z, z\right\rangle \middle| r(k) = i\right) +$$
$$E\left(\left\langle A'_{r(k)} P_{r(k+1)} A_{r(k)} z, z\right\rangle \middle| r(k) = i\right) - V(z,i) =$$
$$E\left(\left\langle \widetilde{A}_{r(k)}(k)' P_{r(k+1)} \widetilde{A}_{r(k)}(k) z, z\right\rangle \middle| r(k) = i\right) +$$
$$2E\left(\left\langle \widetilde{A}_{r(k)}(k)' P_{r(k+1)} A_{r(k)} z, z\right\rangle \middle| r(k) = i\right) +$$
$$A'_i \left(\sum_{j\in S} p_{ij} V(z,j)\right) A_i - V(z,i) =$$

$$E\left(\left\langle A'_{r(k)}P_{r(k+1)}A_{r(k)}z,z\right\rangle\Big|\, r(k)=i\right)-V\left(z,i\right)=$$
$$E\left(\left\langle \widetilde{A}_{r(k)}(k)'P_{r(k+1)}\widetilde{A}_{r(k)}(k)z,z\right\rangle\Big|\, r(k)=i\right)+$$
$$2E\left(\left\langle \widetilde{A}_{r(k)}(k)'P_{r(k+1)}A_{r(k)}z,z\right\rangle\Big|\, r(k)=i\right)-V\left(z,i\right)^2.$$

From the assumption of the lemma we know that

$$E\left(\left\langle \widetilde{A}_{r(k)}(k)'P_{r(k+1)}\widetilde{A}_{r(k)}(k)z,z\right\rangle\Big|\, r(k)=i\right)\underset{k\to\infty}{\to}0$$

and

$$E\left(\left\langle \widetilde{A}_{r(k)}(k)'P_{r(k+1)}A_{r(k)}z,z\right\rangle\Big|\, r(k)=i\right)\underset{k\to\infty}{\to}0.$$

Therefore there exists k_0 such that for all $k\geq k_0$

$$E\left(\left\langle \widetilde{A}_{r(k)}(k)'P_{r(k+1)}\widetilde{A}_{r(k)}(k)z,z\right\rangle\Big|\, r(k)=i\right)+$$
$$2E\left(\left\langle \widetilde{A}_{r(k)}(k)'P_{r(k+1)}A_{r(k)}z,z\right\rangle\Big|\, r(k)=i\right)<\varepsilon\left\|z\right\|^2,$$

where $\varepsilon=\frac{1}{2\max\|P_i\|}$. Consequently, we get

$$\frac{\widetilde{V}\left(z,k,i\right)}{V(z,i)}\leq-\frac{\left\|z\right\|^2}{\langle P_iz,z\rangle}+\varepsilon\leq-\gamma,$$

with $\gamma=\frac{1}{2\max\|P_i\|}$. From the above, we obtain

$$E\left(V\left(z(k+1),r(k+1)\right)\right|z(k)=z,r(k)=i)\leq(1-\gamma)\,EV(z(k),i).$$

Since this inequality is true for all $i\in S$, therefore,

$$EV\left(z(k+1),r(k+1)\right)\leq(1-\gamma)\,EV(z(k),r(k)).$$

Recursively, we have

$$EV\left(z(k+1),r(k+1)\right)\leq(1-\gamma)^{k+1}\,EV(z(0),r(0)).$$

From the definition of V it is also clear that $EV\left(z(k+1),r(k+1)\right)\geq\alpha\left\|z\right\|^2$ for certain positive α. Combining this fact with the last inequality we obtain (12) and (13). ∎

The previous Lemma deals with the MSS of (11), however we will also need results about the ASS of this system. Such a result can be easily obtained from (13) by applying the following observation: For a sequence $y(n)$, $n=1,2,...$ of nonnegative real valued random variables we have

$$E\sum_{n=1}^{\infty}y(n)=\sum_{n=1}^{\infty}Ey(n)$$

and particularly if $\sum_{n=1}^{\infty} Ey(n) < \infty$, then $\sum_{n=1}^{\infty} y(n) < \infty$ a.c. Applying this observation to $y(n) = \|z(n)\|^2$, where $z(n)$ is defined by (11) and having in mind (13) we get the following

COROLLARY 2.1 *Under assumption of Lemma 2.2 with $z(n)$ defined by (11) we have*

$$\sum_{n=1}^{\infty} \|z(n)\|^2 < \infty \ a.c.$$

The next two lemmas contain technical results that will be used in the proof of optimality of adaptive control.

LEMMA 2.3 *Suppose that for each $i \in S$ a sequence $\overline{A}_i(k)$, $k = 0, 1, ...$ of $n \times n$ random matrices is given. Consider a system*

$$z(k+1) = \overline{A}_{r(k)}(k)z(k) + f(k), \tag{14}$$

where $f(k)$ is a sequence of n dimensional random vectors such that

$$E\|f(k)\|^2 < \infty,$$

$(f(0),\ f(1), ...)$ and $\left(\overline{A}_{r(0)}(0),\ \overline{A}_{r(1)}(1),\ ...\right)$ are mutually independent. Suppose that sequence $\left(\overline{A}_{r(0)}(0),\ \overline{A}_{r(1)}(1),\ ...\right)$ satisfies assumptions of Lemma 2.2. Then there exist positive constants c_1 and c_2 such that

$$\sum_{k=0}^{N} E\|z(k)\|^2 \le c_1 \|z_0\|^2 + c_2 \sum_{k=0}^{N} E\|f(k)\|^2 . \tag{15}$$

Proof. We have

$$z(k+1) = \prod_{l=0}^{k} \overline{A}_{r(l)}(l) z_0 + \sum_{l=1}^{k} \prod_{p=l}^{k} \overline{A}_{r(p)}(p) f(p),$$

with the notation

$$\prod_{p=k}^{k} \overline{A}_{r(p)}(p) f(p) = f(k),$$

and therefore

$$\|z(k+1)\| \le \left\| \prod_{l=0}^{k} \overline{A}_{r(l)}(l) \right\| \|z_0\| + \sum_{l=1}^{k} \left\| \prod_{p=l}^{k} \overline{A}_{r(p)}(p) \right\| \|f(p)\| .$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, taking expectation and applying Cauchy-Schwartz inequality, we obtain

$$E\|z(k+1)\|^2 \le 2E\left\|\prod_{l=0}^{k}\overline{A}_{r(l)}(l)\right\|^2 \|z_0\|^2 + 2E\left(\sum_{l=1}^{k}\left\|\prod_{p=l}^{k}\overline{A}_{r(p)}(p)\right\|\|f(p)\|\right)^2 \le$$
$$2E\left\|\prod_{l=0}^{k}\overline{A}_{r(l)}(l)\right\|^2 \|z_0\|^2 + 2E\left[\left(\sum_{l=1}^{k}\left\|\prod_{p=l}^{k}\overline{A}_{r(p)}(p)\right\|^2\right)\left(\sum_{l=1}^{k}\|f(p)\|^2\right)\right] =$$
$$2E\left\|\prod_{l=0}^{k}\overline{A}_{r(l)}(l)\right\|^2 \|z_0\|^2 + 2\left(\sum_{l=1}^{k}E\left\|\prod_{p=l}^{k}\overline{A}_{r(p)}(p)\right\|^2\right)\left(\sum_{l=1}^{k}E\|f(p)\|^2\right). \tag{16}$$

From Lemma 2.2 we have

$$E\left\|\prod_{p=l}^{k}\overline{A}_{r(p)}(p)\right\|^2 \le \beta q^{k-l}.$$

Applying this inequality to (16) we get

$$E\|z(k+1)\|^2 \le 2\beta q^k \|z_0\|^2 + 2\beta\left(\sum_{l=1}^{k}q^{k-l}\right)\left(\sum_{l=1}^{k}E\|f(p)\|^2\right).$$

Because $q < 1$, the last inequality implies (15). ■

Using Corollary 2.1 and following the line of reasoning of the above proof we can show the following:

LEMMA 2.4 *Consider system (14) and suppose that* $\left(\overline{A}_{r(0)}(0),\ \overline{A}_{r(1)}(1),\ \ldots\right)$ *satisfies assumptions of Lemma 2.2. Then there exist nonnegative random variables* c_1 *and* c_2 *such that*

$$\sum_{k=0}^{N}\|z(k)\|^2 \le c_1\|z_0\|^2 + c_2\sum_{k=0}^{N}\|f(k)\|^2. \tag{17}$$

We end this section with definitions of mean square stabilizability and mean square detectability (see Costa, Fragoso, 1995).

DEFINITION 2.2 *System*

$$x(k+1) = A_{r(k)}x(k) + B_{r(k)}u(k),$$

or alternatively $\{A_i, B_i, i \in S\}$ *is called mean square stabilizable if there exists a feedback control* $u(k) = L_{r(k)}x(k)$ *such that the resulting closed loop system*

$$x(k+1) = \left(A_{r(k)} + B_{r(k)}L_{r(k)}\right)x(k)$$

is stochastically stable.

DEFINITION 2.3 *Consider matrices* C_i*,* $i \in S$ *of size* $n \times l$*. The system*

$$x(k+1) = A_{r(k)}x(k)$$
$$y(k) = C_{r(k)}x(k)$$

or alternatively $\{C_i, A_i, i \in S\}$ *is called mean square detectable if there exist matrices* H_i*,* $i \in S$ *such that for any initial conditions* (i_0, x_0) *we have*

$$\lim_{N\to\infty} E\|z(N)\|^2 = 0,$$

where $z(k)$ *is given by*

$$z(k+1) = \left(A_{r(k)} + C_{r(k)}H_{r(k)}\right)z(k).$$

3. Characterization of a class of optimal controls

The control given by (6) is not a unique optimal control for the problem (1), (3). The next theorem describes a large class of controls that are optimal for this problem.

THEOREM 3.1 *Suppose that* $\{A_i, B_i, i \in S\}$ *is mean square stabilizable and* $\left\{\sqrt{Q_i}, A_i, i \in S\right\}$ *is mean square detectable. Let* $G_i(k)$*,* $k = 0, 1, \ldots$ *be a sequence of random* $n \times n$ *matrices such that* $G_i(k)$ *is* $\mathcal{F}_k$*-measurable, there exists a constant* c *such that*

$$\|G_i(k)\| < c \tag{18}$$

and

$$\lim_{k\to\infty} G_i(k) = G_i, \; i \in S \quad a.s. \tag{19}$$

where G_i *are given by (5). Moreover, for each* $i \in S$ *let* $v_i(k)$*,* $k = 0, 1, \ldots$ *be a sequence of independent* n *dimensional random variables such that* $v_i(k)$*,* $r(k)$ *and* $w(k)$ *are mutually independent and*

$$Ev_i(k) = 0, \; i \in S, \; k = 0, 1, \ldots \tag{20}$$

and

$$\lim_{k\to\infty} E\|v_i(k)\|^2 = 0, \quad i \in S. \tag{21}$$

Then the control given by

$$u(k) = -G_{r(k)}(k)x(k) + v_{r(k)}(k) \tag{22}$$

is optimal for problem (1), (3).

Proof. Define a random variable

$$\begin{aligned}\xi(k+1) \quad &= \quad \left\langle Q_{r(k)}x(k), x(k)\right\rangle + \left\langle R_{r(k)}u(k), u(k)\right\rangle \\ &\quad + \left\langle T_{r(k)}x(k+1), x(k+1)\right\rangle - \left\langle P_{r(k)}x(k), x(k)\right\rangle,\end{aligned}$$

where

$$T_{r(k)} = \sum_{j\in S} p_{r(k)\ j}P_j.$$

Using (1) and (4) we obtain

$$\begin{aligned}\xi(k+1) &= \langle (Q_{r(k)} + G^T_{r(k)}(k)R_{r(k)}G_{r(k)}(k) \\ &\quad + L^T_{r(k)}(k)T_{r(k)}L_{r(k)}(k) - P_{r(k)})x(k), x(k)\rangle \\ &\quad + -2\left\langle R_{r(k)}v_{r(k)}(k), G_{r(k)}(k)x(k)\right\rangle \\ &\quad + 2\left\langle T_{r(k)}\left(B_{r(k)}v_{r(k)}(k) + w(k+1)\right), L_{r(k)}(k)x(k)\right\rangle \\ &\quad + \left\langle R_{r(k)}v_{r(k)}(k), v_{r(k)}(k)\right\rangle \\ &\quad + \left\langle T_{r(k)}\left(B_{r(k)}v_{r(k)}(k) + w(k+1)\right), B_{r(k)}v_{r(k)}(k) + w(k+1)\right\rangle\end{aligned} \tag{23}$$

where

$$L_{r(k)}(k) = A_{r(k)} - B_{r(k)}G_{r(k)}(k).$$

Now we will analyze each term in the above sum separately to show that

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{k=0}^{N-1} \xi(k+1) = \sum_{i\in S}\sum_{j\in S} \pi(i)p_{ij}tr\left(P_j\right). \tag{24}$$

From (4) we have

$$P_i = Q_i + L_i^T T_i L_i + G_i R_i G_i,\ \ i \in S,$$

where

$$L_i = A_i - B_i G_i$$

and G_i is given by (5). Therefore

$$\begin{aligned}&Q_{r(k)} + G^T_{r(k)}(k)R_{r(k)}G_{r(k)}(k) + L^T_{r(k)}(k)T_{r(k)}L_{r(k)}(k) - P_{r(k)} = \\ &\quad G^T_{r(k)}(k)R_{r(k)}G_{r(k)}(k) + L^T_{r(k)}(k)T_{r(k)}L_{r(k)}(k) + \\ &\quad -L^T_{r(k)}T_{r(k)}L_{r(k)} - G'_{r(k)}R_{r(k)}G_{r(k)}.\end{aligned}$$

Now (19) implies that

$$\lim_{k\to\infty} L_i(k) = L_i,\ i \in S \text{ a.s.}$$

and consequently

$$\lim_{k\to\infty} \left(Q_i + G_i^T(k)R_iG_i(k) + L_i^T(k)T_iL_i(k) - P_i\right) = 0,\ i \in S \text{ a.s.}$$

and

$$\lim_{k\to\infty} \Big(Q_{r(k)} + G_{r(k)}^T(k)R_{r(k)}G_{r(k)}(k) + L_{r(k)}^T(k)T_{r(k)}L_{r(k)}(k) - P_{r(k)}\Big) = 0 \quad \text{a.s.}$$

Moreover assumptions (18) and (19) guarantee, that there exists constant c_1 such that

$$\left\| Q_{r(k)} + G_{r(k)}^T(k)R_{r(k)}G_{r(k)}(k) + L_{r(k)}^T(k)T_{r(k)}L_{r(k)}(k) - P_{r(k)} \right\| < c_1 \tag{25}$$

and

$$\lim_{k\to\infty} \left\| Q_{r(k)} + G_{r(k)}^T(k)R_{r(k)}G_{r(k)}(k) + L_{r(k)}^T(k)T_{r(k)}L_{r(k)}(k) - P_{r(k)} \right\| = 0. \tag{26}$$

Lemma 2.3 with $\overline{A}_{r(k)}(k) = L_{r(k)}(k)$, and $f(k) = B_{r(k)}v_{r(k)}(k) + w(k+1)$ together with (2) and (21) show that there exists constant c_2 such that

$$\frac{1}{N} E \sum_{k=0}^{N-1} \|x(k)\|^2 < c_2. \tag{27}$$

Finally using (25), (26) and (27) we obtain

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{k=0}^{N-1} \Big\langle \Big(Q_{r(k)} + G_{r(k)}^T(k)R_{r(k)}G_{r(k)}(k) + L_{r(k)}^T(k)T_{r(k)}L_{r(k)}(k) - P_{r(k)} \Big) x(k), x(k) \Big\rangle = 0. \tag{28}$$

From definition of $v_{r(k)}$ and properties of $w(k)$ we obtain

$$E \left\langle R_{r(k)}v_{r(k)}(k), G_{r(k)}(k)x(k) \right\rangle = 0, \tag{29}$$

$$E \left\langle T_{r(k)} \left(B_{r(k)}v_{r(k)}(k) + w(k+1) \right), L_{r(k)}(k)x(k) \right\rangle = 0 \tag{30}$$

and

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{k=0}^{N-1} \left\langle R_{r(k)}v_{r(k)}(k), v_{r(k)}(k) \right\rangle = 0.$$

Finally

$$E\left\langle T_{r(k)}\left(B_{r(k)}v_{r(k)}(k)+w(k+1)\right),B_{r(k)}v_{r(k)}(k)+w(k+1)\right\rangle =$$
$$E\left\langle T_{r(k)}B_{r(k)}v_{r(k)}(k),B_{r(k)}v_{r(k)}(k)\right\rangle +E\left\langle T_{r(k)}w(k+1),w(k+1)\right\rangle$$

and since

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\left\langle T_{r(k)}B_{r(k)}v_{r(k)}(k),B_{r(k)}v_{r(k)}(k)\right\rangle =0$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}E\left\langle T_{r(k)}w(k+1),w(k+1)\right\rangle =\sum_{i\in S}\sum_{j\in S}\pi(i)p_{ij}tr\left(P_{j}\right)$$

we have

$$\lim_{N\to\infty}\frac{1}{N}E\sum_{k=0}^{N-1}E\left\langle T_{r(k)}\left(B_{r(k)}v_{r(k)}(k)+w(k+1)\right),B_{r(k)}v_{r(k)}(k)+w(k+1)\right\rangle$$
$$=\sum_{i\in S}\sum_{j\in S}\pi(i)p_{ij}tr\left(P_{j}\right). \tag{31}$$

When we combine (28), (29), (30) and (3.), we obtain (24). Whence, by the definition of $\xi(k)$ we conclude that

$$\frac{1}{N}E\sum_{k=0}^{N-1}\left[\left\langle Q_{r(k)}x(k),x(k)\right\rangle +\left\langle R_{r(k)}u(k),u(k)\right\rangle\right] =$$

$$\frac{1}{N}E\sum_{k=0}^{N-1}\xi(k+1)+\frac{1}{N}E\sum_{k=0}^{N-1}\left[\left\langle P_{r(k)}x(k),x(k)\right\rangle -\left\langle T_{r(k)}x(k+1),x(k+1)\right\rangle\right] =$$

$$\frac{1}{N}E\sum_{k=0}^{N-1}\xi(k+1)+\frac{1}{N}E\left[\left\langle P_{r(0)}x(0),x(0)\right\rangle -\left\langle T_{r(N-1)}x(N),x(N)\right\rangle\right] +$$

$$\frac{1}{N}E\sum_{k=1}^{N-1}\left[\left\langle P_{r(k)}x(k),x(k)\right\rangle -\left\langle T_{r(k-1)}x(k),x(k)\right\rangle\right] =$$

$$\frac{1}{N}E\sum_{k=0}^{N-1}\xi(k+1)+\frac{1}{N}E\left[\left\langle P_{r(0)}x(0),x(0)\right\rangle -\left\langle T_{r(N-1)}x(N),x(N)\right\rangle\right] +$$

$$\frac{1}{N}E\sum_{k=1}^{N-1}\left[\left\langle\left(P_{r(k)}-T_{r(k-1)}\right)x(k),x(k)\right\rangle\right]. \tag{32}$$

Now observe that (27) implies that

$$\lim_{N\to\infty}\frac{1}{N}E\left[\left\langle P_{r(0)}x(0),x(0)\right\rangle-\left\langle T_{r(N-1)}x(N),x(N)\right\rangle\right]=0, \tag{33}$$

moreover, $T_{r(k-1)}=E\left(P_{r(k)}\middle|\,r(k-1)\right)$ and therefore $E\left(P_{r(k)}-T_{r(k-1)}\right)=0$. Furthermore, random variables $P_{r(k)}-T_{r(k-1)}$ and $x(k)$ are independent given $r(k-1)$, so that

$$E\sum_{k=1}^{N-1}\left[\left\langle\left(P_{r(k)}-T_{r(k-1)}\right)x(k),x(k)\right\rangle\right]=0. \tag{34}$$

Finally, combining (33) and (34) with (32) gives (24). This leads directly to the conclusion of the theorem, because the right hand side of (24) is, according to Theorem 1.1, equal to the minimal value of the cost functional, whereas the left hand side is equal to the value of the cost functional corresponding to the control given by (22). ■

4. Parameter estimation

In this section we briefly describe the weighted least-squares (WLS) algorithm. Consider the following linear regression model

$$y(k+1)=\theta'\varphi(k)+w(k+1), \tag{35}$$

where θ is an unknown parameter matrix , $y(k)$ and $\varphi(k)$ are the observation and the regressor and $w(k)$ is the noise with the properties described in the Introduction.

Fix $\delta>0$, $0<\alpha<1$, and matrix $\theta(0)$ of the same size as θ. Define $P_0=\alpha I$,

$$q(k)=\left\|P_0^{-1}\right\|+\sum_{l=0}^{k}\|\varphi(l)\|,\ \alpha(k)=\frac{1}{\log^{1+\delta}q(k)},$$
$$P(k+1)=P(k)-\frac{P(k)\varphi(k)\varphi'(k)P(k)}{\alpha_k^{-1}+\varphi'(k)P(k)\varphi(k)},$$
$$L(k)=\frac{P(k)\varphi(k)}{\alpha_k^{-1}+\varphi'(k)P(k)\varphi(k)}.$$

With this notation the recursive WLS algorithm has the following form (see Guo, 1991, for details)

$$\theta(k+1)=\theta(k)+L(k)\left(y(k+1)-\theta'(k)\varphi(k)\right).$$

The proof of the next theorem may be found in Guo (1996).

THEOREM 4.1 *The WLS algorithm has the following properties:*
1. $\sum_{l=1}^{k} \|\varphi'(k)(\theta(k)-\theta)\|^2 = o(q(k)) + O(1)$
2. *$\theta(k)$ converges almost surely to a finite random variable $\overline{\theta}$ (not necessarily equal to θ).*

Now we show how the WLS algorithm can be used to estimate coefficients A_i and B_i of (1).

For each $i \in S$ we define the stopping times $(\tau_i(k))_{k\in N}$ by the following recurrent formula

$$\tau_i(1) = \min\{l \geq 0 : r(l) = i\}, \ \tau_i(k+1) = \min\{l > \tau_i(k) : r(l) = i\} \quad (36)$$

and denote

$$\theta_i' = \begin{bmatrix} A_i & B_i \end{bmatrix}, \ \varphi_i(k) = \begin{bmatrix} x(\tau_i(k)) \\ u(\tau_i(k)) \end{bmatrix}. \quad (37)$$

Then, (1) can be rewritten in the form of (35) as

$$x(\tau_i(k)+1) = \theta_i'\varphi_i(k) + w(\tau_i(k)+1) \quad (38)$$

and using the WLS algorithm we can for each $i \in S$ construct a sequence

$$\theta_i'(k) = \begin{bmatrix} \widehat{A}_i(k) & \widehat{B}_i(k) \end{bmatrix}$$

which will be called the WLS estimator of $\theta_i' = \begin{bmatrix} A_i & B_i \end{bmatrix}$. Observe that the assumption about ergodicity of $r(k)$ implies that the sequence is infinite.

Next theorem gives sufficient conditions for $\theta_i(k)$ to be strongly consistent. The proof follows immediately from Lemma 3 of Guo and Chen (1991).

THEOREM 4.2 *For each $i \in S$ let $\overline{v}_i(k)$, $k = 0, 1, \ldots$, be a sequence of independent n dimensional random variables such that $\overline{v}_i(k)$ is independent of $r(k)$ as well as of $w(k)$,*

$$E\overline{v}_i(k) = 0, E\overline{v}_i(k)\overline{v}_i'(k) = I, \ i \in S, \ k = 0, 1, \ldots$$

and put

$$v_i(k) = \frac{\overline{v}_i(k)}{k^{\varepsilon}},$$

where $\varepsilon \in (0, 1/8n)$. Consider system (1) with the control law:

$$u(k) = -G_{r(k)}(k)x(k) + v_{r(k)}(k).$$

Assume that the control is such that

$$q(k) = O(k), \quad (39)$$

for certain $i \in S$ ($q(k)$ depends on i thought (38)). Then the estimator $\theta_i(k)$ is strongly consistent.

5. Adaptive control

The objective of this section is to construct certain adaptive control sequence and to show that it satisfies assumption of Theorem 3.1 and therefore by this theorem it is optimal.

Let us make the following assumption

(A) Suppose that for each $i \in S$ a compact set Ξ_i of pairs (A, B) of matrices is known and the sets are such that $(A_i, B_i) \in \Xi_i$ and for each choice of $(\overline{A}_i, \overline{B}_i) \in \Xi_i$, systems $\left\{\overline{A}_i, \overline{B}_i, i \in S\right\}$ and $\left\{\sqrt{Q_i}, \overline{A}_i, i \in S\right\}$ are stochastically stabilizable and stochastically detectable, respectively.

The meaning of the assumption is that we know the parameters of the system with certain accuracy. If the original system is stochastically stabilizable and stochastically detectable then it is always possible to find a neighborhood $(\Xi_i)_{i\in S}$ of the true parameters such that the assumption (A) is satisfied (see, Czornik, Nawrat, Świerniak, 2002). In the construction of the adaptive control we need this assumption to guarantee that the trajectory of the system is bounded.

Under assumption (A) for each choice of $(\overline{A}_i, \overline{B}_i) \in \Xi_i$ there exists a matrix $\overline{G}_i$ such that the system $\left\{\overline{A}_i - \overline{B}_i\overline{G}_i, i \in S\right\}$ is stable.

For $\theta_i'(k) = \begin{bmatrix} \widehat{A}_i(k) & \widehat{B}_i(k) \end{bmatrix} \in \Xi_i$ denote by $G_i(k)$ the matrix given by (5) with (A_i, B_i) replaced by $(\widehat{A}_i(k), \widehat{B}_i(k))$.

Now the adaptive control is defined by

$$u(k) = \begin{cases} -G_{r(k)}(k)x(k) + v_{r(k)}(k) \text{ if } \begin{bmatrix} \widehat{A}_i(\tau_i(k)) & B_i(\tau_i(k)) \end{bmatrix} \in \Xi_i \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } i \in S \\ -\overline{G}_{r(k)}x(k) + v_{r(k)}(k) \text{ in opposite case} \end{cases} \tag{40}$$

where the random variables $v_i(k)$ are defined in Theorem 4.2

THEOREM 5.1 *Under assumption (A) the adaptive control given by (40) is optimal for system (1) with cost functional (3).*

In the proof of this theorem we will need the following two lemmas.

LEMMA 5.1 *(Czornik, Świerniak, 2002) Suppose that $\{A_i, B_i, i \in S\}$ is stochastically stabilizable and $\left\{\sqrt{Q_i}, A_i, i \in S\right\}$ is stochastically detectable. Let the sequence $(A_i(k), B_i(k))$, such that for each $i \in S$*

$$A_i = \lim_{k\to\infty} A_i(k),\ B_i = \lim_{k\to\infty} B_i(k),$$

Then there exists k_0 such that for all $k \geq k_0$ the coupled Riccati equation

$$P_i(k) = Q_i + (A_i(k) - B_i(k)G_i(k))^T \left(\sum_{j\in S} p_{ij}P_j(k)\right)(A_i(k) - B_i(k)G_i(k)) + G_i'(k)R_iG_i(k),$$

where

$$G_i(k) = \left(R_i + B_i^T(k)\left(\sum_{j\in S} p_{ij}P_j(k)\right)B_i(k)\right)^{-1} B_i^T(k)\left(\sum_{j\in S} p_{ij}P_j(k)\right)A_i(k)$$

has a unique positive semidefinite solution and

$$\lim_{k\to\infty} P_i(k) = P_i,\ i \in S,$$

where P_i is the solutions of (4).

LEMMA 5.2 *Suppose that for certain control u the solution of (1) satisfies the following condition*

$$\sum_{l=1}^{k}\left(\|x(l)\|^2 + \|u(l)\|^2\right) = O(k), \tag{41}$$

then for each $i \in S$ we have

$$\sum_{l=1}^{k}\left(\|x(\tau_i(l))\|^2 + \|u(\tau_i(l))\|^2\right) = O(k). \tag{42}$$

Proof. Fix $i \in S$. From the assumption about $r(k)$ we know that the limit

$$\lim_{k\to\infty} \frac{k}{\tau_i(k)} \tag{43}$$

exists and is greater than 0. We have

$$\frac{\sum_{l=1}^{\tau_i(k)}\left(\|x(l)\|^2 + \|u(l)\|^2\right)}{\tau_i(k)} = \frac{\sum_{l=1}^{\tau_i(k)}\left(\|x(l)\|^2 + \|u(l)\|^2\right)/k}{\tau_i(k)/k} \geq \frac{\sum_{l=1}^{k}\left(\|x(\tau_i(l))\|^2 + \|u(\tau_i(l))\|^2\right)/k}{\tau_i(k)/k}.$$

From the assumption (41) we know that the left hand side of the last inequality is bounded and (43) implies that the denominator in the right hand side is bounded, therefore (42) follows. ■

Proof of Theorem 5.1 From the point 2 of Theorem 4.1 we know that for each $i \in S$ the sequence $\left[\begin{array}{cc}\widehat{A}_i(k) & \widehat{B}_i(k)\end{array}\right]$ converges. Denote by $\overline{\theta}_i' = \left[\begin{array}{cc}\overline{A}_i & \overline{B}_i\end{array}\right]$ the limit. First, we show that $\overline{\theta}_i \in \Xi_i$ for all $i \in S$. Suppose that

$$\overline{\theta}_{i_0}' \notin \Xi_{i_0} \tag{44}$$

for certain $i_0 \in S$. Then, according to (40) the control is

$$u(k) = -\overline{G}_{r(k)}x(k) + v_{r(k)}(k)\,. \tag{45}$$

From the assumption about $\overline{G}_i$ and Lemma 2.4 we conclude that

$$\sum_{l=1}^{k}\left(\|x(l)\|^2 + \|u(l)\|^2\right) = O(k)$$

and by Lemma 5.2 assumption (39) is satisfied and therefore $\overline{\theta}_{i_0} = \theta_{i_0}$, by Theorem 4.2. This is a contradiction to (44). Now rewrite the model (38) as

$$x\left(\tau_i(k)+1\right) = \overline{\theta}_i'\varphi_i(k) + w(\tau_i(k)+1) + \alpha_i(\tau_i(k)), \tag{46}$$

where

$$\alpha_i(\tau_i(k)) = \left(\theta_i - \overline{\theta}_i\right)'\varphi_i(k) = \left(\theta_i - \theta_i(k)\right)'\varphi_i(k) + \left(\theta_i(k) - \overline{\theta}_i\right)'\varphi_i(k).$$

By Theorem 4.1 we conclude that

$$\sum_{l=1}^{k}\|\alpha_i(\tau_i(k))\|^2 = o\left(q(k)\right) + O(1). \tag{47}$$

Since we know that $\overline{\theta}_i \in \Xi_i$ for all $i \in S$ then the control (40) is defined by

$$u(k) = -G_{r(k)}(k)x(k) + v_{r(k)}(k)$$

for sufficiently large k, and therefore (46) takes the following form

$$x\left(\tau_i(k)+1\right) = \left(\overline{A}_i - \overline{B}_iG_i(\tau_i(k))\right)x\left(\tau_i(k)\right) + w(\tau_i(k)+1) + \alpha_i(\tau_i(k)).$$

By Lemma 5.1 and Theorem 1.1 we know that $\overline{A}_i - \overline{B}_iG_i(k)$ converges to a certain matrix $\widetilde{A}_i$ and the system $\left\{\widetilde{A}_i : i \in S\right\}$ is MSS. Now by Lemma 2.4 and (47) we get

$$\sum_{l=1}^{k}\|x(\tau_i(l))\|^2 = O(k) + o\left(q(k)\right)$$

which in light of (45) implies

$$\sum_{l=1}^{k}\left(\|x(\tau_i(l))\|^2 + \|u(\tau_i(l))\|^2\right) = O(k) + o\left(q(k)\right)$$

and consequently

$$q(k) = O(k).$$

The last equality shows that assumptions (39) of Theorem 4.2 are satisfied and therefore $\overline{\theta}_{i_0} = \theta_{i_0}$. Finally, the conclusion of the theorem follows from Theorem 3.1. ■

6. Conclusions

In this paper the adaptive control problem for jump linear system with quadratic cost functional on infinite time interval is solved. The assumptions are that we know certain closed subset of parameters such that the true parameters belong to the set and there is a feedback that stabilizes all systems with coefficients in this set. Moreover, we assume that the state of the Markov chain is perfectly known. Regarding the first of these assumption it seems that it is justified in real-word situation when parameters, although not completely known, are still supposed to be given with some accuracy. As we mentioned, the second assumption could be justified using the sensitivity analysis proposed in Czornik, Nawrat, Świerniak (2002). The assumption about the common stabilizing feedback can be replaced by the stability of the open loop system. In this case in the definition (40) of adaptive control there should be $\overline{G}_{r(k)} = 0$. The proof of optimality remains the same. Under such assumption one of the first results about standard adaptive LQ control have been obtained (see Chen, Guo, 1986). This assumption is very restrictive and its removal is the biggest challenge for further research. Also the perfect observation of the state of the Markov chain is doubtful and in further research it should be replaced by the partial observation of the Markov chain. In overcoming this difficulty the results of from Dufour and Elliott (1998) seem to be promising.

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