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Asymptotic behaviour to a heat equation with a delayed control in the source term

by

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Abstract: A one dimensional heat equation in a semi-infinite medium controlled through a heat source depending on the delayed heat flux at the extremum is studied. By reducing the problem to a delayed Volterra integral equation with a weakly singular kernel, we find conditions on the initial datum and on the source term of the equation to control the asymptotic behaviour of the mean temperatures.

Keywords: heat equation, delayed control, Volterra integral equation with delay.

1. Introduction

The solution u of the non standard initial-boundary value problem for the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = -F(u_x(0,t)), & x > 0,\, t > 0, \\ u(x,0) = h(x), & x > 0, \\ u(0,t) = 0, & t > 0, \end{cases} \tag{1}$$

models the temperature in a semi-infinite slab with an initial temperature profile prescribed by a bounded continuous function h and with its extremum $x = 0$ mantained at zero temperature. The evolution of temperatures in the slab is controlled by the source term $-F(u_x(0,t))$ which depends on the heat flux at $x = 0$ through a certain control function F. Since the source term of the first equation (1) reacts with the *heat flux* instead of the temperature u at the extremum, this problem differs from that one called *thermostat problem* (see Kenmochi and Primicerio, 1988, and the references therein).

Basic properties of the problem (1) were studied in Villa (1996) and Tarzia and Villa (1990); in particular, the global existence and uniqueness of solution is there established for a Lipschitz-continuous control F satisfying the *stabilization conditions*

$$\begin{cases} vF(v) \geq 0, \quad v \in \mathbf{R} \\ F(0) = 0. \end{cases} \tag{2}$$

For this class of control functions and for a bounded continuous initial profile h, it can be proved (Villa, 1986) that the solution u to problem (1) vanishes as $t \uparrow +\infty$.

Results about global existence of solutions to the nonlinear problem (1) can be found in Lions (1969), Racke (1992), Klainexmann (1981), Gwinecki (1995).

A more detailed study of problem (1) begun with the two letters of Berrone (1994) (whose content is partially reproduced in Berrone, Tarzia, Villa (2000), where well-known results on Volterra integral equations were invoked to establish global existence and uniqueness of solution to problem (1) for continuous controls satisfying (2) and, in addition, the precise objective of the control is envisaged. In this last regard, it must be pointed out that the solution u_0 corresponding to problem (1) in the absence of control ($F \equiv 0$) also converges to zero as $t \uparrow +\infty$, so that the interest of a control F satisfying the stabilization conditions (2) is just to increase the rate of convergence to zero as $t \uparrow +\infty$ of the solution u_F of the problem with control F. Concretely, the objective of a stabilizing control F was defined to be

$$\lim_{t \uparrow +\infty} \frac{u_F(x,t)}{u_0(x,t)} = 0 \tag{3}$$

for every $x > 0$. In was shown in Berrone, Tarzia, Villa (2000), on the one hand, by means of examples that not every stabilizing control F fulfills the control condition (3) and, on the other hand, that the following result providing sufficient conditions of controllability holds:

THEOREM 1.1 *If $C_1 \leq h(x) \leq C_2$ for two positive contants C_1, C_2 and if the stabilization conditions (2) hold, the control F is a convex function on $(0, +\infty)$ and $F'(0^+) > 0$; then condition (3) holds.*

Along this paper, a variation of problem (1) is studied in which a delayed response of the heat source replaces the instantaneous one implicit in the problem. The choice of this variation seeks for a realistic improvement of the model described by (1): in a real system, the heat source supposedly reacts not in an instantaneous way to the heat-flux at the extremum of the slab measured at time t but to the heat-flux at an anterior time $t-\tau$. Once this delay is taken into account, problem (1) becomes

$$\begin{cases} u_t(x,t)-u_{xx}(x,t)=-F(u_x(0,t-\tau)), & x>0,\, t>0,\\ u(x,0)=h(x), & x>0,\\ u(0,t)=0, & t>0,\\ u_x(0,t)=\phi(t), & -\tau\le t\le 0. \end{cases} \tag{4}$$

Note that a new condition must be imposed on $u_x(0,t)$ along the "past times" $-\tau\le t\le 0$ in order to uniquely determine the solution u.
The usual size condition

$$|h(x)|\le C_1 e^{C_2 x^\alpha},\ x>0, \tag{5}$$

with $C_1, C_2>0$ and $0\le\alpha<2$, is assumed to hold by the initial temperature profile $h\in\mathcal{C}^0[0,+\infty)$. The case in which $\alpha=0$ (boundedness) and $\alpha=1$ (exponential growth), in (5), will play an important role in the next sections.

Generally speaking, the introduction of the delay τ in the source term causes an oscillating solution and then, its asymptotic properties considerably differ from the solution corresponding to the case $\tau=0$. In this way, the problem of controlling the asymptotic behaviour of the solutions through the function F becomes much more involved. In the next section, as a first step in the study of the changes produced by the introduction of the delay τ, the problem (4) is reduced to solve a Volterra integral equation with delay and some basic differences existing between this equation and the integral equation obtained from (1) are pointed out.

2. Preliminaries

Let us go back, for a moment, to the undelayed problem (1). The analysis of this model relies on the Volterra integral equation with a weakly singular (locally integrable) kernel

$$v(t)=f(t)-\int_0^t \frac{F(v(s))}{\sqrt{\pi(t-s)}}\,ds,\ t>0, \tag{6}$$

which is satisfied by the heat-flux at the extremum $x=0$; i.e., $v(t)=u_x(0,t),\ t>0$. The *forcing function* $f(t)$ in (6) is given by

$$f(t)=\frac{1}{2\sqrt{\pi}t^{3/2}}\int_0^{+\infty}\xi e^{-\xi^2/(4t)}h(\xi)\,d\xi,\ t>0. \tag{7}$$

Equation (6) is obtained in a standard way. Here we want only to sketch this procedure (for more details see Berrone, Tarzia, Villa, 2000). By supposing sufficient regularity of the coefficients and of the initial data, we see that

$u_x(x,t) \equiv v(x,t)$ solves the following problem

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0, & x > 0,\ t > 0, \\ v(x,0) = h'(x), & x > 0, \\ v_x(0,t) = F(v(0,t)), & t > 0. \end{cases} \tag{8}$$

Problem (8) modelizes the temperature in a semi-infinite slab whose extremum is radiating energy according to a general law of radiation $F(v)$ and was extensively studied (Miller, 1971; Saaty, 1967). By following a standard procedure (see, for example, Cannon, 1984), we represent its solution using Green functions, thus obtaining the integral equation (6) with $f(t)$ given by (7). Then, the unique *bounded* solution of problem (1) is expressed in terms of the solution $v(t)$ of equation (6). In fact, recalling that

$$\mathrm{erf}\zeta = \frac{2}{\pi}\int_0^{\zeta} e^{-y^2}\,dy,$$

we have:

$$u(x,t) = u_0(x,t) - \int_0^t \mathrm{erf}\left(\frac{x}{2\sqrt{t-s}}\right) F(v(s))\,ds, \tag{9}$$

where

$$u_0(x,t) = \int_0^{+\infty} G(x,\xi,t)h(\xi)\,d\xi \tag{10}$$

is the (bounded) solution corresponding to problem (1) with $F \equiv 0$ (see Cannon, 1984; Problem 3.3, p. 43). The kernel G under the integral sign in (10) is the Green function corresponding to the quarter plane and it is given by

$$G(x,\xi,t) = K(x-\xi,t) - K(x+\xi,t), \tag{11}$$

where $K(x,t) = (4\pi t)^{-1/2}e^{-x^2/(4t)}$ is the fundamental solution to the one-dimensional heat equation.

For a constant initial temperature $h \equiv h_0$, equation (6) becomes

$$v(t) = \frac{h_0}{\sqrt{\pi t}} - \int_0^t \frac{F(v(s))}{\sqrt{\pi(t-s)}}\,ds,\ \ t > 0, \tag{12}$$

and the control objective (3) is satisfied uniformly on $x > 0$ if and only if the condition

$$\lim_{t\uparrow+\infty}\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds = 0 \tag{13}$$

holds. More precisely, in Berrone (1994) and Berrone, Tarzia, Villa (2000) the following theorem is proved:

Theorem 2.1 *Let $h(x) \equiv h_0 \in \mathbf{R}$ be the initial temperature. Then the condition (3) is satisfied by a control function F if and only if the solution v of the integral equation (6) satisfies*

$$\lim_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds = 0. \tag{14}$$

Remark 2.1 *Let F be a stabilizing control such that the corresponding solution u_1 with an initial datum h_1 satisfies condition (3). If h_2 is any function such that $0 \le h_2 \le h_1$, then also the corresponding solution u_2 satisfies condition (3).*

The cornerstone in the developments contained in Berrone, Tarzia, Villa (2000) is constitued by the monotonicity methods (Gripenberg, Londen, Staffans, 1990, Chapter 20, and Miller, 1971, Chapter 4). In particular, Theorem 1.1 heavily rests on the following general one ensuring the positivity of the solution to equation (6) (Miller, 1971; p. 210).

Theorem 2.2 *Consider the real, scalar non linear Volterra integral equation*

$$v(t) = f(t) - \int_0^t h(t-s) g(s, v(s))\, ds. \tag{15}$$

Suppose $f(t)$ is positive and continuous in $t \in [0, +\infty)$. Let h be positive, continuous and locally L^1 in $0 < t < +\infty$. Suppose $g(t,v)$ is measurable in (t,v) and continuous in v, for $t \ge 0$ and $v \in \mathbf{R}$, with $vg(t,v) \ge 0$ for all (t,v). If

$$f(T)/f(t) \le h(T-s)/h(t-s)$$

whenever $0 \le s < T < t$, then equation (15) has a solution which satisfies $0 \le v(t) \le f(t)$ for all $t \ge 0$.

It should be noted that Theorem 2.2 can be suitably extended to embrace forcing functions like $f(t) = h_0/\sqrt{\pi t}$, occurring in (12).

In an analogous way as (1) leads to the integral equation (6), problem (4) can be reduced to solve for $v(t) = u_x(0,t)$ the following Volterra integral equation with delay and a weakly singular kernel:

$$v(t) = f(t) - \int_0^t \frac{F(v(s-\tau))}{\sqrt{\pi(t-s)}}\, ds, \tag{16}$$

where $f(t)$ is given by (7), with the initial condition

$$v(t) = \phi(t), \;\; -\tau \le t \le 0. \tag{17}$$

As far as the representation of the solution $u(x,t)$ to problem (4) is concerned, it is easily seen that a formula like (9) holds:

$$u(x,t) = u_0(x,t) - \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-s}}\right) F(v(s-\tau))\, ds, \tag{18}$$

where $u_0(x,t)$ is given by (10).

Except for particular selections of the control function F, equation (16) can not be reduced to a delay differential equation. Furthemore, the presence of the singular kernel $(\pi(t-s))^{-1/2}$ and simultaneously the possible presence of singular forcing functions entails the appearance of some difficulties in the application of the usual "step-by-step" method to construct the solution to an initial value problem for that kind of equations (Hale, Verduyn Lunel, 1993; Gyori, Ladas, 1991; Saaty, 1967). In fact, the forcing function f given by (7) may be singular at $t = 0$ for certain initial temperature profiles h (v.g., when $h \equiv h_0$ is constant!) and therefore, divergent integrals can appear in the "step-by-step" construction of the solution to (16) provided that a rapidly increasing, say superlinear, control function F is prescribed. Summarizing, solutions of problem (4) present a blow-up after a finite time unless some restrictions are imposed on data. But what has more important consequences for the analysis of the control problem is the fact that, even if suitable hypotheses on data ensuring global existence of the solution are assumed, a result like Theorem 2.2 is no longer valid for the solution of equation (16). In general, the solution to (16) *changes sign* so that no extension to equation (16) is possible for the monotonicity techniques successfully applied in studying (1). As we will see in a forthcoming section, solutions of (16) are oscillatory even in the most simple situation furnished by linear controls $F(v) = \lambda v$.

In order to partially overcome this difficulty, a different control objective is proposed; namely, the quotient of temperatures in (3) is now replaced by less restrictive conditions on *mean* temperatures. Thus, we look for conditions on the control function F ensuring the existence of the limit $\lim_{t\uparrow+\infty} \widetilde{u}(t)$, where

$$\widetilde{u}(t) = \lim_{x\uparrow+\infty} \left(\frac{1}{x}\int_0^x u(\xi,t)\, d\xi\right),\ t > 0, \tag{19}$$

provided that there exists the limit $\lim_{t\uparrow+\infty} \widetilde{u}_0(t)$ with

$$\widetilde{u}_0(t) = \lim_{x\uparrow+\infty} \left(\frac{1}{x}\int_0^x u_0(\xi,t)\, d\xi\right),\ t > 0. \tag{20}$$

The paper is organized as follows. In Section 3 basic results on existence and uniqueness for the solution to (16) are proved. A theorem furnishing estimates for the solution to problem in the case of a monotone control as well as a result on existence and uniqueness for the solution to (4) is also given there. In Section 4, we consider the problem of controlling mean temperatures through the function

F. In the first place, some hypotheses are made on the initial profile h entailing the existence of the limit (20). After this, some conditions that guarantee the existence of limits (19) are found. A subsection is then devoted to the case of a linear control, a case in which the Laplace transform becomes a useful tool. Monotone controls are considered in the final subsection, where a class of suitable controls being able to stabilize mean temperatures is specified.

3. Basic results

In this section we study the basic properties of the problem (16)-(17) where f is given by (7). At the end of the section, an existence and uniqueness result for problem (4) will be derived from these properties.

To begin with, we recall some simple remarks on the behaviour of the forcing function f. For a proof we refer to Berrone (1994) and Berrone, Tarzia, Villa (2000).

THEOREM 3.1 *Let h be a bounded function and f given by (7). Then we have*

$$|f(t)| \leq \frac{\|h\|_\infty}{2\sqrt{\pi}t^{3/2}} \int_0^{+\infty} \xi e^{-\xi^2/(4t)}\, d\xi = \frac{\|h\|_\infty}{\sqrt{\pi t}},\ t > 0. \tag{21}$$

More generally, if for a positive constant C_1 we have $|h(x)| \leq C_1 x^\alpha,\ x \geq 0$, $\alpha \geq 0$ then there exists another positive contant C_2 such that

$$|f(t)| \leq C_2 t^{(\alpha-1)/2},\ t > 0.$$

If $h \in \mathcal{C}^0[0, +\infty)$, then we have

$$\lim_{t \downarrow 0} \left[\sqrt{\pi t} f(t)\right] = h(0).$$

Furthermore, if $h \in \mathcal{C}^{1,\alpha}[0, +\infty)$, then there exists a positive constant C such that

$$\left|\sqrt{\pi t} f(t) - h(0)\right| \leq \sqrt{\pi t}\left(|h'(0)| + Ct^{\frac{\alpha}{2}}\right),\ t > 0.$$

Now, a result on global existence and uniqueness of the solution to the initial value problem (16)-(17) is proved. The assumptions we made on data cover the case of a forcing function having a weak singularity at the origin. In the sequel of the paper we turn around the basic setting fixed by these assumptions.

THEOREM 3.2 *Assume that the following conditions are satisfied by the data of the initial value problem (16)-(17):*

i) $\phi \in \mathcal{C}^0[-\tau, 0]$;
ii) $f \in \mathcal{C}^0(0, +\infty) \cap L^p(0, 1),\ (1 \leq p \leq +\infty)$;

iii) *there exist two constants* $C > 0$ *and* $0 \leq \alpha \leq 1$ *such that*

$$|F(v)| \leq C\,|v|^{\alpha}\,,\ v \in \mathbf{R}. \tag{22}$$

Then, there exists a unique function v *defined on* $[-\tau, +\infty)$ *solving (16)-(17) which is continuous at every point* $t \in [-\tau, +\infty)$ *with the possible exception of a discrete set of points of the form* $t = n\tau,\ n \in \mathbf{N_0}$.

Proof. The proof follows from a "step-by-step" construction of the solution like that employed to show existence and uniqueness of solution to a delay differential equation (Gyori, Ladas, 1991; Hale, Verduyn Lunel, 1993; Saaty, 1967).

Assuming that $t \in ((n-1)\tau, n\tau]$ for a certain $n \in \mathbf{N_0}$ and splitting the integral in (16), we can write

$$\begin{aligned} v(t) = f(t) &- \textstyle\sum_{k=1}^{n-1} \int_{(k-1)\tau}^{k\tau} \frac{F(v(s-\tau))}{\sqrt{\pi(t-s)}}\,ds \\ &+ \textstyle\int_{(n-1)\tau}^{t} \frac{F(v(s-\tau))}{\sqrt{\pi(t-s)}}\,ds,\ t \in ((n-1)\tau, n\tau], \end{aligned} \tag{23}$$

Introducing the notation $v_n(t) = v(t+n\tau),\ t \in (-\tau, 0],\ (n \in \mathbf{N_0})$, in equation (23), we obtain

$$\begin{aligned} v_n(t) = f(t+n\tau) &- \textstyle\sum_{k=1}^{n-1} \int_{-\tau}^{0} \frac{F(v_{k-1}(s))}{\sqrt{\pi(t-s+(n-k)\tau)}}\,ds \\ &+ \textstyle\int_{-\tau}^{t} \frac{F(v_{n-1}(s))}{\sqrt{\pi(t-s)}}\,ds,\ t \in (-\tau, 0]. \end{aligned} \tag{24}$$

Now, starting from $v_0 \equiv \phi$ and iteratively using expression (24), a sequence $\{v_n\}_{n=0}^{+\infty}$ of functions defined on the interval $(-\tau, 0]$ is constructed. Every function of this sequence turns out to be continuous; furthermore, for every $n \in \mathbf{N_0}$, there exists $p_n \geq 1$ such that $v_n \in L^{p_n}(-\tau, 0)$. Both assertions will now be proved by induction. In fact, $v_0 \equiv \phi$ is continuous in $[-\tau, 0]$ by hypotheses i) and assuming that $v_k \in L^{p_k}(-\tau, 0)$ is continuous on $(-\tau, 0],\ k = 0, 1, \ldots, n-1$, from (24) we deduce

$$|v_n(t)| \leq M_n + \left| \int_{-\tau}^{t} \frac{F(v_{n-1}(s))}{\sqrt{\pi(t-s)}}\,ds \right|,\ t \in (-\tau, 0], \tag{25}$$

where M_n is a positive constant. The integral of the second member of (25) is the convolution product of $F \circ v_{n-1} \in L^{p_{n-1}/\alpha}(-\tau, 0)$ ($F \circ v_{n-1} \in L^{\infty}(-\tau, 0)$, provided that $\alpha = 0$) and $t \mapsto (\pi t)^{-1/2} \in L^1(-\tau, 0)$. Since $\frac{\alpha}{p_{n-1}} + 1 \geq 1$, the Young's theorem (Wheeden, Zygmund, 1977; p. 146) ensures that, as a function of t, this integral belongs to $L^{p_n}(-\tau, 0)$, with

$$p_n = \begin{cases} \left(\frac{\alpha}{p_{n-1}} + 1 - 1\right)^{-1} = \frac{p_{n-1}}{\alpha}, & 0 < \alpha \leq 1 \\ +\infty, & \alpha = 0, \end{cases} \quad ,$$

and, in view of the inductive hypothesis, it turns out to be a continuous function as well.

To end the proof, a function v is constructed by "gluing" the tracts v_n one each other. Formally, for every $n \in \mathbf{N_0}$ and $t \in ((n-1)\tau, n\tau]$, we define

$$v(t) = v_n(t - n\tau).$$

Of course, we also set $v(-\tau) = v_0(-\tau)$. It is not difficult to see that this function is the unique solution to the initial value problem (16)-(17) and that it has the regularity specified in the statement of the theorem. ∎

Remark that the solution to the initial value problem (16)-(17), constructed in Theorem (3.2) is left-continuous on $[-\tau, +\infty)$. However, *finite* discontinuities at points $t = n\tau,\ n \in \mathbf{N_0}$, can really appear. This is obviously true for $t = 0$ when the forcing function f is singular at $t = 0$, but this singularity can be "dragged to the right" in the step-by-step process of construction: the case corresponding to a linear control function $F(v) = \lambda v$ and to the forcing function $f(t) = h_0/\sqrt{\pi t}$ illustrates this fact. For simplicity, assume that the initial data $\phi \equiv 0$; then we have

$$v(t) = \begin{cases} 0, & t \in [-\tau, 0] \\ h_0/\sqrt{\pi t}, & t \in (0, \tau] \end{cases},$$

and, for $t \in (\tau, 2\tau]$,

$$\begin{aligned} v(t) &= \frac{h_0}{\sqrt{\pi t}} - \frac{\lambda h_0}{\pi} \int_\tau^t \frac{ds}{\sqrt{s-\tau}\sqrt{t-s}} \\ &= \frac{h_0}{\sqrt{\pi t}} - \lambda h_0. \end{aligned}$$

Thus, $t = \tau$ is a point of discontinuity of the solution v and it is easy to see that there are no other discontinuities. Indeed, the same behaviour is generally exhibited by the solution of problem (16)-(17) corresponding to a forcing function f satisfying $|f(t)| \le Mt^{-\gamma},\ 0 < t < 1$, and to a Lipschitz-continuous control function F with $F(0) = 0$.

Remark 3.1 *We emphasize the fact that restriction (22) on the control function F can be omitted from the statement of Theorem (3.2) provided that the forcing function f is continuous from the right at $t = 0$; i.e., for $f \in \mathcal{C}^0[0, +\infty)$. In fact, every integral in (24) turns out to be convergent and the "step-by-step" construction works in this case furnishing a solution v continuous on $(0, +\infty)$. Note that if h is continuous and $|h(x)| \le Cx^\alpha$ with $\alpha > 1$ or if $h \in \mathcal{C}^{1,\alpha}[0, +\infty)$ and $h(0) = h'(0) = 0$, then Theorem (3.1) ensures that $f \in \mathcal{C}^0[0, +\infty)$ and $f(0) = 0$.*

Remark 3.2 *Observe that if we take assumptions* **i)** *and* **ii)** *of Theorem 3.2, but we omit* **iii)**, *i.e. we do not consider any growth assumption on F, Theorem*

3.2 is false, as is shown by taking $h(x) = h_0 > 0$ *and* $F(v) = v^3$, $\phi(t)$ *any continuous function on* $[-\tau, 0]$. *By constructing "step-by-step" the solution* $v(t)$, *for* $t \in (0, \tau]$ *we find*

$$v(t) \equiv v_1(t) = \frac{h_0}{\sqrt{\pi t}} - \int_0^t \frac{[\phi(s-\tau)]^3}{\sqrt{\pi(t-s)}} ds$$

and, for $t \in (\tau, 2\tau]$,

$$v(t) \equiv v_2(t) = \frac{h_0}{\sqrt{\pi t}} - \int_0^\tau \frac{\phi^3(s-\tau)}{\sqrt{\pi(t-s)}} ds - \int_\tau^t \frac{v_1^3(s-\tau)}{\sqrt{\pi(t-s)}} ds. \tag{26}$$

Replacing expression of $v_1(t)$ *in (26) we see that one of the terms in the resulting expression is given by*

$$\int_\tau^t \frac{h_0^3}{(\sqrt{\pi(s-\tau)})^3 \sqrt{\pi(t-s)}} ds = \int_0^{t-\tau} \frac{h_0^3}{\sqrt{\pi s^3} \sqrt{\pi(t-\tau-s)}} ds \equiv +\infty,$$

which shows that the solution exhibits a blow-up at $t = \tau$.

Our next result provides a way of obtaining estimates for the solution to the initial value problem (16)-(17) when the control function is non-decreasing and $f \in \mathcal{C}^0(0, +\infty) \cap L^p(0, 1)$ (compare with Saaty, 1967, p. 280; Walter, 1970, p. 38).

THEOREM 3.3 *Let* v *be the solution to problem (16)- (17) with data satisfying, apart from conditions* **i)** *and* **ii)** *of Theorem 3.2, the following ones:*

iii) F *is Hölder-continuous with exponent* $0 \le \alpha \le 1$, $F(0) = 0$ *and* $F(u) \le F(v)$ *whenever* $u \le v$.

Furthermore, let v_1, v_2 *be two real functions defined on* $[-\tau, +\infty)$ *and everywhere continuous with the possible exception of points of the form* $t = n\tau$, *such that*

iv) $v_1(t) \le \phi(t) \le v_2(t), \ t \in [-\tau, 0]$,

v) *for* $0 < T \le +\infty$ *the inequalities*

$$\begin{cases} v_1(t) \le f(t) - \int_0^t \frac{F(v_2(s-\tau))}{\sqrt{\pi(t-s)}} ds \\ v_2(t) \ge f(t) - \int_0^t \frac{F(v_1(s-\tau))}{\sqrt{\pi(t-s)}} ds \end{cases}, \ t \in (0, T]; \tag{27}$$

are satisfied. Then the inequalities

$$v_1(t) \le v(t) \le v_2(t), \tag{28}$$

hold also for $t \in (0, T]$.

Proof. First note that assumptions **iii)** contains conditions **iii)** of Theorem 3.2, so the existence of a unique solution v to problem (16)-(17) is ensured. Now, the proof proceeds by induction. Take for instance the case in which $T = +\infty$. If $t \in (0, \tau]$, from **iii), iv)** and **v)** we obtain

$$v_1(t) \leq f(t) - \int_0^t \frac{F(v_2(s-\tau))}{\sqrt{\pi(t-s)}}\, ds \leq f(t) - \int_0^t \frac{F(\phi(s-\tau))}{\sqrt{\pi(t-s)}},$$

and

$$v_2(t) \geq f(t) - \int_0^t \frac{F(v_1(s-\tau))}{\sqrt{\pi(t-s)}}\, ds \geq f(t) - \int_0^t \frac{F(\phi(s-\tau))}{\sqrt{\pi(t-s)}}\, ds = v(t).$$

Then, assuming that inequalities (28) hold for $t \in [-\tau, n\tau)$, a new application of **iii)** and **v)** shows that they hold also for $t \in [-\tau, (n+1)\tau)$. This completes the induction.

When $T < +\infty$, there exists $N \in \mathbf{N}$ such that $T \in ((N-1)\tau, N\tau]$ and we arrive to the conclusion by finite induction. ∎

Observe that this result can be extended to general monotone control functions F when $f \in C^0[0, +\infty)$. A criterion for positivity of the solution for problem (16)-(17) is furnished by the following Corollary of Theorem 3.3.

Corollary 3.1 *If, apart from the conditions* **i)** *and* **ii)** *of Theorem 3.2 and* **iii)** *of Theorem 3.3, there exists a non-negative function p such that*
i) $0 \leq \phi(t) \leq p(t),\ t \in [-\tau, 0]$;
ii) *for* $t > 0$,

$$\int_0^t \frac{F(p(s-\tau))}{\sqrt{\pi(t-s)}}\, ds \leq f(t) \leq p(t); \tag{29}$$

then, the inequalities

$$0 \leq v(t) \leq p(t),\ t > 0,$$

hold for the solution $v(t)$ to problem (16)-(17).

Proof. Since inequalities (29) are inequalities (27) with $v_1 = 0$ and $v_2 = p$, the proof immediately follows from Theorem 3.3. ∎

Conditions **i)** and **ii)** of Corollary 3.1 naturally lead us to consider the solution p to the problem

$$\begin{cases} p(t) = \varphi(t) + \int_0^t \frac{F(p(s-\tau))}{\sqrt{\pi(t-s)}}\, ds, & t > 0 \\ p(t) = \phi(t), & -\tau \leq t \leq 0, \end{cases} \tag{30}$$

for non-negative functions ϕ and φ. In terms of the function φ and the solution p, the inequalities (29) take the form

$$p(t) - \varphi(t) \leq f(t) \leq p(t),\ t > 0.$$

We finish this section with a result on existence and uniqueness of solution to problem (4) which is based on Theorem (3.2).

THEOREM 3.4 *Assume that the following conditions*

i) $\phi \in \mathcal{C}^0[-\tau, 0]$;

ii) *h is a bounded continuous function on* $(0, +\infty)$;

iii) *there exist two constants* $C > 0$ *and* $0 \le \alpha \le 1$ *such that* $|F(v)| \le C\,|v|^\alpha\,,\ v \in \mathbf{R}$,

are fulfilled by the data of the problem (4). Then, the problem has a unique bounded solution with the following regularity: $u(x,t),\ u_x(x,t),\ u_{xx}(x,t) \in C^0(\mathbf{R}^+ \times (0,T))$, *for all* $T > 0$, $u_t(x,t) \in C^0(\mathbf{R}^+ \times (0,T))$, *with the possible exception of a discrete set of points of the form* $t = n\tau,\ n \in \mathbf{N_0}$.

Proof. It is deduced from condition **ii)** and estimate (21) that condition **ii)** of Theorem 3.2 is satisfied. This fact, together with conditions **i)** and **ii)**, enable us to conclude that Theorem 3.2 is applicable and then, a unique global solution v exists for equation (16). In this way, the unique bounded solution to Problem 4 is given in terms of v by formula (18). ■

4. Controlling mean temperatures

As stated in the introduction, we look for conditions on the function F in order to control, when possible, mean temperatures of the slab. Concretely, we would like to define

$$\widetilde{u}(t) = \lim_{x\uparrow+\infty}\left(\frac{1}{x}\int_0^x u(\xi,t)\,d\xi\right), \quad \widetilde{u}_0(t) = \lim_{xx\uparrow+\infty}\left(\frac{1}{x}\int_0^x u_0(\xi,t)\,d\xi\right),\ t > 0, \tag{31}$$

where u and u_0 are the solutions of problem (4) with control function F and $F \equiv 0$, respectively. However, the limits (31) do not exist in general and a preliminary study of them is to be our first order of business.

4.1. Mean temperatures

First of all, we rewrite expression (10) for u_0 in the alternative forms

$$\begin{aligned} u_0(x,t) &= \int_{-\infty}^{+\infty} K(x-\xi,t)\overline{h}(\xi)\,d\xi = \int_{-\infty}^{+\infty} K(\xi,t)\overline{h}(x-\xi)\,d\xi \\ &= \int_{-\infty}^{+\infty} K(\xi,t)\overline{h}(x+\xi)\,d\xi, \end{aligned} \tag{32}$$

where $\overline{h}$ is the odd extension of h to $\mathbf{R}$. From the first and the last member of (32) we derive

$$\begin{aligned}\frac{1}{x}\int_0^x u_0(\xi,t)\,d\xi &= \int_{-\infty}^{+\infty} K(\eta,t)\left(\frac{1}{x}\int_0^x \overline{h}(\xi+\eta)\,d\xi\right)d\eta \\ &= \int_{-\infty}^{+\infty} K(\eta,t)\left(\frac{1}{x}\int_\eta^{x+\eta} \overline{h}(\xi)\,d\xi\right)d\eta. \end{aligned} \tag{33}$$

Now, for a locally integrable function $h:\mathbf{R}\to\mathbf{R}$, we define its "means from the right" as

$$\overline{M}h(\eta) = \limsup_{x\uparrow+\infty}\left(\frac{1}{x}\int_\eta^{\eta+x} h(\xi)\,d\xi\right), \tag{34}$$

$$\underline{M}h(\eta) = \liminf_{x\uparrow+\infty}\left(\frac{1}{x}\int_\eta^{\eta+x} h(\xi)\,d\xi\right). \tag{35}$$

The means (from the right) $\overline{M}h$ and $\underline{M}h$ will be called *superior* and *inferior mean (from the right) of* h, respectively, and as it will be immediately seen, they do not really depend on η; i.e., they are (possibly infinite) constants. In fact, for $\eta,\eta_0\in\mathbf{R}$, we have

$$\begin{aligned}\overline{M}h(\eta+\eta_0) &= \limsup_{x\uparrow+\infty}\frac{1}{x}\int_{\eta+\eta_0}^{\eta+\eta_0+x} h(\xi)\,d\xi \\ &= \limsup_{x\uparrow+\infty}\left(\frac{\eta_0+x}{x}\frac{1}{\eta_0+x}\int_\eta^{\eta+\eta_0+x} h(\xi)\,d\xi - \frac{1}{x}\int_\eta^{\eta+\eta_0} h(\xi)\,d\xi\right) \\ &= \limsup_{x\uparrow+\infty}\left(\frac{1}{\eta_0+x}\int_\eta^{\eta+\eta_0+x} h(\xi)\,d\xi\right) \\ &= \overline{M}h(\eta),\end{aligned}$$

and a similar calculation holds with $\underline{M}h$ instead of $\overline{M}h$. A chain of inequalities involving means from the right is provided by the following result.

LEMMA 4.1 *For every locally integrable real funtion h satisfying the growth condition (5), the inequalities*

$$\liminf_{x\uparrow+\infty} h(x) \le \underline{M}h \le \underline{M}u_0(\cdot,t) \le \overline{M}u_0(\cdot,t) \le \overline{M}h \le \limsup_{x\uparrow+\infty} h(x) \tag{36}$$

hold for every $t>0$.

Proof. From the definition, we see that

$$\underline{M}h \le \overline{M}h, \tag{37}$$

for every locally integrable function h. Moreover, inequality (37) can be enlarged by observing that

$$\overline{M}h = \limsup_{x\uparrow+\infty} \int_0^1 h(x\xi)\,d\xi \le \limsup_{x\uparrow+\infty} h(x) \tag{38}$$

and, similarly,

$$\underline{M}h \ge \liminf_{x\uparrow+\infty} h(x). \tag{39}$$

Take for instance the inequality (38). The inequality is trivially true when $\limsup_{x\uparrow+\infty} h(x) = +\infty$ and therefore, we can assume that $\limsup_{x\uparrow+\infty} h(x) = L < +\infty$. For a given $\varepsilon > 0$ there exists an $x_0 > 0$ such that $h(x) < L + \varepsilon$ for every $x > x_0$, so that

$$\begin{aligned} \frac{1}{x}\int_0^x h(\xi)\,d\xi &= \frac{1}{x}\int_0^{x_0} h(\xi)\,d\xi + \frac{1}{x}\int_{x_0}^x h(\xi)\,d\xi \\ &\le \frac{1}{x}\int_0^{x_0} h(\xi)\,d\xi + \left(1 - \frac{x_0}{x}\right)(L+\varepsilon)\,, \end{aligned}$$

whence

$$\limsup_{x\uparrow+\infty} \frac{1}{x}\int_0^x h(\xi)\,d\xi \le L + \varepsilon.$$

This proves inequality (38) and the proof of (39) is similar. In short, we can write

$$\liminf_{x\uparrow+\infty} h(x) \le \underline{M}h \le \overline{M}h \le \limsup_{x\uparrow+\infty} h(x). \tag{40}$$

Now, recalling that $K \ge 0$ and $\int_{-\infty}^{+\infty} K(\eta,t)\,d\eta = 1$ and then taking oscillation limits in (33), an argument as the previous one produces

$$\underline{M}h \le \underline{M}u_0(\cdot,t) \le \overline{M}u_0(\cdot,t) \le \overline{M}h. \tag{41}$$

■

REMARK 4.1 *The following simple example shows that inequalities in the chain (36) can be strict. Let $h(x) \equiv \sin x + x\cos x$; then, for every $t > 0$, we obtain*

$$\begin{aligned} \liminf_{x\uparrow+\infty} h(x) &= -\infty < -1 = \underline{M}h < -e^{-t} = \underline{M}u_0(\cdot,t) \\ &< e^{-t} = \overline{M}u_0(\cdot,t) < 1 = \overline{M}h < +\infty = \limsup_{x\uparrow+\infty} h(x). \end{aligned}$$

Remark 4.2 *From Lemma 4.1, several sufficient conditions ensuring the existence of* $\widetilde{u}_0(t) = \lim_{x\uparrow+\infty}\left(\frac{1}{x}\int_0^x u_0(\xi,t)\,d\xi\right)$ *can be deduced. This occurs, for example, when the condition* $-\infty < \underline{M}h = \overline{M}h < +\infty$ *or the stronger one* $|h(+\infty)| < +\infty$ *hold. Under the first assumption, we find*

$$\widetilde{u}_0(t) \equiv \widetilde{u}_0 \equiv \lim_{x\uparrow+\infty}\left(\frac{1}{x}\int_0^x h(\xi)\,d\xi\right) \equiv: Mh, \tag{42}$$

while the identity $\widetilde{u}_0(t) \equiv \widetilde{u}_0 \equiv h(+\infty)$ *is true under the second one. Note that, since* $h \in \mathcal{C}^0[0,+\infty)$*, the boundedness of* h *is implied by the condition* $|h(+\infty)| < +\infty$*. For an initial temperature profile* $h \in L^1(\mathbf{R}^+)$*, we have* $\widetilde{u}_0(t) \equiv \lim_{x\uparrow+\infty}\left|\frac{1}{x}\int_0^x h(\xi)\,d\xi\right| \le \lim_{x\uparrow+\infty}\left(\frac{1}{x}\int_0^{+\infty}|h(\xi)|\,d\xi\right) = 0$*. On the other hand, for a constant initial profile* $h(x) \equiv h_0$ *we have* $-\infty < h(+\infty) = h_0 < +\infty$ *and therefore* $\widetilde{u}_0(t) \equiv h_0 > 0$.

Indeed, what motivate our consideration of mean temperatures is the possibility of recovering control conditions so simple as was (13) in the absence of delay. In clarifying this claim, now we show a series of useful expressions for

$$\widetilde{u}(t) = \lim_{x\uparrow+\infty}\left(\frac{1}{x}\int_0^x u(\xi,t)\,d\xi\right)$$

in terms of solution $v(t)$ of equation (16).

Theorem 4.1 *Assume that the initial temperature* h *satisfies* $|Mh| < +\infty$*, where* M *is defined in (42). Then, for every* $t > 0$ *we have*

i)

$$\widetilde{u}(t) = \widetilde{u}_0 - \int_0^t F(v(s-\tau))\,ds$$

and

ii)

$$\widetilde{u}(t) = \widetilde{u}_0 - \int_0^t \frac{f(s)}{\sqrt{\pi(t-s)}}\,ds + \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,.$$

Proof. We integrate both members of the representation formula (18) to obtain

$$\begin{aligned}\int_0^x u(\xi,t)\,d\xi &= \int_0^x u_0(\xi,t)\,d\xi - \int_0^x\int_0^t \operatorname{erf}\left(\frac{\xi}{2\sqrt{t-s}}\right)F(v(s-\tau))\,ds\,d\xi\\ &= \int_0^x u_0(\xi,t)\,d\xi - \int_0^t\left(\int_0^x \operatorname{erf}\left(\frac{\xi}{2\sqrt{t-s}}\right)d\xi\right)F(v(s-\tau))\,ds,\end{aligned}$$

whence, taking into account that

$$\lim_{x\uparrow+\infty}\left(\frac{1}{x}\int_0^x \operatorname{erf}\left(\frac{\xi}{2\sqrt{t-s}}\right)d\xi\right) = 1,$$

and that $\lim_{x\uparrow+\infty}\left(x^{-1}\int_0^x u_0(\xi,t)\,d\xi\right)=\widetilde{u}_0$, we derive expression **i)**.

Next, by applying the Abel transformation, $v(t)\to\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}ds$, (Cannon, 1984; Miller, 1971), to both members of (16) we get

$$\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds=\int_0^t \frac{f(s)}{\sqrt{\pi(t-s)}}\,ds-\int_0^t F(v(s-\tau))\,ds,\ t>0. \tag{43}$$

Expression **ii)** is quickly derived from (43) and expression **i)**. ■

REMARK 4.3 *From* **i)** *of Theorem (4.1) and Remark (4.2) we obtain some sufficient conditions for the existence of* $\widetilde{u}(t)$.

It is our aim to discuss the behaviour when $t\uparrow+\infty$ of expression **ii)** of the previous theorem. In the first place, a calculation involving Laplace transforms shows that

$$\begin{aligned}\int_0^t \frac{f(s)}{\sqrt{\pi(t-s)}}\,ds &= \int_0^t \frac{1}{\sqrt{\pi(t-s)}}\left(\frac{1}{2\sqrt{\pi}s^{3/2}}\int_0^{+\infty}\xi e^{-\xi^2/(4s)}h(\xi)\,d\xi\right)ds\\ &= 2\int_0^{+\infty}K(\xi,t)h(\xi)\,d\xi. \end{aligned}\tag{44}$$

Now, by integrating by parts and making a suitable change of variable, we obtain

$$\begin{aligned}2\int_0^{+\infty}K(\xi,t)h(\xi)\,d\xi &= \int_0^{+\infty}\frac{e^{-\xi^2/(4t)}}{\sqrt{\pi t}}\,d\left(\int_0^\xi h(\eta)\,d\eta\right)\\ &= \int_0^{+\infty}\frac{\xi e^{-\xi^2/(4t)}}{2\sqrt{\pi t^3}}\left(\int_0^\xi h(\eta)\,d\eta\right)d\xi\\ &= \int_0^{+\infty}\frac{2\sqrt{t}\xi e^{-\xi^2}}{2\sqrt{\pi t^3}}\left(\int_0^{2\sqrt{t}\xi} h(\eta)\,d\eta\right)2\sqrt{t}d\xi\\ &= \frac{4}{\sqrt{\pi}}\int_0^{+\infty}\xi^2 e^{-\xi^2}\frac{1}{2\sqrt{t}\xi}\left(\int_0^{2\sqrt{t}\xi} h(\eta)\,d\eta\right)d\xi\end{aligned}$$

and assuming that there exists $Mh=\widetilde{u}_0$, the dominated convergence theorem gives

$$\begin{aligned}&\lim_{t\uparrow+\infty}\left(2\int_0^{+\infty}K(\xi,t)h(\xi)\,d\xi\right)=\\ &= \frac{4}{\sqrt{\pi}}\int_0^{+\infty}\xi^2 e^{-\xi^2}\lim_{t\uparrow+\infty}\left(\frac{1}{2\sqrt{t}\xi}\left(\int_0^{2\sqrt{t}\xi} h(\eta)\,d\eta\right)\right)d\xi\\ &= \frac{4}{\sqrt{\pi}}\int_0^{+\infty}\xi e^{-\xi^2}\widetilde{u}_0\,d\xi=\widetilde{u}_0.\end{aligned}\tag{45}$$

From Theorem 4.1 and (44), (45) we see that

$$\widetilde{u}_0 - \liminf_{t\uparrow+\infty} \int_0^t F(v(s-\tau))\,ds = \limsup_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds$$

and

$$\widetilde{u}_0 - \limsup_{t\uparrow+\infty} \int_0^t F(v(s-\tau))\,ds = \liminf_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds;$$

in particular, the two limits

$$\lim_{t\uparrow+\infty} \int_0^t F(v(s))\,ds, \quad \lim_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds \tag{46}$$

exist or cease to exist in a simultaneous way. Note that the existence of the first limit is the same as the existence of $\int_0^{+\infty} F(v(s))\,ds$ as an improper integral. Since condition (2) is supposed to be satisfied by the control function F, a sufficient condition ensuring the existence of $\int_0^{+\infty} F(v(s))\,ds$ is that the solution $v(t)$ to equation (16) should be *ultimately positive* (*negative*); i.e., that there exists a $t_0 \geq 0$ such that $v(t) \geq 0$ (≤ 0) for $t > t_0$.

Summarizing the previous discussion, we state the following:

THEOREM 4.2 *Assume that there exists one of the limits (46); then, there exist the other one and the equalities*

$$\begin{aligned}\lim_{t\uparrow+\infty} \widetilde{u}(t) &= \widetilde{u}_0 - \int_{-\tau}^0 F(\phi(s))\,ds - \int_0^{+\infty} F(v(s))\,ds \\ &= \lim_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds\end{aligned}$$

hold for the limit of the mean temperature $\widetilde{u}(t)$. A sufficient condition in order that the limits (46) exist is that the solution $v(t)$ to equation (16) should be ultimately positive (negative).

Proof. See the discussion above. ∎

If the solution $v(t)$ does not change sign for $t > t_0$ then, from Theorem 4.2, the existence of (possible infinite) limits of the mean temperature $\widetilde{u}(t)$ is ensured and we can say that the control exhibits a satisfactory behaviour. Now, assume that $v(t) \geq 0$ for $t > t_0$; then, for $t \geq 2t_0$, we have

$$\begin{aligned}v(t) &\leq \frac{\|h\|_\infty}{\sqrt{\pi t}} - \int_0^{t_0} \frac{F(v(s-\tau))}{\sqrt{\pi(t-s)}}\,ds \\ &\leq \frac{\|h\|_\infty}{\sqrt{\pi t}} + \frac{1}{\sqrt{\pi(t-t_0)}} \int_0^{t_0} |F(v(s-\tau))|\,ds \\ &\leq \frac{\|h\|_\infty + \sqrt{2}\int_0^{t_0} |F(v(s-\tau))|\,ds}{\sqrt{\pi t}}.\end{aligned}$$

Hence, the behaviour of an ultimately positive solution $v(t)$ of equation (16) is controlled, for t sufficiently large, by a function of the form $Ct^{-1/2}$, ($C > 0$). Next we prove that $\lim_{t\uparrow+\infty} \widetilde{u}(t)$ exists provided that the solution $v(t)$ to equation (16) decreases in a sufficiently rapid way when $t \to +\infty$.

THEOREM 4.3 *Let $C > 0$ and $0 \leq \delta < 1/2$ be two constants such that*

$$|v(t)| \leq \frac{C}{t^{\delta+1/2}}, \ t > t_0;$$

then,

i) *if $\delta > 0$, then $\lim_{t\uparrow+\infty} \widetilde{u}(t) = 0$;*
ii) *if $\delta = 0$, then $\liminf_{t\uparrow+\infty} \widetilde{u}(t)$ and $\limsup_{t\uparrow+\infty} \widetilde{u}(t)$ are both finite. Furthermore, if $\lim_{t\uparrow+\infty} \left(\sqrt{t}v(t)\right) = C_0$, then*

$$\lim_{t\uparrow+\infty} \widetilde{u}(t) = \sqrt{\pi}C_0.$$

Proof. Since the solution $v(t)$ is piecewise continuous for $t > \tau$ and integrable in $(0, \tau)$, we have $\int_0^{t_0} |v(s)| \, ds < +\infty$ and therefore, for $t > t_0$, we can write

$$\begin{aligned} \left| \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}} ds \right| &\leq \int_0^{t_0} \frac{|v(s)|}{\sqrt{\pi(t-s)}} ds + \int_{t_0}^t \frac{|v(s)|}{\sqrt{\pi(t-s)}} ds \\ &\leq \frac{\int_0^{t_0} |v(s)| \, ds}{\sqrt{\pi(t-t_0)}} + \frac{C}{\sqrt{\pi}} \int_{t_0}^t \frac{ds}{s^{\delta+1/2}(t-s)^{1/2}} \\ &\leq \frac{\int_0^{t_0} |v(s)| \, ds}{\sqrt{\pi(t-t_0)}} + \frac{C}{\sqrt{\pi}} \int_0^t \frac{ds}{s^{\delta+1/2}(t-s)^{1/2}} \\ &= \frac{\int_0^{t_0} |v(s)| \, ds}{\sqrt{\pi(t-t_0)}} + \frac{CB(1/2-\delta, 1/2)}{\sqrt{\pi}t^{\delta}}, \end{aligned} \tag{47}$$

where B denotes the Beta function.

Now, if $0 < \delta < 1/2$, the inequalities (47) show that

$$\lim_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}} ds = 0$$

and the result follows from Theorem 4.2. When $\delta = 0$, inequalities (47) imply the boundedness of the integral $\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}} ds$ for $t \to +\infty$ and hence, by using Theorem 4.1-**ii)**, we infer the boundedness of $\widetilde{u}(t)$ when $t \to +\infty$.

Finally, assuming that $\lim_{t\uparrow+\infty}\left(\sqrt{t}v(t)\right)=C_0$, for $0<t_0<t$, we have

$$\begin{aligned}
&\left|\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds-\sqrt{\pi}C_0\right| = \\
&= \left|\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds - C_0\int_0^t \frac{ds}{\sqrt{s}\sqrt{\pi(t-s)}}\right| \\
&\leq \int_0^{t_0}\frac{|v(s)-C_0/\sqrt{s}|}{\sqrt{\pi(t-s)}}\,ds+\int_{t_0}^t \frac{|\sqrt{s}v(s)-C_0|}{\sqrt{s}\sqrt{\pi(t-s)}}\,ds \\
&\leq \frac{1}{\sqrt{\pi(t-t_0)}}\int_0^{t_0}\left|v(s)-C_0/\sqrt{s}\right|\,ds+\int_{t_0}^t \frac{|\sqrt{s}v(s)-C_0|}{\sqrt{s}\sqrt{\pi(t-s)}}\,ds.
\end{aligned}$$

Hence, fixing an $\varepsilon>0$ and taking $t_0>0$ such that $|\sqrt{s}v(s)-C_0|<\varepsilon/\sqrt{\pi}$, we obtain

$$\begin{aligned}
&\left|\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds-\sqrt{\pi}C_0\right| = \\
&\leq \frac{1}{\sqrt{\pi(t-t_0)}}\int_0^{t_0}\left|v(s)-\frac{C_0}{\sqrt{s}}\right|\,ds+\varepsilon\int_{t_0}^t \frac{ds}{\sqrt{\pi s}\sqrt{\pi(t-s)}} \\
&\leq \frac{1}{\sqrt{\pi(t-t_0)}}\int_0^{t_0}\left|v(s)-\frac{C_0}{\sqrt{s}}\right|\,ds+\varepsilon\int_{0}^t \frac{ds}{\sqrt{\pi s}\sqrt{\pi(t-s)}} \\
&= \frac{1}{\sqrt{\pi(t-t_0)}}\int_0^{t_0}\left|v(s)-\frac{C_0}{\sqrt{s}}\right|\,ds+\varepsilon,
\end{aligned}$$

and therefore

$$\lim_{t\uparrow+\infty}\left|\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\,ds-\sqrt{\pi}C_0\right|\leq\varepsilon$$

hence our claim follows by the arbitrariness of ε. ■

All the material in this subsection was oriented to provide a suitable objective for the control F. As it will be seen in the next subsection, solutions $v(t)$ to problem (16)-(17) may be oscillatory even for a linear control function. This "bad behaviour" causes the mean temperature $\widetilde{u}(t)$ to oscillate so that the existence of $\lim_{t\uparrow+\infty}\widetilde{u}(t)$ can not be expected in general. Then, a reasonable condition to be satisfied by a given control function F is just to ensure the existence of such a limit of the mean temperatures of the slab. The main developments of the two remaining section are aimed to find sufficient conditions in order for this objective to be satisfied.

4.2. The linear control function

For a linear control function F, equation (16) can be solved by means of the Laplace transform. In the sequel, we show that very basic properties of this transformation are useful in studying the associated control problem.

After setting $F(v) = \lambda v,\ v \in \mathbf{R}$, $(\lambda > 0)$, the initial value problem (16)-(17) becomes

$$\begin{cases} v(t) = f(t) - \lambda \int_0^t \frac{v(s-\tau)}{\sqrt{\pi(t-s)}}\, ds, & t > 0, \\ v(t) = \phi(t), & -\tau \le t \le 0. \end{cases} \tag{48}$$

To solve (48), Laplace transform can be used like in solving an initial value problem for linear delay differential equations (Hale, Verduyn Lunel, 1993; Gyori, Ladas, 1991; Saaty, 1967). As a first step in proving this claim, the following lemma provides an estimate for the solution v to problem (16)-(17) for a Lipschitz continuous control function F.

LEMMA 4.2 *Let v be the solution to the initial value problem, F being a Lipschitz continuous function with Lipschitz constant L, h bounded-continuous on* $[0, +\infty)$ *and $\phi \in \mathcal{C}^0[-\tau, 0]$. Then v satisfies the following inequality*

$$|v(t)| \le \frac{A}{\sqrt{t}} + B + L^2 \int_0^t e^{L^2(t-s)} \left(\frac{A}{\sqrt{s}} + B \right) ds,\ t > 0, \tag{49}$$

where A, B are two suitable positive constants.

Proof. Suppose that h is a bounded continuous function on $[0, +\infty)$ and that $|F(v)| \le L|v|$. From (16) and (21) we first deduce

$$\begin{aligned} |v(t)| &\le \frac{\|h\|_\infty}{\sqrt{\pi t}} + \int_0^t \frac{|F(\phi(s-\tau))|}{\sqrt{\pi(t-s)}}\, ds \\ &\le \frac{\|h\|_\infty}{\sqrt{\pi t}} + \frac{2L}{\sqrt{\pi}} \|\phi\|_\infty \sqrt{\tau},\ 0 < t \le \tau. \end{aligned} \tag{50}$$

Moreover, for $t > 0$ we have

$$|v(t)| \le \frac{\|h\|_\infty}{\sqrt{\pi t}} + L \int_0^t \frac{|v(s-\tau)|}{\sqrt{\pi(t-s)}}\, ds \tag{51}$$

and, after applying the Abel transformation to both members of this inequality, we arrive at the estimate

$$\int_0^t \frac{|v(s)|}{\sqrt{\pi(t-s)}}\, ds \le \|h\|_\infty + L \int_0^t |v(s-\tau)|\, ds,\ t > 0,$$

which, for $t > \tau$, can be written in the form

$$\int_0^{t-\tau} \frac{|v(s)|}{\sqrt{\pi(t-\tau-s)}}\, ds \leq \|h\|_\infty + L\int_0^{t-\tau} |v(s-\tau)|\, ds,\ t>\tau. \tag{52}$$

For $t > \tau$, we conclude from (51) and (52) that

$$\begin{aligned}
|v(t)| &\leq \frac{\|h\|_\infty}{\sqrt{\pi t}} + L\int_0^{\tau} \frac{|\phi(s-\tau)|}{\sqrt{\pi(t-s)}}\, ds + L\int_0^{t-\tau} \frac{|v(s-\tau)|}{\sqrt{\pi(t-\tau-s)}}\, ds \\
&\leq \frac{\|h\|_\infty}{\sqrt{\pi t}} + \frac{2L}{\sqrt{\pi}}\|\phi\|_\infty \sqrt{\tau} + L\left(\|h\|_\infty + L\int_0^{t-\tau} |v(s-\tau)|\, ds\right) \\
&\leq \frac{\|h\|_\infty}{\sqrt{\pi t}} + \frac{2L}{\sqrt{\pi}}\|\phi\|_\infty \sqrt{\tau} + L\left(\|h\|_\infty + L\int_0^{t} |v(s-\tau)|\, ds\right) \\
&\leq \frac{\|h\|_\infty}{\sqrt{\pi t}} + \frac{2L}{\sqrt{\pi}}\|\phi\|_\infty \sqrt{\tau} + L\left(\|h\|_\infty + L\|\phi\|_\infty \tau + L\int_0^{t} |v(s)|\, ds\right).
\end{aligned} \tag{53}$$

Inequalities (50) and (53) show that there exist two positive constants A, B such that

$$|v(t)| \leq \frac{A}{\sqrt{t}} + B + L^2\int_0^t |v(s)|\, ds,\ t>0,$$

and applying Gronwall's Lemma we finally obtain the result. ■

Now we prove that a Laplace transform is admitted by the solutions to (48).

Theorem 4.4 *Let v be the solution to the initial value problem (48) with h bounded-continuous on $[0,+\infty)$ and $\phi \in \mathcal{C}^0[-\tau,0]$. Then v admits a Laplace tranform.*

Proof. To the solution v of the initial value problem (48) we apply the result of Lemma (4.2). Thus, inequality (49) holds for the solution $v(t)$, whence we deduce that v is locally L^1 on $[0,+\infty)$ as well as an exponential growth of v when $t \uparrow +\infty$. In this way, the existence of the Laplace transform of the solution v to problem (48) turns out to be a consequence of a well know existence result for this transform. ■

As usual, we denote by $\widehat{g}$ and $\overset{\vee}{g}$ the Laplace transform and the inverse Laplace transform of the function g, respectively. In order to compute $\widehat{v}$, we proceed to transform (48) as follows:

$$\widehat{v}(s) = \widehat{f}(s) - \frac{\lambda e^{-\tau s}}{\sqrt{s}}\left(\int_{-\tau}^0 \phi(t)e^{-st}\, dt + \widehat{v}(s)\right),$$

whence

$$\widehat{v}(s)=\frac{\widehat{f}(s)-\frac{\lambda e^{-\tau s}}{\sqrt{s}}\int_{-\tau}^{0}\phi(t)e^{-st}\,dt}{1+\frac{\lambda e^{-\tau s}}{\sqrt{s}}},\ s>0.$$

In particular, for h bounded-continuous on $[0,+\infty)$, the inequality $\left|\widehat{f}(s)\right|\le\|h\|_\infty/\sqrt{s}$ shows that the transform $\widehat{f}(s)$ converges at least for $s>0$. Thus we find

$$\widehat{v}(s)=\frac{\widehat{f}(s)-\frac{\lambda e^{-\tau s}}{\sqrt{s}}\int_{-\tau}^{0}\phi(t)e^{-st}\,dt}{1+\frac{\lambda e^{-\tau s}}{\sqrt{s}}}=\frac{\sqrt{s}\widehat{f}(s)-\lambda e^{-\tau s}\int_{-\tau}^{0}\phi(t)e^{-st}\,dt}{\sqrt{s}+\lambda e^{-\tau s}},\ s>0. \tag{54}$$

Let us illustrate with some simple examples how the expression of the Laplace transform $\widehat{v}$ given by (54) can be used to compute the solution v.

In the case $h(x)\equiv h_0$, $\phi(t)\equiv 0$, we have $f(t)=h_0/\sqrt{\pi t}$ and (54) gives

$$\widehat{v}(s)=\frac{h_0}{\sqrt{s}}\left(1+\lambda\frac{e^{-\tau s}}{\sqrt{s}}\right)^{-1}. \tag{55}$$

To compute the inverse transform of the second member of (55) we proceed in a formal way as follows:

$$\frac{h_0}{\sqrt{s}}\left(1+\lambda\frac{e^{-\tau s}}{\sqrt{s}}\right)^{-1}=h_0\sum_{n=0}^{+\infty}\frac{(-\lambda)^n}{s^{(n+1)/2}}e^{-n\tau s}.$$

Then, taking into account that $\left(s^{-(n+1)/2}\right)^{\vee}(t)=t^{(n-1)/2}/\Gamma\left((n+1)/2\right)$ and $\left(e^{-n\tau s}\right)^{\vee}(t)=\delta(t-n\tau),\ n\ge 0$, the convolution theorem enables us to write

$$\begin{aligned} v(t) &= h_0\left(\sum_{n=0}^{+\infty}\frac{(-\lambda)^n}{s^{(n+1)/2}}e^{-n\tau s}\right)^{\vee}(t)\\ &= h_0\sum_{n=0}^{+\infty}\frac{(-\lambda)^n t^{(n-1)/2}}{\Gamma\left((n+1)/2\right)}*\delta(t-n\tau)\\ &= h_0\sum_{n=0}^{+\infty}\frac{(-\lambda)^n}{\Gamma\left((n+1)/2\right)}\left(t-n\tau\right)^{(n-1)/2}H(t-n\tau) \end{aligned} \tag{56}$$

where $H(t)$ is the Heaviside step function. Of course, the validity of expression (56) may be tested by direct substitution in equation (48). Now, attending to the values $v(N\tau)$ of the solution $v(t)$ at the points $t_N = N\tau,\ N \in \mathbf{N}$, from (56) we deduce

$$\begin{aligned} v(N\tau) &= h_0 \sum_{n=0}^{N-1} \frac{(-\lambda)^n}{\Gamma\left((n+1)/2\right)} (N\tau - n\tau)^{(n-1)/2} \\ &= \frac{h_0}{\sqrt{\tau}} \sum_{n=0}^{N-1} \frac{(-\lambda\sqrt{\tau})^n}{\Gamma\left((n+1)/2\right)} (N-n)^{(n-1)/2}. \end{aligned}$$

For every $\lambda > 0$, numerical experiences reveal that the behaviour of the sequence $\{v(N\tau)\}$ is oscillatory. It should be emphasized that this oscillatory behaviour does not occur in the absence of delay ($\tau = 0$). In fact, it was proved in Berrone, Tarzia, Villa (2000) that the inequalities $0 \le v(t) \le 1/\sqrt{\pi t},\ t > 0$, are satisfied by the solution to the problem with $\tau = 0$. Now, using (55) we can compute

$$\begin{aligned} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds &= \left(\frac{1}{\sqrt{s}}\,\widehat{v}(s)\right)^{\vee}(t) \\ &= \left(\frac{h_0}{s}\left(1 + \lambda\frac{e^{-\tau s}}{\sqrt{s}}\right)^{-1}\right)^{\vee}(t) \\ &= h_0\left(\sum_{n=0}^{+\infty} \frac{(-\lambda)^n}{s^{n/2+1}} e^{-n\tau s}\right)^{\vee}(t) \\ &= h_0 \sum_{n=0}^{+\infty} \frac{(-\lambda)^n}{\Gamma\left(n/2+1\right)} (t-n\tau)^{n/2}\, H(t-n\tau), \end{aligned}$$

so that the behaviour of this integral when $t \uparrow +\infty$ is oscillatory as well.

In the previous subsection we have seen that an ultimately positive solution $v(t)$ of equation (16) is bounded from above by a function of the form $Ct^{-1/2}$ and, by Theorem 4.3, this fact entails that the oscillation limits of the mean temperature $\widetilde{u}(t)$ are both finite. As the following theorem shows, the situation is better when the control function is linear.

Theorem 4.5 *Suppose that the initial temperature h satisfies $|Mh| < +\infty$ and that the control is a linear function $F(v) \equiv \lambda v,\ (\lambda > 0)$. If the solution $v(t)$ to problem (48) is ultimately positive, then*

$$\lim_{t\uparrow+\infty} \widetilde{u}(t) = 0.$$

Proof. From Theorem (4.2), we see that it is sufficient to prove that

$$\lim_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds = 0.$$

Suppose that $v(t) \geq 0$, for $t > t_0$. From (48) we obtain that, if $t > \max\{t_0, \tau\}$,

$$0 \leq v(t) = f(t) - \lambda \int_0^t \frac{v(s-\tau)}{\sqrt{\pi(t-s)}}\, ds \tag{57}$$

$$= f(t) - \lambda \int_0^\tau \frac{\phi(s-\tau)}{\sqrt{\pi(t-s)}} - \lambda \int_0^{t-\tau} \frac{v(s)}{\sqrt{\pi(t-\tau-s)}}. \tag{58}$$

Taking into account that $f(t) \to 0$ and $\int_0^\tau \frac{\phi(s-\tau)}{\sqrt{\pi(t-s)}} \to 0$, if $t \uparrow \infty$, we obtain

$$\limsup_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds \leq 0. \tag{59}$$

Moreover

$$\int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds = \int_0^{t_0} \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds + \int_{t_0}^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds, \tag{60}$$

The first integral on the right hand side of (60) tends to zero if $t \uparrow +\infty$, the second one is a positive function of t, hence

$$\liminf_{t\uparrow+\infty} \int_0^t \frac{v(s)}{\sqrt{\pi(t-s)}}\, ds \geq 0,$$

which, together (59), gives the result. ■

Next, Corollary 3.1 is combined with the previous theorem in order to obtain a precise result of controllability.

THEOREM 4.6 *Suppose, apart from the assumptions of Theorem 4.5, that*
i) $\phi(t) \geq 0,\ -\tau \leq t \leq 0$, *and*
ii) $p(t) - \varphi(t) \leq f(t) \leq p(t),\ t > 0$, *where* $p(t)$ *solves the problem*

$$\begin{cases} p(t) = \varphi(t) + \lambda \int_0^t \frac{p(s-\tau)}{\sqrt{\pi(t-s)}}\, ds, & t > 0, \\ p(t) = \phi(t), & -\tau \leq t \leq 0, \end{cases} \tag{61}$$

for a given positive function φ.
Then we have

$$\lim_{t\uparrow+\infty} \widetilde{u}(t) = 0.$$

Proof. From the remark after the proof of Corollary 3.1 and from conditions **i)** and **ii)**, we see that the solution $v(t)$ to (48) is non-negative for $t > 0$ and therefore, from Theorem 4.5 we have the result. ■

4.3. Nonlinear monotone controls

In this last subsection, a concrete class of control functions is identified which allows a suitable control of mean temperatures in problem (4). The method we use in identifying this class is mainly based on the estimates furnished by Theorem 3.3 for the solutions to equation (16) and, even if our developments seem to be of a very specific nature, they admit an ample generalization.

As it was seen in the example after Theorem 4.4, the solutions to equation (16) may present great amplitude oscillations and it was this fact what destroy any attemp to control the mean temperatures of the slab. In order to avoid ascillations, we implement a twofold strategy. On the one hand, we impose on the control function F the condition of reacting only when a certain "threshold flux" v_0 is exceeded, i.e.:

i) $F(v) = 0$ if $|v| \leq v_0$.

On the other hand, we ask of F to satisfy a condition of "saturation", i.e.,

ii) F is a bounded function: $|F(v)| \leq M,\ v \in \mathbf{R}$.

Indeed, these conditions should be as a rule fulfilled by realistic controls, so that we consider the class $\mathcal{F}(v_0, M)$ of continuous control functions F with F non-decreasing, $vF(v) \geq 0,\ v \in \mathbf{R}$, and satisfying conditions **i)** and **ii)** above.

Our claim of controllability rests on the following lemma.

LEMMA 4.3 *Let $F \in \mathcal{F}(v_0, M)$ be where v_0 and M are positive constants such that $\pi v_0^2 - 16\tau M^2 > 0$ and let h be a bounded function with*

$$\|h\|_\infty \leq \frac{\pi v_0^2}{8M} - 2\tau M; \tag{62}$$

then there exists a positive constant A such that

$$|v(t)| \leq \frac{A}{\sqrt{\pi t}},\ t > 0,$$

for the corresponding solution $v(t)$ to problem (16)-(17).

Proof. Let us define two functions v_1 and v_2 as follows:

$$v_1(t) = \begin{cases} \phi(t), & -\tau \leq t \leq 0 \\ -A/\sqrt{\pi t}, & t > 0 \end{cases},\ v_2(t) = \begin{cases} \phi(t), & -\tau \leq t \leq 0 \\ A/\sqrt{\pi t}, & t > 0 \end{cases},$$

where A is a positive constant. We have to determine a suitable value for A so that the inequalities

$$v_1(t) \leq v(t) \leq v_2(t),\ t > 0,$$

hold for the solution $v(t)$ to problem (16)-(17).

By Theorem 3.3, it is sufficient to show that

$$\begin{cases} -\frac{A}{\sqrt{\pi t}} \leq f(t) - \int_0^t \frac{F(v_2(s-\tau))}{\sqrt{\pi(t-s)}}\,ds \\ \frac{A}{\sqrt{\pi t}} \geq f(t) - \int_0^t \frac{F(v_1(s-\tau))}{\sqrt{\pi(t-s)}}\,ds \end{cases}, \ t > 0. \tag{63}$$

Now, taking into account that h is bounded and using Theorem 3.1, it is not difficult to see that inequalities (63) will be satisfied provided that

$$\frac{A}{\sqrt{\pi t}} \geq \frac{\|h\|_\infty}{\sqrt{\pi t}} + M\int_0^t \frac{ds}{\sqrt{\pi(t-s)}},\ \ 0 < t \leq T, \tag{64}$$

and

$$\frac{A}{\sqrt{\pi t}} \geq \frac{\|h\|_\infty}{\sqrt{\pi t}} + M\int_0^T \frac{ds}{\sqrt{\pi(t-s)}},\ \ t > T, \tag{65}$$

where

$$T = \tau + \frac{A^2}{\pi v_0^2} \tag{66}$$

is the smallest positive constant such that

$$F(v_i(t-\tau)) = 0,\ \ t \geq T, i = 1, 2.$$

Note that also the other properties of the control function F have been employed in reducing inequalities (63) to the simpler ones (64) and (65).

Now, a simple computation show us that (64) and (65) are both fulfilled provided that

$$A \geq \|h\|_\infty + 2MT$$

or, from (66),

$$A \geq \|h\|_\infty + 2M\left(\tau + \frac{A^2}{\pi v_0^2}\right).$$

The existence of a positive constant A satisfying this last inequality is guaranteed by the hypotheses (62). In fact, the quadratic equation

$$A = \|h\|_\infty + 2M\left(\tau + \frac{A^2}{\pi v_0^2}\right) \tag{67}$$

has a positive solution provided that its discriminant

$$1 - 4\frac{2M}{\pi v_0^2}\left(\|h\|_\infty + 2M\tau\right) \geq 0;$$

i.e., $\|h\|_\infty \leq \pi v_0^2/\left(8M\right) - 2\tau M$. This completes the proof. ■

We are now able to state our main result.

Theorem 4.7 *Assume that the hypotheses of Lemma 4.3 hold for a given initial temperature profile h such that $|Mh| < +\infty$. Then, the mean temperature $\widetilde{u}(t)$ satisfies*

$$\lim_{t\uparrow+\infty} \widetilde{u}(t) = \widetilde{u}_0 - \int_0^T F(v(s-\tau))\,ds$$

where T is given by expression (66) (in which A can be taken as the smallest root of equation (67)).

Proof. From Lemma 4.3 we obtain $|v(t)| \le A/\sqrt{\pi t},\ t > 0$, and therefore, $F(v(t-\tau)) = 0,\ t \ge T = \tau + A^2/\left(\pi v_0^2\right)$. Then, using Theorem 4.1, we obtain

$$\lim_{t\uparrow+\infty} \widetilde{u}(t) = \lim_{t\uparrow+\infty} \left(\widetilde{u}_0 - \int_0^t F(v(s-\tau))\,ds \right) = \widetilde{u}_0 - \int_0^T F(v(s-\tau))\,ds.$$

■

By means of this result we can give a class of control functions $F(v)$ which guarantees, under suitable assumptions on the initial datum $h(x)$ the existence of the limit, as $t \to +\infty$, of the mean temperatures.

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