Control and Cybernetics

vol. **32** (2003) No. 4

Reachable sets concept - a general abstract analysis

by

Tomasz Terlikowski

Institute of Geophysics
Polish Academy of Sciences
Ksiecia Janusza 64, Warszawa 01-452, Poland
e-mail: tom@igf.edu.pl

Abstract: A control problem over the infinite time horizon for periodical system is considered. Our aim is to analyze a special concept of solving problems of this type, based on the known idea of *reachable sets*. We try to consider this concept in a more general manner than it was done in earlier works and to find what is really essential in it. The algorithm corresponding to the proposed general description is presented.

Keywords: infinity, periodicity, reachable set.

1. Introduction

The reachable sets concept has been introduced a few decades ago, see Bertsekas and Rhodes (1971, 1972), Glover et al. (1971). In some papers on this subject another terminology was used (*periodically invariant*, *viable sets*), see Blanchini et al. (1993), Karbowski (1999). We found that there is something common and essential in all those approaches, but it has not been explicitly and clearly formulated. All the papers quoted are often loaded with many particular details concerning special features of each case.

In the present paper we are going to consider the concept in as general terms as possible. That is why we try to apply the formulations using the logical and set-theoretical language only. This should not be strange at all, however, realizing that this language has been accepted in mathematical sciences for almost 200 years and that it makes possible to see clearly many apparently different things. This will be sufficiently clear, we hope, via an example, which we consider in the sequel. The main tasks to be achieved in this paper are the following:

First, we formulate in Section 2 the *periodic* and *infinite* horizon control problem. The proposed formulation comprises all the cases, to which the reachable sets concept can be applied. This is really a decisive and essential step for further considerations. Although this formulation is apparently similar to the classical one, it is essentially different in fact, since it uses the elements of the *power set* $\mathbf{2^X}$ of the state space as decision variables (instead of the states themselves), see the examples in Section 3.

Secondly, we discuss the methodological constructions built within the reachable sets concept. The basic notions are introduced, the theorems are proposed and demonstrated in detail in Section 3. At the same time, the respective necessary assumptions are formulated.

Finally, the respective computational algorithm is theoretically analyzed in Section 4. Let us underline at once the generality of the presented approach. The formulation of control problem, of the assumptions admitted and finally of some properties concerning the concept itself (the respective theorems), have this general, abstract character.

2. Problem formulation

The present formulation is applicable to a large class of problems, due to the use of an abstract scheme expressed by means of elementary logic and set theory.

What is important in the proposed formulation, it is that the problem is *periodic* and *infinite*. Moreover, the required condition to be satisfied has the special form, with the general quantifier $\forall n$ at the beginning and the same relation for each period i.e. for each $n \in N$; where $N = \{0, 1, \ldots\}$ is the set of all non-negative integers.

There are two groups of variables, having the sense of *state sets* and *controls* in our problem. Suppose that we are given a set $\mathbf{X}$, called *state space* and a set $\mathbf{M}$ representing the *control space*. Let us denote by X the family of all subsets of $\mathbf{X} : \mathsf{X} = \mathbf{2^X}$. For a given $\boldsymbol{p}(x, m)$ being *sentential function (predicate)* of two free variables and $\mathbf{F}$ being a *mapping* from $\mathsf{X} \times \mathbf{M}$ into X, we consider the following *infinite problem*:

Find a sequence $\{\mathbf{m}_n\}$, $\mathbf{m}_n \in \mathbf{M}$ and a sequence $\{X_n\}$, $X_n \in \mathsf{X}$, $n = 0, 1, \ldots,$

defined by the state equation: $X_{n+1} = \mathbf{F}(X_n, \mathbf{m}_n),\ n = 0, 1, \ldots \quad (1)$

that satisfy the condition: $\forall n \in N : \boldsymbol{p}(X_n, \mathbf{m}_n). \quad (2)$

The problem (1)-(2) will be reformulated now into a more logically concise form. Introducing two auxiliary variables $\boldsymbol{x}$ and $\boldsymbol{\mu}$, representing an initial state set X_0 and a sequence of controls $\{\mathbf{m}_n\}$ respectively, we can formulate the problem as follows:

Find $\boldsymbol{x} \in \mathsf{X}$ and $\boldsymbol{\mu} \in \mathbf{M}^N$ that satisfy the following formula $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$:

$$\exists \{\mathbf{m}_n\} \in \mathbf{M}^N, \{X_n\} \in \mathsf{X}^N : [\{\mathbf{m}_n\} = \boldsymbol{\mu}] \wedge [X_0 = \boldsymbol{x}] \wedge$$
$$[\forall\, n \in N : X_{n+1} = \mathbf{F}(X_n, \mathbf{m}_n)] \wedge [\forall\, n \in N : \boldsymbol{p}(X_n, \mathbf{m}_n)]. \quad (3)$$

Note first that formula $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$ can be satisfied *only* by $\boldsymbol{x} \in \mathsf{X}$ and $\boldsymbol{\mu} \in \mathbf{M}^N$. Indeed, if $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$ is true, then, according to (3) there exists a set $X_0 \in \mathsf{X}$, such that $X_0 = \boldsymbol{x}$ and a sequence $\mathbf{m}_n \in \mathbf{M}^N$, such that $\{\mathbf{m}_n\} = \boldsymbol{\mu}$.

Taking into account this remark, our problem will be formulated shortly as follows:

$$\text{Find a pair } (\boldsymbol{x}, \boldsymbol{\mu}) \text{ that satisfies formula } \boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu}) \tag{4}$$

where $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$ is defined in (3). This is exactly this type of problem, to which *the reachable sets concept can* be applied.

One can see that the above form follows that of Terlikowski (2002) for a min-max optimization problem. It has been obtained in the paper quoted after a suitable formal manipulation (see chapter "The regular concise form of the problem").

2.1. Transferring the classical problem into the power set of the state space

It will be shown now how to transformulate the classical control problem, e.g. that of keeping the state in a given set, to the form presented above.

A dynamic system is considered described by the following discrete time state equation, over the infinite time horizon:

$$x_{t+1} = f_t(x_t, m_t, z_t), \quad t = 0, 1, \ldots$$

where $x_t \in \mathbf{X} \subseteq \mathbb{R}^n$ is the *process state* value at instant t; $m_t \in M \subseteq \mathbb{R}^m$ - the *control* value and $z_t \in Z \subseteq \mathbb{R}^z$ - *disturbance* value within the *stage* t, i.e. between time instants t and $t+1$. $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^z$ are n, m, z - dimensional Euclidean spaces respectively; f_t is a function, $f_t : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^z \to \mathbb{R}^n$; $n, m, z \in \{0, 1, \ldots\}$, i.e. the set of all natural numbers.

The following instantaneous constraints are imposed, noted in a unified form as: $(x_t, m_t, z_t) \in W_t \subseteq X \times M \times Z$, $t = 0, 1, \ldots$, where W_t is a given *nonempty* set, depending on t.

It is assumed that we deal with the following, *periodicity* relations in our control process: $\forall\, t : f_t = f_{t+T}$, $\forall\, t : W_t = W_{t+T}$. Thus, each *period* consists of T "similar" intervals called *stages*, numbered with indices $i = 0, \ldots, T{-}1$. The idea is to transfer the problem from the state space $\mathbf{X}$ into the space of *state sets*, $\mathbf{2^X}$.

We use the following notation and definitions: $\mathsf{M} = \{(R_0, \ldots, R_{T-1}) : \forall i = 0, 1, \ldots, T{-}1 \quad R_i \in M^{\mathbf{X}}\}$ is the set of all T-element sequences of control laws. We denote by $\{K^i\}_0^T$ the sequence of $T+1$ mappings $K^i : \mathbf{2^X} \times \mathsf{M} \to \mathbf{2^X}$: $K^0(X, m) = \mathbf{X}$; $K^{i+1}(X, m) = \{y : \exists x, z : [x \in K^i(X, m) \wedge (x, R_i(x), z) \in W_i \wedge y = f_i(x, R_i(x), z)]\}$, $i = 0, 1, \ldots, T-1$.

The mapping K^{i+1}, $i = 0, \ldots, T-1$ determines the set of states reached by the system at the end of i-th stage, given a set X of initial states (at the beginning of the first period, i.e. at time instant 0), and a sequence of control rules $m = (R_0, \ldots, R_{T-1})$ within this period, provided that every disturbance value is such that constraints are satisfied. The last mapping, corresponding to the T-th stage, is denoted by $F : F(X, m) = K^T(X, m)$ and it determines the set of all states reached by the system at the end of the first period, for any initial state $x_0 \in X$ and the control m.

Next, the sequence of T relations $P_i \subseteq \mathbf{2^X} \times \mathsf{M}, i = 0, 1, \ldots, T-1$, is defined as follows: $P_i(X, m) \equiv \{\forall x \in K^i(X, m)\ \exists z\ (x, R_i(x), z) \in Wi\}$. Note that the condition in this formula expresses the requirement for the pair $(x, R_i(x))$ to belong to the projection of the set W_i on the product of state and control spaces $\mathbb{R}^n \times \mathbb{R}^m$. As it is seen, $P_i(X, m)$ signifies that for the control m and for any state reached at the beginning of the i-th stage - all the constraints imposed for i-th stage are satisfied. One may say that every relation P_i, $i = 0, \ldots, T-1$, describes those constraints, which concern only the i-th stage of the first period,

Finally, we define the relation $\mathbf{P}$ as the conjunction of all relations from the sequence $\{P_i\}_0^{T-1}$:

$$\mathbf{P}(X, m) \equiv \{X \subseteq \mathbf{X} \wedge m \in \mathsf{M} \wedge [\forall\, i = 0, \ldots, T-1 : P_i(X, m)\}.$$

Note that, due to periodicity of the control process, $\mathbf{P}$ describes those constraints that concern the stages $t = nT, nT + 1 \ldots, (n + 1)T - 1$, for any $n = 0, 1, \ldots,$ that is, for any period of the infinite control process. Finally, with the aid of this notation and due to the *periodicity* of control process, we may note informally the problem as the infinite conjunction:

$$P(X_0, m_0) \wedge P(X_1, m_1) \wedge \cdots \wedge P(X_n, m_n) \wedge \ldots$$

In this conjunction, m_n is a finite T-element sequence of control laws in the n-th period, that is $m_n = (R_{nT}, \ldots, R_{(n+1)T-1})$, and the state sets X_n (determining the states at the beginning of n-th period) are defined recursively: $X_{n+1} = F(X_n, m_n)$, $n = 0, 1, \ldots,$ with a given initial condition.

Our problem can be therefore be noted in the following *concise regular* form:

$$\begin{aligned}&\text{Find an initial state set } \mathbf{X}_0 \subseteq \mathbf{X} \text{ and a sequence } \{m_n\},\\ &\text{with } m_n \in \mathsf{M}, n = 0, 1, \ldots, \text{ such that } \forall n \in N : \mathbf{P}(X_n, m_n)\\ &\text{subject to: } X_0 = \boldsymbol{X}_0,\ X_{n+1} = F(X_n, m_n),\ n = 0, 1, \ldots .\end{aligned}$$

3. The basic notions and theorems in reachable sets concept

The idea of *reachable sets* has been applied to solve e.g. the min-max optimization problem for linear system case in Glover et al. (1971), Blanchini et al. (1993). The present analysis, although applicable to an essentially larger class of problems, needs, however, some additional assumptions. They are in any way easily satisfied in the classical case of a problem described through a recursive state transformation, as considered in all earlier works.

The following two abstract and general conditions concern the *set-algebraic* properties of $\boldsymbol{p}$ and function $\mathbf{F}$:

$$\forall X,\ X' \in \mathsf{X}\ \ \forall \mathbf{m} \in \mathsf{M} : \{ (\, (\boldsymbol{p}(X, \mathbf{m}) \wedge X' \subseteq X) \rightarrow \boldsymbol{p}(X', \mathbf{m})\,) \wedge \boldsymbol{p}(\emptyset, \mathbf{m})\,)\} \tag{A}$$

$$\forall X,\ X' \in \mathsf{X}\ \ \forall \mathbf{m} \in \mathsf{M} : \{(X' \subseteq X \rightarrow \mathbf{F}(X', \mathbf{m}) \subseteq \mathbf{F}(X, \mathbf{m})) \wedge (\mathbf{F}(\emptyset, \mathbf{m}) = \emptyset)\} \tag{B}$$

The symbol $\alpha \rightarrow \beta$ means here and thereafter the *implication* and $\alpha \equiv \beta$ the *equivalence*, if α and β are *sentential expressions*.

Condition (A) states that the relation determined in X by $\boldsymbol{p}(\bullet, \mathbf{m})$, for any fixed $\mathbf{m} \in \mathsf{M}$, is *invariant* with respect to the inclusion relation $\subseteq$. The first part of (B) states that $\mathbf{F}(\bullet, \mathbf{m})$, with any fixed $\mathbf{m} \in \mathsf{M}$, is a mapping $\mathsf{X} \rightarrow \mathsf{X}$ *homomorphic* for the inclusion relation $\subseteq$ in $\mathbf{X}$, i.e. the relation $\subseteq$ between two sets is conserved in their images by $\mathbf{F}(\bullet, \mathbf{m})$.

The form of (A), (B) is in fact very general. We use such a form to capture many different cases and to stress that this is the only important thing (and not many other specific features of the state equation etc.), which is relevant from the point of view of the considered concept of reachable sets. The general formal language is ideal to obtain such a methodological gain. Assumptions (A) and (B) occur to be quite naturally applicable to the optimization problem considered in Terlikowski (1997) and Karbowski (1999). Indeed, these assumptions are satisfied if $\boldsymbol{p}$ and $\mathbf{F}$ result from a usual description in the state space and $\boldsymbol{p}$ results from a *safety-type* description of control variants, Terlikowski (2002). Note that only the cases of this type are considered in the majority of works in this field. The basic theorems characterizing a special class of solutions will be now proved under two above assumptions.

The first theorem introduces the crucial notion of the concept: the *reachability*, as well as a special class of solutions called *reachable stationary solutions*. By *stationary solution* we mean such a pair $(X, \{\mathbf{m}_n\})$, which is a solution of (3) and, moreover, $\{\mathbf{m}_n\}$ is a *constant* (*time invariant*) *sequence of controls*, that is: $\mathbf{m}_n = m$ for all $n = 0, 1, \ldots$. This sequence will be usually noted by $\{\bar{\bar{\boldsymbol{m}}}\}$.

THEOREM 3.1 *Let the pair* (X, m) *satisfy the following reachability condition:*

$$\boldsymbol{p}(X, m) \wedge [\mathbf{F}(X, m) \subseteq X]. \tag{5}$$

Then, under assumptions (A) *and* (B)*, the pair* $(X, \{\bar{\bar{\boldsymbol{m}}}\})$ *is a solution of problem (3).*

Proof. Let $(X, \{\bar{\bar{\boldsymbol{m}}}\})$ satisfy (5). We consider the sequence $\{X_n\}$, $X_0 = X$, being the solution of equation (1), for the constant sequence $\{\mathbf{m}_n\}$ such that $\mathbf{m}_n = m$, $n = 0, 1, \ldots$. We will prove that $(X_0, \{\bar{\bar{\boldsymbol{m}}}\})$ is a solution of (3).

First, it is evident that the formula $\forall n \in N : X_{n+1} = \mathbf{F}(X_n, \mathbf{m}_n)$ occurring in (3) is satisfied by the sequences $\{X_n\}$ and $\{\mathbf{m}_n\} = \{\bar{\bar{\boldsymbol{m}}}\}$. In particular, we have: $X_0 = X \in \mathsf{X}$, $\mathbf{m}_0 \in \mathsf{M}$ and $X_n \in \mathsf{X}$ for all $n = 0, 1, \ldots$. We will prove by induction that $\forall n \geq 0 : X_n \subseteq X$. Indeed, we have $X_0 = X \subseteq X$ and supposing $X_n \subseteq X$ for some $n \geq 0$, we get by assumption (B): $X_{n+1} = \mathbf{F}(X_n, m) \subseteq \mathbf{F}(X, m)$. The latter, due to $\mathbf{F}(X, m) \subseteq X$ in (5), implies: $X_{n+1} \subseteq X$. Then, since we have $\boldsymbol{p}(X, m)$ by (5), we see that assumption (B) implies $\boldsymbol{p}(X_n, m)$. This means that $\boldsymbol{p}(X_n, \mathbf{m}_n)$ holds for every $n \geq 0$. Hence, $\{X_n\}$ and $\{\mathbf{m}_n\} = \{\bar{\bar{\boldsymbol{m}}}\}$ satisfy formula $\forall n \in N : \boldsymbol{p}(X_n, \mathbf{m}_n)$ in (3), so the pair $(X, \{\bar{\bar{\boldsymbol{m}}}\})$ satisfies formula $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$. ■

DEFINITION 3.1 *Any pair (X, m) satisfying (5) is called reachable pair and X is then called reachable set. The corresponding pair $(X, \{\bar{\bar{\boldsymbol{m}}}\})$ is called reachable stationary solution of problem (3).*

According to Theorem 3.1, the reachability concept allows us to transfer in a way the infinite formulation (3), as well as the infinite control sequence $\{\mathbf{m}_n\}$, into a finite form. Indeed, (3) is now replaced by a *finite* formula (5) and $\{\mathbf{m}_n\}$ is replaced by a *finite-wise*, i.e. constant, *stationary* sequence $\{\bar{\bar{\boldsymbol{m}}}\}$. In the sequel we shall be interested mainly in reachable stationary solutions. We shall formulate now the third assumption which, together with (A) and (B), enables one to find *effectively* a solution of reachability condition (5). The following crucial definition will be applied to the problem under consideration.

For a given set $X \in \mathsf{X}$, let us denote by $\mathbf{R}(\mathrm{X})$ the following family of sets:

$$\mathbf{R}(X) = \{Y : \exists\, \mathbf{m}\, [\boldsymbol{p}(Y, \mathbf{m}) \wedge \mathbf{F}(Y, \mathbf{m}) \subseteq X]\}\,. \tag{6}$$

By virtue of assumptiona (A) and (B), the family $\mathbf{R}(X)$ has the following important property:

$$\forall X, Y, Y' \in \mathsf{X} : [(Y \in \mathbf{R}(X)) \wedge (Y' \subseteq Y)] \rightarrow [Y' \in \mathbf{R}(X)]\,, \tag{7}$$

which states that any subset of an element of $\mathbf{R}(X)$ is also an element of $\mathbf{R}(X)$. Observe that the sets Y belonging to $\mathbf{R}(X)$ are *inclusion inverse* elements to X, by mapping $\mathbf{F}$. However, it is not only function $\mathbf{F}$, but also formula $\boldsymbol{p}(\boldsymbol{x}, m)$ that intervenes in definition (6). Thus, we should rather say that the sets from the family $\mathbf{R}(X)$ are *inclusion $\boldsymbol{p}$-inverse* elements to X, by mapping $\mathbf{F}$.

We shall use the following definition:

DEFINITION 3.2 *For a given family of sets* $\mathbf{E}$*, we call largest set of* $\mathbf{E}$ *the set* $\max \subseteq \mathbf{E}$ *defined as follows:*

$$[X = max \subseteq \mathbf{E}] \equiv [X \in \mathbf{E} \wedge (\forall X' \in \mathbf{E} : X' \subseteq X)]. \tag{8}$$

The above formula is understood as a conditional definition of functional term $max \subseteq$.

The latter notions, $\mathbf{R}(\mathsf{X})$ and $max \subseteq \mathbf{E}$, appear in the following definition of *mapping* $\mathbf{D}$:

$$[Y = \mathbf{D}(X)] \equiv [(X \in \mathsf{X}) \wedge (Y = X \cap max \subseteq \mathbf{R}(X))]. \tag{9}$$

The third assumption, stating the existence of the *larges "inclusion* $\boldsymbol{p}$*-inverse"* element by mapping $\mathbf{F}$, is the following:

$$\forall X \in \mathsf{X} : [\mathbf{R}(X) \neq \emptyset] \rightarrow [\exists\, \mathbf{Z} : \mathbf{Z} = max \subseteq \mathbf{R}(X)]. \tag{C}$$

Note that the family $\mathbf{R}(\mathsf{X})$ is *never empty* due to assumption $\boldsymbol{p}(\emptyset, \mathbf{m})$ in (A) and $\mathbf{F}(\emptyset) = \emptyset$ in (B). It includes at least one element, the empty set $\emptyset$. Assumption (C) is then equivalent to: $\forall X \in \mathsf{X}\, \exists\, \mathbf{Z} : \mathbf{Z} = \max \subseteq \mathbf{R}(X)$.

Hence, under (A), (B), (C), the mapping $\mathbf{D}$ is defined correctly and $\mathbf{D} \in \mathsf{X}^{\mathsf{X}}$. The second, crucial theorem concerns the problem of effective solution of reachability condition (5).

THEOREM 3.2 *The following potential reachability condition for a set* X*:*

$$\exists \mathbf{m} :\ \boldsymbol{p}(X, \mathbf{m}) \wedge [\mathbf{F}(X, \mathbf{m}) \subseteq X] \tag{10}$$

which states that the pair $(X, \mathbf{m})$ *satisfies reachability condition (5) with some* $\mathbf{m}$*, is equivalent, under assumptions* (A)*,* (B)*,* (C)*, to the equation:*

$$X = \mathbf{D}(X). \tag{11}$$

Proof. Let us denote by $\mathbf{W}(X, Y, \mathbf{m})$ the following formula: $\boldsymbol{p}(Y, \mathbf{m}) \wedge \mathbf{F}(X, \mathbf{m}) \subseteq X$ which occurs in definition (6) of family $\mathbf{R}(X)$. We have then, by this definition: $\mathbf{R}(X) = \{Y : \exists \mathbf{m} : \mathbf{W}(X, Y, \mathbf{m})\}$. Suppose that X' satisfies equation (11); then, by definition (9) of $\mathbf{D} :\ X' \in \mathsf{X}$ and $X' = X' \cap \max \subseteq \mathbf{R}(X')$. The second relation means that a set Y' exists, namely $Y' = \max \subseteq \mathbf{R}(X')$, such that:

$$Y' \in \mathbf{R}(X') \quad \text{and} \quad X' = X' \cap Y', \tag{12}$$

the latter equation being equivalent to $X' \subseteq Y'$.

We will show that (12) implies that (10) is satisfied for X' i.e. that: $\exists \mathbf{m} : \mathbf{W}(X', X', \mathbf{m})$. Indeed, according to the first part of (12), Y' satisfies the formula $\exists \mathbf{m} : \mathbf{W}(X', Y, \mathbf{m})$ with free variable Y. That is, there exists $\boldsymbol{m}$ such that $\mathbf{W}(X', Y', \boldsymbol{m})$. Hence, we have: $\boldsymbol{p}(Y', \boldsymbol{m}) \wedge \mathbf{F}(Y', \boldsymbol{m}) \subseteq X'$ and, due to the second part of (12), $X' \subseteq Y'$. Thus, by assumption (A) we get: $\boldsymbol{p}(X', \boldsymbol{m})$. On the other hand, by (B): $\mathbf{F}(X', \boldsymbol{m}) \subseteq \mathbf{F}(Y', \boldsymbol{m})$, hence: $\mathbf{F}(X', \boldsymbol{m}) \subseteq X'$.

As it is seen, the *potential reachability condition* is really satisfied for X'. Inversely, let us suppose that X' satisfies (10). This means that formula $\mathbf{W}(X, Y, \boldsymbol{m})$ is satisfied, with some m, by the triple $(X', X', \boldsymbol{m})$. Then, $X' \subseteq \mathsf{X}$ and, by definition of $\mathbf{W} : X' \in \mathbf{R}(X')$. Thus, by assumption (C), $X' \subseteq \max \subseteq \mathbf{R}(X')$. But the latter is equivalent to $X' = X' \cap \max \subseteq \mathbf{R}(X')$, that is, to equation (11). ∎

This theorem allows us to realize the process of searching for a solution of the *potential reachability condition* (10) in a convenient, algorithmic way: namely, by solving equation (11). Having found any set X, a solution of (11), we obtain a *reachable stationary solution* of problem (3), simply by finding any m such that the pair (X, m) satisfies reachability condition (5)(Theorem 3.1).

Let us consider some simple examples.

EXAMPLE 3.1 Given the discrete time state equation: $x_{n+1} = x_n - m_n(x_n) + z_n$ with the constraints: $z_n \in Z = [\,0, \frac{1}{2}\,]$, $m_n(x_n) \in U = [\,0, 1\,]$, one searches for a set $X_0 \subseteq \mathbf{R}$ and the control rules $m_n(x_n) : \mathbf{R} \to \mathbf{R}$ such that the condition: $x_n \in \bar{\mathbf{X}} = [0, 1]$ be satisfied for all $n = 0, 1, \ldots$.

(1) According to the Bertsekas' procedure aimed at keeping the state within a given set X, we construct backward in time the sequence $\{X_{-i}\}$ of states:

$$X_{-0} = X, \quad X_{-i-1} = X \cap \{x : \exists u \in U \quad \forall z \in Z \quad (x - u + z) \in X_{-i}\}. \qquad \text{(E1)}$$

Then we take the intersection of all sets X_{-i}, $\cap\{X_{-i} : i = 0, 1, \ldots\}$ as the sought state set X_0. In our problem we put $X = \bar{\mathbf{X}}$ and, after respective calculations, we get immediately that

$$\bar{\mathbf{X}} \cap \{x : \exists u \in U \quad \forall z \in Z \quad (x - u + z) \in \bar{\mathbf{X}}\} \qquad \text{(E2)}$$

is the sought solution X_0 for our problem. Moreover, for every interval $X = [X_{\min}, X_{\max}] \subseteq \bar{\mathbf{X}}$ such that $|X| = X_{\max} - X_{\min} \geq |Z| = \frac{1}{2}$, we get a solution $X_0 = X$ as well, just by substituting $\bar{\mathbf{X}}$ with X in (E2). The set defined in (E2) is the largest of all those solutions X_0. The formula from (E2): $\forall z \in Z\ (x - m_n(x) + z) \in X$ determines every admissible control law m_n, for any interval X mentioned above.

(2) One can also use the approach of the present paper. First, we should reformulate the problem so as to express it in the space $\mathbf{2^R}$ of the state sets. We get the formulation (3) with $\mathbf{M} = \mathbf{R^R}$ and

$$\begin{aligned} &\boldsymbol{p}(X, \boldsymbol{m}) \equiv \big[(X \subseteq \bar{\mathbf{X}} \wedge \mathbf{m} \in \mathbf{U^R})\big]\,; \\ &\mathbf{F}(X, \mathbf{m}) = \{(x - u + z) : x \in X, z \in Z, u = \mathbf{m}(x)\}, \end{aligned} \qquad \text{(E3)}$$

with $\mathbf{m}$ a function on the state set X, $\mathbf{m} \in \mathbf{M}$. Note that $\mathbf{M}$ represents the set of all control rules.

As far as reachable solutions are concerned (see Definition 1), we must, according to Theorem 3.2, analyse the family $\mathbf{R}(X)$, see (6). In the present case we get:

$$\mathbf{R}(X) = \{Y :\ \exists \mathbf{m}\ Y \subseteq \bar{\mathbf{X}} \wedge \mathbf{m} \in U^{\mathbf{R}} \wedge \mathbf{F}(Y, \mathbf{m}) \subseteq X\}$$

so, using (E3) we obtain:

– for every interval $X = [X_{\min}, X_{\max}] \subseteq \bar{\mathbf{X}}$ such that $|X| = X_{\max} - X_{\min} \geq |Z| = \frac{1}{2}$, $\mathbf{R}(X)$ is the family of all subsets of X, (E4)

– $\mathbf{R}(X) = \{\emptyset\}$ for X not satisfying the above condition.

Thus, max $\subseteq \mathbf{R}(X) = X$ and hence, (see 9), $\mathbf{D}(X) = X$ for any interval X satisfying (E4), what means, Theorem 3.2, that every such X is a reachable set. By Theorem 3.1 any set $X_0 = X$ is therefore a solution of our problem with stationary, i.e. constant with respect to time control rule. ∎

It is now worth considering the important question of *generality* of the presented reachability approach. The next theorem answers partially this question.

THEOREM 3.3 *If there exists the largest set X^* of the family of all X such that $(X, \boldsymbol{m})$ is a solution of problem (3), that is:*

$$X^* = max \subseteq \{X : \exists \boldsymbol{\mu}\ \boldsymbol{P}(X, \boldsymbol{\mu})\}, \tag{13}$$

then, under assumptions (A), (B), (C), the set X^* is (the largest) reachable set. Thus, due to Theorem 3.2:

$$X^* = max \subseteq \{X : X = \mathbf{D}(X)\}. \tag{14}$$

Proof. One easily proves by induction, using the regular properties of formula $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$ and applying a respective transnumeration of sequences, that if a pair $(X, \{\mathbf{m}_n\})$ is a solution of problem (3), then the "shifted" pair $(\mathbf{F}(X, \mathbf{m}_0), \{\mathbf{m'}_n\})$ is also a solution of this problem. Here $\mathbf{m}_0$ is the first element of $\{\mathbf{m}_n\}$ and $\{\mathbf{m'}_n\}$ is such a sequence that $\mathbf{m'}_n = \mathbf{m}_{n+1}$ for every $n = 0, 1, \ldots$. Let the pair $(X^*, \{\mathbf{m}_n\})$, where X^* is defined by (13), satisfy formula $\boldsymbol{P}(\boldsymbol{x}, \boldsymbol{\mu})$ and let $\mathbf{m}_0$ be the first element of the sequence $\{\mathbf{m}_n\}$. We shall prove that the pair $(X^*, \mathbf{m}_0)$ satisfies reachability condition (9).

It follows from the above that $\mathbf{F}(X^*, \mathbf{m}_0)$ satisfies the condition: $\exists \mu\ \boldsymbol{P}(X, \mu)$, occurring in (13). Therefore, since X^* is the largest of all sets satisfying this condition, there must be: $\mathbf{F}(X^*, \mathbf{m}_0) \subseteq X^*$. Evidently, we have also: $\boldsymbol{p}(X^*, \mathbf{m}_0)$. Thus, the reachability condition (9) is really satisfied by $(X^*, \mathbf{m}_0)$.

Then, we conclude from Theorem 3.2 that X^* satisfies equation (11), that is: $X^* = \mathbf{D}(X^*)$. Consider now any X' satisfying equation (11). By Theorem 3.2, X' satisfies the *potential reachability condition* (10), that is the pair (X', m) satisfies reachability condition (5) for some m. Therefore, it follows from Theorem 3.1 that $(X', \{\bar{\bar{\boldsymbol{m}}}\})$ is a solution of problem (3); hence, X' must be, by definition (13), included in X^*. This means, together with $X^* = \mathbf{D}(X^*)$, that X^* is *the largest* set satisfying (11). Thus (14) holds. ∎

Theorem 3.3 (together with 3.1 and 3.2) justify, in a sense, the presented approach. It is clear that if we confine ourselves to *reachable stationary solutions* only, this is not an essential restriction. It suffices that the largest set existed, (13), for which the original problem (3) has a solution.

The Theorems 3.1 and 3.3 are close to the results obtained in Blanchini et al. (1993), and in Karbowski (1999). In both papers considerations focused on particular cases only. In the first one it was a linear problem, while in Karbowski (1999) a min-max multiobjective problem was considered. A general approach similar to that presented here has been discussed in Terlikowski (1997), but in a very concise form, without some details and proofs.

Let us consider another example.

EXAMPLE 3.2 In order to show the capacities of our approach, we consider a very simple case of an infinite process that does not use at all any control. Namely, we take the process:

$$X_{i+1} = \mathbf{F}(X_i), \quad i = 0, 1, \ldots \tag{E5}$$

of subsequent self transformations of sets $X_i \in \mathsf{X} = \mathbf{2}^{\mathbf{R}}$ being subsets of the real numbers space $\mathbf{R}$.

The formula $\boldsymbol{p}$ has the form: $\boldsymbol{p}(X) \equiv (X \subseteq [0,1])$, for any $X \subseteq \mathbf{R}$. Considering the problem (3) in this case, we can not apply the classical idea of recursive state sets since we do not consider a transformation of state values, but of their sets as a whole. We shall apply the general approach developed in Section 3 (Theorems 3.1 and 3.2).

Denote $\mathbf{C} = [c_{\min}, c_{\max}] = [\frac{1}{4}, \frac{3}{4}]$ and $a_X = \inf(X)$, $b_X = \sup(X)$ for any $X \subseteq \mathbf{R}$. Let the function $\mathbf{F} : \mathbf{2}^{\mathbf{R}} \to \mathbf{2}^{\mathbf{R}}$ be defined as follows: $\mathbf{F}(\emptyset) = \emptyset$ and for any

$$X \subseteq \mathbf{C} : \mathbf{F}(X) = [a_X + \frac{1}{4}(b_X - a_X),\ b_X - \frac{1}{4}(b_X - a_X)]; \tag{E6}$$

having represented any $X \subseteq \mathbf{R}$ as the sum $X = X_{\inf} \cup \bar{X} \cup X_{\sup}$ where $X_{\inf} = X \cap (-\infty, c_{\min})$, $X_{\sup} = X \cap (c_{\max}, +\infty)$ and $\bar{X} \subseteq \mathbf{C}$, we let:

$$\mathbf{F}(X) = \mathbf{F}_{\inf}(X_{\inf}) \cup \mathbf{F}(\bar{X}) \cup \mathbf{F}_{\sup}(X_{\sup}) \tag{E7}$$

and

$$\mathbf{F}_{\inf}(X_{\inf}) = [a_X - \frac{1}{4}(b_X - a_X), \quad b_X + \frac{1}{4}(c_{\min} - b_X)], \tag{E8}$$

$$\mathbf{F}_{\sup}(X_{\sup}) = [a_X - \frac{1}{4}(a_X - c_{\max}), \quad b_X + \frac{1}{4}(b_X - a_X)], \tag{E9}$$

where a_X, b_X denote infimum/supremum of respective sets: $X_{\inf}$ in (E8), $X_{\sup}$ in (E9).

As it can be seen, the consecutive sets X_i are *contracted* or *expanded* according to the state equation (E5), during the infinite process. The obtained problem, which consists in satisfying the constraints $\boldsymbol{p}(X_n)$ on the infinite time horizon, can be considered as a particular case of (3), where e.g. the set M is a singleton. Observe that we have no possibility of applying directly the classical Bertsekas' approach here, since the subsequent state sets X_i are not transformed according to any state equation of the form $x_{i+1} = f_i(x_i, m_i, z_i)$.

To find and discuss the solution, we first verify by a simple calculation that assumptions (A), (B), (C) are satisfied. Then we observe what follows:

1. For any $X \subseteq \mathbf{C}$ the set X belongs itself to the family $\mathbf{R}(X) = \{Y : \mathbf{F}(Y) \subseteq X\}$, due to definition (E6) of $\mathbf{F}$. Moreover, X is the largest set of this family: $X = \max \subseteq \mathbf{R}(X)$.
   Therefore, see (9): $\mathbf{D}(X) = X \cap \max \subseteq \mathbf{R}(X) = X$. This means, according to Theorem 3.2, that any set $X \subseteq \mathbf{C}$ is a reachable set. Due to Theorem 3.1, it is then a solution of our problem. It is easy to verify that $\mathbf{C}$ is the largest of those reachable solutions.
2. For $X \not\subset \mathbf{C}$ the situation is different. We may limit the analysis to the case $X \subseteq [0, 1]$ taking into account the condition $\boldsymbol{p}(X)$ that should be satisfied by $X_0 = X$, see (3). For example, there is no any nonempty set $X \subseteq [0, c_{\min})$, which would be reachable, since any (and so, the largest) $Y \in R(X)$ must be strictly included in X. This follows from the expansion property of F, see (E8). In consequence, the reachability condition $X = \mathbf{D}(X)$ is never satisfied in such a case. The same concerns $X(c_{\max}, 1]$ and, more generally, any $X \not\subset \mathbf{C}$.

Finally, X is a reachable solution if and only if $X \subseteq \mathbf{C}$. After all, it can be easily verified by (E7)-(E9) that every solution must satisfy condition $X \subseteq \mathbf{C}$. Due to Theorem 3.3 the set X^* being the largest of all solutions, is a reachable solution, what has been else stated above in 1. ■

Note that we deal in the above example with a specific (not general) case when the set of all solutions of problem (3) is identical with the set of reachable solutions.

4. Algorithm for finding a reachable solution

Three theorems given in Section 3 constitute a sufficient background for finding a solution of problem (3) in an effective algorithmic way. The corresponding algorithm seeks a *reachable set* of initial states by finding a *fixed point* of mapping $\mathbf{D}$, i.e. such a set X that $\mathbf{D}(X) = X$, see Theorem 3.2.

A sequence $\{\bar{\mathbf{X}}_k\}$ of state sets is determined as follows: for a chosen $\bar{\mathbf{X}}_0 \in \mathsf{X}$ treated as the starting set, one calculates the sequence of sets $\{\bar{\mathbf{X}}_k\}$:

$$\bar{\mathbf{X}}_{k+1} = \mathbf{D}(\bar{\mathbf{X}}_k), \quad k = 0, 1, \dots \; , \tag{15}$$

where $\mathbf{D}(X) = X \cap \max \subseteq \mathbf{R}(X)$.

The algorithm terminates the iteration (15) at some step ϖ. The set $\bar{\mathbf{X}}_\varpi$, denoted by $\tilde{\mathbf{X}}$, is the *final result* of the algorithm. We would like $\tilde{\mathbf{X}}$ to be a good approximation of a *fixed point* of $\mathbf{D}$. The controls m corresponding to each set $\bar{\mathbf{X}}_k$, such as to satisfy the *reachability condition* (5), are also determined when calculating $\mathbf{D}(\bar{\mathbf{X}}_k)$. In particular, at the final step of the algorithm one determines m such that the pair $(\tilde{\mathbf{X}}, m)$ satisfies the condition:

$$\boldsymbol{p}(\tilde{\mathbf{X}}, m) \wedge [\mathbf{F}(\tilde{\mathbf{X}}, m) \subseteq \tilde{\mathbf{X}}]. \tag{16}$$

The main question to be considered concerns the relationship between the sequence $\{\bar{\mathbf{X}}_k\}$ and the fixed points of $\mathbf{D}$. The respective *convergence conditions* for the sequence $\{\bar{\mathbf{X}}_k\}$ in the space $\mathbf{2}^{\mathsf{X}}$ have then to be fulfilled, so that the set $\tilde{\mathbf{X}}$ be a good approximation of such a fixed point, in an appropriately chosen topology. However, questions of algorithm convergence are not strictly analyzed in this paper.

The concept makes use of the *contraction mapping* idea. The following, obvious *contraction property*: $\forall X \in \mathsf{X} : \mathbf{D}(X) \subseteq X$, that follows from the definition (9) of $\mathbf{D}$, is crucial there. Thereby, the sequence $\{\bar{\mathbf{X}}_k\}$, (15), is a descending one: $\bar{\mathbf{X}}_{k+1} \subseteq \bar{\mathbf{X}}_k, \quad k = 0, 1, \ldots$.

This fact suggests the following idea for our consideration. We consider the set $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$, being the product of all sets $\bar{\mathbf{X}}_k$:

$$\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0}) = \cap\{\bar{\mathbf{X}}_\mathbf{k} : \ \mathbf{k} = \mathbf{0}, \mathbf{1}, \ldots\} \tag{17}$$

as a *theoretical representation* of $\tilde{\mathbf{X}}$. Hence, instead of considering an unknown set $\tilde{\mathbf{X}}$ and another unknown reachable set X being a fixed point of $\mathbf{D}$, one considers the set $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$, well defined by the algorithm. Instead of considering the convergence of $\{\bar{\mathbf{X}}_k\}$ to one of those unknown sets, we consider its convergence to $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$. But a new question, that of the reachability of $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$, arises.

Let us assume that the following supposition holds:

$$\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0}) = \mathbf{D}(\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})), \tag{18}$$

which states that $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ is a fixed point of mapping $\mathbf{D}$.

However, this crucial supposition is not provable under the assumptions (A), (B), (C) only. Some *continuity* type assumptions are now also needed. We propose the following abstract form of a new assumption, expressed similarly to (C), in terms of the mapping $\mathbf{R}$, see (6).

Suppose that $\bar{\bar{\mathbf{X}}}_\mathbf{0} \in \mathsf{X}$ is such that the sequence $\{\bar{\mathbf{X}}_k\}$, $\bar{\mathbf{X}}_{k+1} = \mathbf{D}(\bar{\mathbf{X}}_k)$, $k = 0, 1, \ldots$ satisfies the following relation:

$$\cap\{\mathbf{R}(\bar{\mathbf{X}}_k) : \bar{\mathbf{X}}_k \in \{\bar{\mathbf{X}}_k\}\} \subseteq \mathbf{R}(\cap\{\bar{\mathbf{X}}_k : \bar{\mathbf{X}}_k \in \{\bar{\mathbf{X}}_k\}\}). \tag{D}$$

This assumption has the form of a certain *continuity* definition. Indeed, the inverse inclusion $\supseteq$ in (D) holds, under (A), (B), (C), simply by logical rules of transposition of quantifiers $\forall$ and $\exists$. Thus, relationship (D) is equivalent to the following:

$$\cap\{\mathbf{R}(\bar{\mathbf{X}}_k) : \bar{\mathbf{X}}_k \in \{\bar{\mathbf{X}}_k\}\} = \mathbf{R}(\cap\{\bar{\mathbf{X}}_k : \bar{\mathbf{X}}_k \in \{\bar{\mathbf{X}}_k\}\}). \tag{19}$$

As it is shown in Terlikowski (2002) for the min-max optimization problem, some standard topological assumptions are sufficient for (D).

The following two theorems present the basic relations existing between the reachable sets and the algorithmically defined set $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$.

THEOREM 4.1 *Suppose that assumptions (A), (B) and (C) are satisfied. Then, for any set* $\bar{\mathbf{X}}_0$ *satisfying assumption (D), the set* $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ *verifies equation (18), i.e.:* $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0}) = \mathbf{D}(\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0}))$.

Proof. Using notation $\bar{\mathbf{X}} = \bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ and $X_k = \bar{\mathbf{X}}_k$, we have evidently: $\mathbf{D}(\bar{\mathbf{X}}) \subseteq \bar{\mathbf{X}}$. To prove that also $\bar{\mathbf{X}} \subseteq \mathbf{D}(\bar{\mathbf{X}})$, it suffices to show that $\bar{\mathbf{X}} \in \mathbf{R}(\bar{\mathbf{X}})$. In fact, $\bar{\mathbf{X}} \subseteq X_k,\ k = 1, 2, \ldots$, thus, by (15): $\bar{\mathbf{X}} \subseteq \mathbf{D}(X_k),\ k = 0, 1, \ldots$. Then, by (5) we obtain: $\bar{\mathbf{X}} \subseteq \max \subseteq \mathbf{R}(X_k)$ and hence, by property (7) we get: $\bar{\mathbf{X}} \in \mathbf{R}(X_k)$ for any $k = 0, 1, \ldots$. Thus $\bar{\mathbf{X}} \in \cap\{\mathbf{R}(X_k) : \ k = 0, 1, \ldots\}$. Therefore, by assumption (D): $\bar{\mathbf{X}} \in \mathbf{R}(\cap\{X_k : \ k = 0, 1, \ldots\}) = \mathbf{R}(\bar{\mathbf{X}})$. This means that $\bar{\mathbf{X}} \subseteq (\bar{\mathbf{X}} \cap \max \subseteq \mathbf{R}(\bar{\mathbf{X}})) = \mathbf{D}(\bar{\mathbf{X}})$. ■

Thus, under assumptions (A)-(D), the *theoretical representation* $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ of the algorithm *result* $\tilde{\mathbf{X}}$ is the sought reachable set.

Of course, an appropriate *topology* in X should be still selected, such that the sequence $\{\bar{\mathbf{X}}_k\}$ *converges* to its product $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$. The set $\tilde{\mathbf{X}} = \bar{\mathbf{X}}_\varpi$ will be then an approximation of $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$, that is: $\tilde{\mathbf{X}} \approx \bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$, in this topology. Finally, $\tilde{\mathbf{X}}$ will provide, with the respective control m, compare (16), an approximate solution to problem (3).

Let us underline that the above algorithm starts with any set $\bar{\mathbf{X}}_0 \in \mathsf{X}$ satisfying (D) and, of course, iteration (15) may lead to the empty set $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ for some $\bar{\mathbf{X}}_0$. We shall now briefly consider the case of the *largest* reachable sets in relation to $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$.

THEOREM 4.2 *Let (A), (B), (C) be satisfied and let* $\bar{\mathbf{X}}_0$ *be any subset of the state space,* $\bar{\mathbf{X}}_0 \in \mathsf{X}$*. Then, for any reachable set* X *included in* $\bar{\mathbf{X}}_0$ *the set* X *is included in* $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$*. If, moreover, (D) holds for* $\bar{\mathbf{X}}_0$*, then* $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ *is the largest reachable set included in* $\bar{\mathbf{X}}_0$.

Proof. Before proving the first asertion, let us show that: $\mathbf{A} \subseteq \mathbf{B} \rightarrow \mathbf{D}(\mathbf{A}) \subseteq \mathbf{D}(\mathbf{B})$ for any $\mathbf{A}, \mathbf{B} \in \mathsf{X}$ (*monotonicity* of $\mathbf{D}$).

Indeed, we have: $\mathbf{D}(\mathbf{A}) = \mathbf{A} \cap \max \subseteq \mathbf{R}(\mathbf{A})$ and $\mathbf{D}(\mathbf{B}) = \mathbf{B} \cap \max \subseteq \mathbf{R}(\mathbf{B})$.

First, we find that $\mathbf{R}(\mathbf{A}) \subseteq \mathbf{R}(\mathbf{B})$ if $\mathbf{A} \subseteq \mathbf{B}$. This follows from the definition (6) of family $\mathbf{R}$, under assumptions (A) and (B). Hence, we have $\max \subseteq \mathbf{R}(\mathbf{A}) \subseteq \max \subseteq \mathbf{R}(\mathbf{B})$, that together with $\mathbf{A} \subseteq \mathbf{B}$, implies $\mathbf{D}(\mathbf{A}) \subseteq \mathbf{D}(\mathbf{B})$.

It will be shown now, using the induction principle, that any $X \subseteq \bar{\mathbf{X}}_0$, satisfying reachability condition (5) is included in each set $\bar{\mathbf{X}}_k$ generated by algorithm (15). Suppose that $X \subseteq \bar{\mathbf{X}}_k$ for some $k \geq 0$. Then, by monotonicity of $\mathbf{D} : \mathbf{D}(X) \subseteq \mathbf{D}(\bar{\mathbf{X}}_k)$. By Theorem 3.2 we have $X = \mathbf{D}(X)$, if X is a reachable set. It is then evident that $X \subseteq \mathbf{D}(\bar{\mathbf{X}}_k)$ and since, by definition of algorithm (15), $\mathbf{D}(\bar{\mathbf{X}}_k) = \bar{\mathbf{X}}_{k+1}$, there is $X \subseteq \bar{\mathbf{X}}_{k+1}$. We have thereby proved that $\forall k \geq 0 : \ X \subseteq \bar{\mathbf{X}}_k$, what means, by definition (17), that $X \subseteq \bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$. Due to Theorem 4.1, $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ is a reachable set, if (D) is satisfied with $\bar{\mathbf{X}}_0$, and if (A), (B), (C) hold. But due to the first part of this theorem, any reachable set $\mathbf{X} \subseteq \bar{\mathbf{X}}_0$ is included in $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$. Since $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ is one of such sets (we have $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0}) \subseteq \bar{\mathbf{X}}_\mathbf{0}$), thus $\bar{\bar{\mathbf{X}}}(\bar{\mathbf{X}}_\mathbf{0})$ is the largest reachable set included in $\bar{\mathbf{X}}_0$. ■

Let us consider finally the set $\bar{\bar{\mathbf{X}}}(\mathsf{X})$ where X denotes the whole state space. This is, under assumptions (A), (B), (C), the largest set resulting from the algorithm (15), by monotonicity of the operator $\mathbf{D}$. But, as it results from Theorem 4.2, it is also the largest of all reachable sets, if X satisfies (D). Moreover, according to Theorem 3.3, it is the largest set at all, as far as solutions of problem (3) are concerned, if such a largest set exists.

5. Conclusions

The main tasks of this paper were twofold:

1. to develop the concept of *reachable set*s for a very general statement of a class of *periodic* control problems defined over the *infinite* time horizon (Section 3),
2. to analyze an algorithmic realization of the above general scheme (Section 4).

The formulation of periodic, infinite control problem, proposed in Section 2 is much more general than those considered in the previous works in this field. This formalism allows us to capture a large class of control problems and, due to its transparency and generality, to find and clearly present the basic mathematical facts concerning the reachable sets concept. This analysis, in particular the formulation of sufficient general assumptions (A)-(D), used within the presented five theorems, is the original contribution of the author. It is shown elsewhere, Terlikowski (2002), that all the sufficient applicability conditions (A), (B), (C), (D) are quite easily fulfilled for an important class of optimal min-max control problems. We obtain then the complete solution of the problem. In particular, an effective algorithm fitting the scheme presented in Section 4 occurs to be entirely applicable. Note that computers available today are suitable for implementation of such an algorithm which operates on sets.

We would like to emphasize, however, that the special optimization problems (e.g. the min-max ones) are not the main subject of the present paper. Our attention is focused in fact on the theoretical aspects of the considered *reachable sets concept.*

Acknowledgments. The author wishes to express his special thanks to Dr. A. Karbowski from the Warsaw University of Technology and to Dr. P. Rowiński from the Institute of Geophysics, Polish Academy of Sciences in Warsaw, for their helpful cooperation and discussions.

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