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The order reduction and robust D-stability analysis of discrete uncertain time-delay systems by time-scale separation

by

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Abstract: In this paper, the robust D-stability problem for discrete uncertain multiple time-delay systems is examined on the basis of their such reduced-order models (the slow and fast subsystems), which are obtained by time-scale separation method. Under the condition that the slow and fast subsystems of the nominal system are both $D(\alpha, r)$-stable, a D-stability criterion for the slow and fast subsystem of the original uncertain system is first derived. A delay-dependent criterion in terms of spectral radius is then proposed to guarantee the robust D-stability of the original uncertain system subject to structured perturbations. Moreover, the criterion for the robust asymptotic stability of the system subject to structured perturbations can be obtained from the proposed robust D-stability criterion. A numerical example is provided to illustrate our main results.

Keywords: D-stability, two-time-scale system, time delay, time-scale separation, structured perturbation.

Notations

S_0^n	:	set $\{0, 1, 2, \cdots, n\}$,
S_1^n	:	set $\{1, 2, \cdots, n\}$,
$\|A\|$	:	$[\,\|a_{ij}\|\,]$, with matrix $A = [a_{ij}]$
$A \le B$	:	$a_{ij} \le b_{ij}$ for all i, j with $A = [a_{ij}]$ and $B = [b_{ij}]$,
$\rho(A)$	:	spectral radius of matrix A,
$\|\|A\|\|$	:	spectral norm of matrix A,
x_s	:	state vector of slow subsystem,
x_f	:	state vector of fast subsystem.

1. Introduction

Multiple time-scale phenomena which exist in many physical systems, complicate the stability analysis of the systems. Fortunately, the singular perturbation method provides us with a powerful tool for the order reduction and separation of time scales; see, for example, Kokotovic et al. (1986), Khalil (1989), Hsiao and Hwang (1996), Sun et al. (1996), Mukaidani and Mizukami (1997), Shi et al. (1998), and the references therein. An important procedure within this method is that the multiple time-scale systems are modeled as singularly perturbed systems. Typical singularly perturbed systems include high-gain control systems, armature-controlled DC motors, tunnel diode circuits, nonlinear time-invariant RLC networks, control system of an airplane, etc. (Kokotovic et al., 1986). A key to the analysis of multiple time-scale systems lies in the construction of the reduced-order systems. Then, the dynamics of the full-order system can be inferred from that of the reduced-order systems.

On the other hand, robust stability of time-delay systems has been considered over the years because time delays are commonly encountered in many engineering systems. For example, feedback systems controlled by computer have time delays due to the execution of many numerical operations. Besides, chemical processes, turbojet engines, long transmission lines, electric networks, hydraulic systems, and pneumatic systems, etc., all have time delays. Their existence frequently causes the instability of the system. Moreover, most of the dynamic systems contain some uncertainties that may arise, for example, from modeling errors or linearization approximation (Phoojaruenchanachai et al., 1998). As these uncertainties are almost inevitable for any modeling of real systems, the problem of maintaining the stability of a system subject to uncertainties has been of consideration to researchers over the years (Wu and Mizukami, 1994; Lien et al., 1998). Furthermore, due to the existence of uncertainties, the poles cannot be placed precisely at a specific location. Therefore, assigning the poles in a specific region instead of a specific location is more practical. Consequently, to achieve the various aspects of system performance, the technique of pole-assignment has been considered during the past years; see, for example, Rachid (1990), Chou (1991), Lee et al. (1992), Su and Shyr (1994), Hsiao et al. (2000), and the references therein.

Recently, there have been many related research papers on the analysis and control design of the two-time-scale systems. Li and Li (1992) proposed a frequency-domain stability criterion for discrete two-time-scale systems in which the time delay and uncertain perturbation have not yet been considered. Trinh and Aldeen (1995) proposed a criterion of asymptotic stability for singularly perturbed systems with multiple time delays, but the delay terms were treated as the perturbations of the nominal system. Hsiao et al. (2000) investigated the D-stabilization problem of discrete singularly perturbed systems in which the factors of time delay and uncertainty have not been taken into consideration. However, the factors of uncertainties and time delays do exist in most of the dynamic systems. Consequently, it is practical to take them into consideration. This is due not only to theoretical interests but also to the relevance of these phenomena in the field of control engineering applications. It is the purpose of this paper to investigate the robust D-stability problem of uncertain discrete multiple time-delay two-time-scale systems.

2. Problem formulation

Consider the following uncertain discrete two-time-scale system with multiple time delays, which is modeled as a singularly perturbed system:

$$x_1(k+1) = \sum_{i=0}^{n}(A_{1i} + \Delta A_{1i})x_1(k-h_i) + \varepsilon\sum_{i=0}^{n}(A_{2i} + \Delta A_{2i})x_2(k-h_i), \tag{1a}$$

$$x_2(k+1) = \sum_{i=0}^{n}(A_{3i} + \Delta A_{3i})x_1(k-h_i) + \varepsilon\sum_{i=0}^{n}(A_{4i} + \Delta A_{4i})x_2(k-h_i), \tag{1b}$$

where $x_1(k) \in \Re^{n_s}$ and $x_2(k) \in \Re^{n_f}$, i.e., the dimensionality of the system (1) is $n_s + n_f$. The small positive parameter ϵ is called singular perturbation parameter, which often occurs naturally due to the presence of small parameters in various physical systems. For instance, it may represent the machine reactance or transient in voltage regulators in a power system model, the time constant of the driver and the actuator in an industrial control system, and it may be due to fast neutrons in a nuclear reactor model. It can be seen that, when the singular perturbation parameter ϵ is sufficiently small, the system (1) will have two clusters of poles: one cluster of poles close to the origin and another close to the poles of subsystem $x_1(k+1) = \sum_{i=0}^{n}(A_{1i} + \Delta A_{1i})x_1(k-h_i)$.

Consequently, the system (1) possesses two-time-scale property, i.e., it possesses slow and fast behavior in its response. Since the method of time-scale separation is used for the analysis of the system (1), the state of system (1) is decomposed into two parts, $x_1(k)$ and $x_2(k)$ corresponding to the slow and fast behavior, respectively. The discrete system (1) is referred to as the C-model in Naidu and Rao (1985). Moreover, A_{1i}, A_{2i}, A_{3i} and A_{4i}, $i \in S_0^n$, are constant matrices with appropriate dimensions, ΔA_{ji}, are perturbation matrices with the structured perturbations $|\Delta A_{ji}| \leq D_{ji}$, where D_{ji} are constant matrices with positive elements and are assumed to be known, $h_0 = 0$, and h_i, $i \in S_1^n$, are pair-wise different positive integers, the discrete instants k are non-negative integers.

REMARK 1 *System (1) is a discrete-time system with time delay characterized via a delay difference equation. According to the system augmentation approach, (1) can be converted into a higher-order delay-free equation $x(k+1) = (A + \Delta A)x(k)$. This is done as follows. Let $h = max_{i=1}^{n} h_i$. Then $x_1(k)$, $x_1(k-1)$,*

...., $x_1(k-h)$, $x_2(k)$, $x_2(k-1)$, ..., $x_2(k-h)$, may be selected as the states of the delay-free system.

However, note that there are altogether $(h+1)(n_s+n_f)$ state variables in the system. Consequently, the system matrix A is of dimension $(h+1)(n_s+n_f)\times(h+1)(n_s+n_f)$. Take, for example, $h=5$, $n_s=3$, and $n_f=3$, then A is of dimension 36×36, which is a big matrix. However, in system (1), only 3×3 matrices are involved. Consequently, stability robustness analysis becomes quite troublesome if we use the converted delay-free system.

The goal of this paper is to find a D-stability criterion for the system (1) by using the time-scale separation method. Throughout this paper, we use the following definition for the D-stability property of discrete multiple time-delay systems:

DEFINITION 1 *(Wang and Wang, 1995): Let α and r be real numbers, such that $r>0$ and $|\alpha|+r<1$. A discrete linear time-invariant time-delay system $x(k+1)=\sum_{i=0}^{n}(A_i+\Delta A_i)x(k-h_i)$ is said to be robust $D(\alpha,r)$-stable if, for all values of ΔA_i from a given region, all poles of the system are within the disk $D(\alpha,r)$ centered at $(\alpha,0)$ with radius r. In other words, the solutions of its characteristic equation satisfy $|z-\alpha|<r$.*

DEFINITION 2 *The multiple time-delay system $x(k+1)=\sum_{i=0}^{n}\Psi_i x(k-i)$ is $D(\alpha,r)$-stable if and only if $|\det[zI-\Psi(z)]|>0$, $|z-\alpha|\geq r$, where $\Psi(z)=\sum_{i=0}^{n}\Psi_i z^{-i}$.*

Before the main results are derived, the nominal system of the uncertain system (1) is first introduced as follows.

$$x_1(k+1)=\sum_{i=0}^{n}A_{1i}x_1(k-h_i)+\varepsilon\sum_{i=0}^{n}A_{2i}x_2(k-h_i), \tag{2a}$$

$$x_2(k+1)=\sum_{i=0}^{n}A_{3i}x_1(k-h_i)+\varepsilon\sum_{i=0}^{n}A_{4i}x_2(k-h_i). \tag{2b}$$

According to the time-scale separation method in Mahmoud (1982) and Saksena et al. (1984), the slow subsystem and fast subsystem of (2) can be derived as follows.

2.1. The slow subsystem of the nominal system

The two-time-scale property of a singularly perturbed system is characterized by the presence of slow and fast modes. The dynamics of the fast mode of the system dynamics, which is active only in a short period (fast transient period), will appear at first. The dynamics of system states $x_1(k)$ and $x_2(k)$ in the fast transient period are called the fast components of $x_1(k)$ and $x_2(k)$, respectively.

The stability problem for the fast mode of the system (1), will be discussed in the following sections. If the fast mode is stable, the fast component of $x_2(k)$ is significant only during the fast transient period. After the period, the system dynamics reaches its slow mode and the fast component of $x_2(k)$ is negligible. Consequently, the variables $x_1(k)$ and $x_2(k)$ approach their slow components, i.e., the quasi-steady states $x_s(k)$ and $\bar{x}_2(k)$, respectively. The concept of quasi-steady states is introduced by Kokotovic et al. (1986) to indicate the rapid convergence of the systems states to their slow components after a short fast transient period.

Consequently, in order to obtain the nominal slow subsystem, we replace $x_2(k-h_i)$ and $x_1(k)$ by $\bar{x}_2(k)$ and $x_s(k)$, respectively. Then the nominal discrete singularly perturbed system (2) can be reduced to the following slow subsystem:

$$x_s(k+1) = \sum_{i=0}^{n} A_{1i} x_s(k-h_i) + \varepsilon \sum_{i=0}^{n} A_{2i} \bar{x}_2(k), \tag{3a}$$

$$\bar{x}_2(k+1) = \sum_{i=0}^{n} A_{3i} x_s(k-h_i) + \varepsilon \sum_{i=0}^{n} A_{4i} \bar{x}_2(k). \tag{3b}$$

From (3), we obtain the slow subsystem as

$$x_s(k+1) = \sum_{i=0}^{n} M_i \cdot x_s(k-h_i), \tag{4a}$$

where

$$M_i = A_{1i} + \varepsilon \left(\sum_{j=0}^{n} A_{2j} \right) \cdot \left(I - \varepsilon \sum_{j=0}^{n} A_{4j} \right)^{-1} A_{3i}. \tag{4b}$$

Let

$$B_j(z) = \sum_{i=0}^{n} A_{ji} z^{-h}, \quad j \in S_1^4. \tag{5}$$

Taking the z-transform of the slow subsystem (4), we have

$$\begin{aligned} X_s(z) &= \left(zI - \sum_{i=0}^{n} M_i z^{-h_i} \right)^{-1} z \cdot x_s(0) = [zI - K(z)]^{-1} z \cdot x_s(0) \\ &= G_{ns}(z) \cdot z \cdot x_s(0), \end{aligned} \tag{6a}$$

where

$$K(z) = B_1(z) + \varepsilon \left(\sum_{i=0}^{n} A_{2i} \right) \cdot \left(I - \epsilon \sum_{i=0}^{n} A_{4i} \right)^{-1} B_3(z), \tag{6b}$$

$$G_{ns}(z) = [zI - K(z)]^{-1}. \tag{6c}$$

If the slow subsystem (4) is $D(\alpha, r)$-stable, then

$$|\det [G_{ns}^{-1}(z)]| > 0, \quad |z - \alpha| \geq r.$$

2.2. The fast subsystem of the nominal system

During the fast transient period, the quasi-steady states $x_s(k)$ and $\bar{x}_2(k)$ are assumed to be constant. Thus, we may approximate $x_s(k - h_i)$ and $\bar{x}_2(k + 1)$ by $x_s(k)$ and $\bar{x}_2(k)$, respectively. Moreover, by view of the nominal system (2) and the slow subsystem (4), it can be seen that $x_1(k)$ approaches $x_s(k)$ in (4) as ϵ approaches zero. Therefore, if ϵ is sufficiently small, it is reasonable to approximate $x_1(k)$ by $x_s(k)$, i.e., $x_1(k) \approx x_s(k)$. According to the above discussion, we are in position to derive the fast subsystem of the nominal system (2). Denote $x_f(k)$ as the fast component of $x_2(k)$. We have $x_f(k) = x_2(k) - \bar{x}_2(k)$ and

$$x_f(k + 1) = x_2(k + 1) - \bar{x}_2(k + 1) = x_2(k + 1) - \bar{x}_2(k). \tag{7}$$

According to (2b) and (3b), we obtain the fast subsystem from (7) as

$$x_f(k + 1) = \varepsilon \sum_{i=0}^{n} A_{4i} x_f(k - h_i). \tag{8}$$

Taking the z-transform of the fast subsystem (8), we get

$$X_f(z) = \left(zI - \varepsilon \sum_{i=0}^{n} A_{4i} z^{-h} \right)^{-1} z \cdot x_f(0) = G_{nf}(z) \cdot z \cdot x_f(0), \tag{9a}$$

where

$$G_{nf}(z) = [\, zI - \varepsilon B_4(z) \,]^{-1}. \tag{9b}$$

If the fast subsystem (8) is $D(\alpha, r)$-stable, then

$$|\det[G_{nf}^{-1}(z)]| > 0 \quad |z - \alpha| \geq r.$$

In the following derivation, we assume that the nominal slow subsystem (4) and fast subsystem (8) are both $D(\alpha, r)$-stable.

3. The slow and fast subsystem of the uncertain system

In this section, the stability of the slow and fast subsystems of the uncertain system is examined. Some useful lemmas for further derivations are introduced as follows.

LEMMA 1 *(Hsiao et al., 2001) For any matrix* $A \in R^{m\times m}$*, if* $\rho(A) < 1$*, then* $|\det(I \pm A)| > 0$ *.*

LEMMA 2 *(Su and Shyr, 1994)*
The zero state of the system $x(k+1) = \sum_{i=0}^{n} M_i x(k-i)$ *is asymptotically stable if and only if* $|\det[zI - M(z)]| > 0$, $|z| \geq 1$*, where* $M(z) = \sum_{i=0}^{n} M_i z^{-i}$*.*

LEMMA 3 *(Chou, 1991) For any* $m \times m$ *matrices* A*,* B*, and* C*, if* $|B| < C$*, then*

(a) $\rho(AB) \leq \rho(\,|A| \cdot |B|\,) \leq \rho(\,|A| \cdot C\,)$

(b) $\rho(A+B) \leq \rho(\,|A+B|\,) \leq \rho(\,|A| + |B|\,) \leq \rho(|A| + C)$*.*

LEMMA 4 *(John, 1967) If* $f(z)$ *is analytic in a bounded domain* Ψ *and continuous in the closure of* Ψ*, then* $|f(z)|$ *takes its maximum on the boundary of* Ψ*.*

3.1. The slow subsystem of the uncertain system

Before investigating the D-stability of the uncertain slow subsystem, we introduce a lemma.

LEMMA 5 *If*

$$\left\| \varepsilon \left(\sum_{i=0}^{n} A_{4i} \right) \cdot |\eta| \right\| < 1 \tag{10}$$

holds, then we have

$$\left| \left(I - \varepsilon \sum_{i=0}^{n} A_{4i} - \epsilon \sum_{i=0}^{n} \Delta A_{4i} \right)^{-1} \right| \leq \xi \tag{11a}$$

where

$$\xi = |\eta| \cdot \left[I - \left(\varepsilon \sum_{i=0}^{n} D_{4i} \right) \cdot |\eta| \right]^{-1} \tag{11b}$$

with

$$\eta = \left(I - \varepsilon \sum_{i=0}^{n} D_{4i} \right)^{-1} \tag{11c}$$

Proof. From (11c), we have

$$\left(I - \varepsilon \sum_{i=0}^{n} A_{4i} - \varepsilon \sum_{i=0}^{n} \Delta A_{4i} \right)^{-1} = \eta \cdot \left[I - \varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta \right]^{-1}. \tag{12}$$

Since

$$\left\| \varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta \right\| \leq \left\| \varepsilon \left(\sum_{i=0}^{n} D_{4i} \right) \cdot |\eta| \right\| < 1, \tag{13}$$

we obtain

$$\left[I - \varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta \right]^{-1} = I + \varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta + \left[\varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta \right]^{2} + \left[\varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta \right]^{3} + \cdots$$

Hence, we have

$$\left| \left[I - \varepsilon \left(\sum_{i=0}^{n} \Delta A_{4i} \right) \cdot \eta \right]^{-1} \right| \leq \left[I - \varepsilon \left(\sum_{i=0}^{n} D_{4i} \right) \cdot |\eta| \right]^{-1}. \tag{14}$$

Thus, the inequality (11a) holds in view of (12) and (14). This completes the proof. ■

Let

$$B_{j\Delta}(z) = \sum_{i=0}^{n} (A_{ji} + \Delta A_{ji}) z^{-h_i}, \ j \in S_1^4. \tag{15}$$

Similarly to the derivation of (6a) and according to (1), we get

$$X_s(z) = G_{\Delta s}(z) \cdot z \cdot x_s(0) \tag{16a}$$

where

$$\begin{aligned} G_{\Delta s}(z) \quad = \quad & \Big\{ zI - \Big[B_{1\Delta}(z) + \varepsilon \Big[\sum_{i=0}^{n} (A_{2i} + \Delta A_{2i}) \Big] \\ \cdot \quad & \Big[I - \varepsilon \sum_{i=0}^{n} (A_{4i} + \Delta A_{4i}) \Big]^{-1} B_{3\Delta}(z) \Big] \Big\}^{-1}. \end{aligned} \tag{16b}$$

From Definition 2, the uncertain slow subsystem (16) is $D(\alpha, r)$-stable if the following inequality holds:

$$|\det[\, G_{\Delta s}^{-1}(z) \,]| > 0, \quad |z - \alpha| \geq r. \tag{17}$$

LEMMA 6 *If the nominal slow subsystem is $D(\alpha, r)$-stable, i.e. $|\det[G_{ns}^{-1}(z)]| > 0$, $|z - \alpha| \geq r$ then the uncertain slow subsystem (16) is $D(\alpha, r)$-stable if the following inequality holds:*

$$\rho\left[\,\Xi(\beta)\cdot\left|\,G_{ns}(\alpha+r\beta^{-1})\,\right|\right]<1,\quad |\beta|\le 1. \tag{18a}$$

where

$$\begin{aligned}\Xi(\beta) \quad = \quad & \left[\varepsilon\left(\left|\sum_{i=0}^{n}A_{2i}\right|+\sum_{i=0}^{n}D_{2i}\right)\cdot\xi\cdot\left(\varepsilon\sum_{i=0}^{n}D_{4i}\right)+\varepsilon\left(\sum_{i=0}^{n}D_{2i}\right)\right]\\ & \cdot\left\{\left|\eta\cdot B_3(\alpha+r\beta^{-1})\right|+|\eta|\cdot\sum_{i=0}^{n}\left[D_{3i}\cdot\left|(\alpha+r\beta^{-1})^{-h_i}\right|\right]\right\}\\ & +\sum_{i=0}^{n}\left[D_{1i}\cdot\left|(\alpha+r\beta^{-1})^{-h_i}\right|\right]+\varepsilon\cdot\left|\left(\sum_{i=0}^{n}A_{2i}\right)\cdot\eta\right|\\ & \cdot\sum_{i=0}^{n}\left[D_{3i}\cdot\left|(\alpha+r\beta^{-1})^{-h_i}\right|\right].\end{aligned} \tag{18b}$$

Proof. Since

$$\begin{aligned}\left[I-\varepsilon\sum_{i=0}^{n}(A_{4i}+\Delta A_{4i})\right]^{-1} \quad = \quad & \eta+\left(I-\varepsilon\sum_{i=0}^{n}A_{4i}-\varepsilon\sum_{i=0}^{n}\Delta A_{4i}\right)^{-1}\\ & \cdot\quad\left(\varepsilon\sum_{i=0}^{n}\Delta A_{4i}\right)\cdot\eta,\end{aligned}$$

according to (6c), (16b) can be rewritten as

$$G_{\Delta s}(z)=[zI-K(z)-\Delta K(z)]^{-1}=[G_{ns}^{-1}(z)-\Delta K(z)]^{-1} \tag{19a}$$

where

$$\begin{aligned}\Delta K(z) \quad = \quad & \sum_{i=0}^{n}\Delta A_{1i}z^{-h_i}+\varepsilon\left(\sum_{i=0}^{n}A_{2i}\right)\cdot\eta\cdot\left(\sum_{i=0}^{n}\Delta A_{3i}z^{-h_i}\right)\\ + \quad & \left\{\varepsilon\left[\sum_{i=0}^{n}(A_{2i}+\Delta A_{2i})\right]\cdot\left[I-\varepsilon\sum_{i=0}^{n}A_{4i}-\varepsilon\sum_{i=0}^{n}\Delta A_{4i}\right]^{-1}\left(\varepsilon\sum_{i=0}^{n}\Delta A_{4i}\right)\right.\\ + \quad & \left.\varepsilon\sum_{i=0}^{n}\Delta A_{2i}\right\}\cdot\eta\cdot\left[B_3(z)+\sum_{i=0}^{n}\Delta A_{3i}z^{-h_i}\right].\end{aligned} \tag{19b}$$

Consequently, we have

$$|\,\det[G_{\Delta s}^{-1}(z)]\,|=|\,\det[I-\Delta K(z)\cdot G_{ns}(z)]\,|\cdot|\,\det[G_{ns}^{-1}(z)]|.$$

Since $|\,\det[G_{\Delta s}^{-1}(z)]|>0$, $|z-\alpha|\ge r$, the examination of (17) is equivalent to the verification of the following inequality:

$$|\det[I - \Delta K(z) \cdot G_{ns}(z)]| > 0, \quad |z - \alpha| \geq r. \tag{20}$$

Let $\beta^{-1} = (z - \alpha)/r$, i.e., $z = \alpha + r\beta^{-1}$. Then, (20) can be rewritten as

$$|\det[I - \Delta K(\alpha + r\beta^{-1}) \cdot G_{ns}(\alpha + r\beta^{-1})]| > 0, \quad |\beta| \geq 1. \tag{21}$$

Due to (19b) and Lemma 5, $\Delta(K\alpha + r\beta^{-1})$ satisfies

$$|\Delta K(\alpha + r\beta^{-1})| \leq \Xi(\beta), \quad |\beta| \leq 1, \tag{22}$$

By Lemma 3, this yields

$$\rho[\Delta K(\alpha + r\beta^{-1}) \cdot G_{ns}(\alpha + r\beta^{-1})] \leq \rho[\Xi(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})|]$$

Then, according to Lemma 1, (21) holds if the inequality (18a) is satisfied. Hence, the uncertain slow subsystem (16) is $D(\alpha, r)$-stable in view of (17) and Definition 2. This completes the proof. ■

3.2. The fast subsystem of the uncertain system

Similarly to the derivation of (9) and according to (1), we have

$$X_f(z) = G_{\Delta f}(z) \cdot z \cdot x_f(0), \tag{23a}$$

where

$$G_{\Delta f}(z) = [zI - \varepsilon B_{4\Delta}(z)]^{-1}. \tag{23b}$$

Therefore, the uncertain fast subsystem (23) is $D(\alpha, r)$-stable, if the following inequality holds:

$$|\det[G_{\Delta f}^{-1}]| > 0, \ |z - \alpha| \geq r. \tag{24}$$

LEMMA 7 *If the nominal fast subsystem is $D(\alpha, r)$-stable, i.e., $|\det[G_{\Delta f}^{-1}]| > 0$, $|z - \alpha| \geq r$, then the uncertain fast subsystem (23) is $D(\alpha, r)$-stable if the following inequality holds:*

$$\rho\left\{\left[\varepsilon \sum_{i=0}^{n} D_{4i} \cdot \left|(\alpha + r\beta^{-1})^{-h_i}\right|\right] \cdot \left|G_{nf}(\alpha + r\beta^{-1})\right|\right\} < 1, \quad |\beta| \leq 1. \tag{25}$$

Proof. Making use of (9b) and (15), we obtain that $G_{\Delta f}^{-1}(z)$ satisfies

$$|\det[G_{\Delta f}^{-1}(z)]| = \left|\det\left[I - \left(\varepsilon \sum_{i=0}^{n} \Delta A_{4i} z^{-h_i}\right) \cdot G_{nf}(z)\right]\right| \cdot |\det[G_{nf}^{-1}(z)]|.$$

Since $|\det[G_{\Delta f}^{-1}]| > 0,\ |z-\alpha| \geq r$, the following inequality:

$$\left| \det \left[I - \left(\varepsilon \sum_{i=0}^{n} \Delta A_{4i} z^{-h_i} \right) \cdot G_{nf}(z) \right] \right| > 0, \quad |z-\alpha| \geq r,$$

i.e.

$$\left| \det \left\{ I - \varepsilon \cdot \left[\sum_{i=0}^{n} \Delta A_{4i} \cdot (\alpha + r\beta^{-1})^{-h_i} \right] \cdot G_{nf}(\alpha + r\beta^{-1}) \right\} \right| > 0, \quad |\beta| \leq 1, \tag{26}$$

implies (24). According to Lemma 3, it can be easily shown that

$$\rho \left\{ \left[\varepsilon \sum_{i=0}^{n} \Delta A_{4i} \cdot (\alpha + r\beta^{-1})^{-h_i} \right] \cdot G_{nf}(\alpha + r\beta^{-1}) \right\}$$

$$\leq \rho \left\{ \left[\varepsilon \sum_{i=0}^{n} \Delta D_{4i} \cdot |(\alpha + r\beta^{-1})^{-h_i}| \right] \cdot |G_{nf}(\alpha + r\beta^{-1})| \right\}.$$

By Lemma 1, if (25) holds, then the inequality (26) is satisfied and hence the uncertain fast subsystem (23) is $D(\alpha, r)$-stable in view of (24) and Definition 2. This completes the proof. ■

4. Stability criteria of the original system

In this section, we will introduce the main result for the $D(\alpha, r)$-stability of the original uncertain system (1). The D-stability property of the original system can be inferred from that of its reduced-order model, i.e. the slow and fast subsystems. Taking the z-transform of the uncertain discrete singularly perturbed system (1) and according to (15), we have

$$X_2(z) = G_{\Delta f}(z) B_{3\Delta}(z) X_1(z) + G_{\Delta f}(z) x_2(0) \tag{27}$$

and

$$X_1(z) = z\Phi(z) x_1(0) + \varepsilon z \Phi(z) B_{2\Delta}(z) G_{\Delta f}(z) x_2(0), \tag{28a}$$

where

$$\Phi(z) = \{ zI - B_{1\Delta}(z) - \varepsilon B_{2\Delta}(z) G_{\Delta f}(z) B_{3\Delta}(z) \}^{-1}. \tag{28b}$$

From (27) and (28), it is obvious that if $G_{\Delta f}(z)$ and $\Phi(z)$ are both $D(\alpha, r)$-stable (i.e., all poles of $G_{\Delta f}(z)$ and $\Phi(z)$ are within the disk $D(\alpha, r)$), then the uncertain discrete singularly perturbed system (1) is $D(\alpha, r)$-stable.

THEOREM 1 *Suppose that the nominal slow subsystem (4) and fast subsystem (8) are $D(\alpha, r)$-stable, and the following inequalities hold for all $|\beta| \leq 1$:*

$$\left\| \Xi(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})| \right\| < 1, \tag{29}$$

$$\left\| \left[\varepsilon \sum_{i=0}^{n} \Delta D_{4i} \cdot |(\alpha + r\beta^{-1})^{-h_i}| \right] \cdot |G_{nf}(\alpha + r\beta^{-1})| \right\| < 1. \tag{30}$$

Then the uncertain discrete singularly perturbed system (1) is $D(\alpha, r)$-stable if

$$\rho\{\Gamma(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})| \cdot [I - \Xi(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})|]^{-1}\} < 1, \quad |\beta| \le 1, \tag{31a}$$

where

$$\begin{aligned} \Gamma(\beta) \quad = \quad & \left\{ \varepsilon \left(\left| \sum_{i=0}^{n} A_{2i} \right| + \sum_{i=0}^{n} D_{2i} \right) \xi + \varepsilon \left(|B_2(\alpha + r\beta^{-1})| + \sum_{i=0}^{n} D_{2i} \right) \right. \\ & \cdot \ |G_{nf}(\alpha + r\beta^{-1})| \\ & \cdot \ \left. \left[I - \varepsilon \left(\sum_{i=0}^{n} D_{4i} \cdot |(\alpha + r\beta^{-1})^{-h_i}| \right) \cdot |G_{nf}(\alpha + r\beta^{-1})| \right]^{-1} \right\} \\ & \cdot \ \left[|B_3(\alpha + r\beta^{-1})| + \sum_{i=0}^{n} D_{3i} \cdot |(\alpha + r\beta^{-1})^{-h_i}| \right]. \end{aligned} \tag{31b}$$

Proof. By virtue of (15), (16b), and (28b), $\Phi(z)$ is given by

$$\Phi(z) = [G_{\Delta s}^{-1}(z) - \Delta\Phi(z)]^{-1} \tag{32a}$$

where

$$\begin{aligned} \Delta\Phi(z) \quad = \quad & -\varepsilon \left[\sum_{i=0}^{n} (A_{2i} + \Delta A_{2i}) \right] \cdot \left[I - \varepsilon \sum_{i=0}^{n} (A_{4i} + \Delta A_{4i}) \right]^{-1} B_{3\Delta}(z) \\ + \quad & \varepsilon \sum_{i=0}^{n} (A_{2i} + \Delta A_{2i}) \cdot z^{-h_i} \cdot G_{\Delta f}(z) B_{3\Delta}(z). \end{aligned} \tag{32b}$$

From Definition 2, $\Phi(z)$ is $D(\alpha, r)$-stable, if the following inequality holds:

$$|\det[G_{\Delta s}^{-1}(z) - \Delta\Phi(z)]| > 0, \quad |z - \alpha| \ge r. \tag{33}$$

Moreover, we have

$$|\det[G_{\Delta s}^{-1}(z) - \Delta\Phi(z)]| = |\det[I - \Delta\Phi(z)\cdot G_{\Delta s}(z)]| \cdot \det[G_{\Delta s}^{-1}(z)]|.$$

It is obvious that if (29) and (30) hold, implying (18) and (25), then $G_{\Delta s}(z)$ and $G_{\Delta f}(z)$ are both $D(\alpha, r)$-stable. Hence we have $|\det[G_{\Delta s}^{-1}(z)]| > 0$, $|z-\alpha| \geq r$. Consequently, if $|\det[I - \Delta\Phi(z)\cdot G_{\Delta s}(z)]| > 0$, $|z-\alpha| \geq r$, i.e.

$$|\det[I - \Delta\Phi(\alpha + r\beta^{-1})\cdot G_{\Delta s}(\alpha + r\beta^{-1})]| > 0, \quad |\beta| \leq 1, \tag{34}$$

then the inequality (33) holds. Similarly as in the derivation of (11a), if (29) holds, then we have

$$|G_{\Delta s}(\alpha + r\beta^{-1})| \leq |G_{ns}(\alpha + r\beta^{-1})| \cdot \left[I - \Xi(\beta)\cdot |G_{ns}(\alpha + r\beta^{-1})|\right]^{-1} \tag{35}$$

for all $|\beta| \leq 1$, in view of (19) and (22). Similarly, if (30) is satisfied, then we get

$$|G_{\Delta f}(\alpha + r\beta^{-1})| \leq |G_{nf}(\alpha + r\beta^{-1})| \cdot \left\{I - \varepsilon\left[\sum_{i=0}^{n} D_{4i}|(\alpha + r\beta^{-1})^{-h_i}|\right]\cdot |G_{nf}(\alpha + r\beta^{-1})|\right\}^{-1}. \tag{36}$$

Due to (11a), (32b), and (36), it can be shown that

$$|\Delta\Phi(\alpha + r\beta^{-1})| \leq \Gamma(\beta), \quad |\beta| \leq 1. \tag{37}$$

As a consequence of (35) and (37), we have

$$\rho[\Delta\Phi(\alpha + r\beta^{-1})\cdot G_{\Delta s}(\alpha + r\beta^{-1})] \leq \rho\left\{\Gamma(\beta)\cdot |G_{ns}(\alpha + r\beta^{-1})| \cdot [\, I - \Xi(\beta)\cdot |G_{ns}(\alpha + r\beta^{-1})|\,]^{-1}\right\}.$$

According to Lemma 1, if (31a) holds, then (34) is obtained. Thus we have that $\Phi(z)$ is $D(\alpha, r)$-stable in view of (33). Hence, the uncertain discrete singularly perturbed system (1) is $D(\alpha, r)$-stable. This completes the proof. ■

REMARK 2 *It is obvious that* $|\eta| \leq |G_{ns}(z)|$ *in view of (9b) and (11c). Thus, if the inequality (30) holds, then (10) and hence (11a) hold. In other words, inequality (30) implies Lemma 5.*

COROLLARY 1 *Suppose that the nominal slow subsystem (4) and fast subsystem (8) are asymptotically stable and the following inequalities hold for all* $|\beta| \leq 1$*:*

$$\left\| \tilde{\Xi}(\beta)\cdot |G_{ns}(\beta^{-1}|\right\| < 1, \quad \left\| \left(\varepsilon \sum_{i=0}^{n} D_{4i}\cdot |\beta^{h_i}|\right)\cdot |G_{nf}(\beta^{-1})|\right\| < 1,$$

where

$$\tilde{\Xi}(\beta) = \left[\varepsilon\left(\left|\sum_{i=0}^{n} A_{2i}\right| + \sum_{i=0}^{n} D_{2i}\right) \cdot \xi \cdot \left(\varepsilon \sum_{i=0}^{n} D_{4i}\right) + \varepsilon\left(\sum_{i=0}^{n} D_{2i}\right)\right]$$

$$\cdot \left\{\left|\eta \cdot B_3\left(\beta^{-1}\right)\right| + |\eta| \cdot \sum_{i=0}^{n}(D_{3i} \cdot |\beta^{h_i}|)\right\} + \sum_{i=0}^{n}(D_{1i} \cdot |\beta^{h_i}|)$$

$$+ \varepsilon \cdot \left|\left(\sum_{i=0}^{n} A_{2i}\right) \cdot \eta\right| \cdot \sum_{i=0}^{n}(D_{3i} \cdot |\beta^{h_i}|)$$

Then the uncertain discrete singularly perturbed system (1) is asymptotically stable if

$$\rho\left\{\tilde{\Gamma}(\beta) \cdot \left|G_{ns}(\beta^{-1})\right| \cdot \left[I - \tilde{\Xi}(\beta) \cdot \left|G_{ns}(\beta^{-1})\right|\right]^{-1}\right\} < 1, \quad |\beta| \le 1,$$

where

$$\begin{aligned}\tilde{\Gamma}(\beta) \quad = \quad & \left\{\varepsilon\left(\left|\sum_{i=0}^{n} A_{2i}\right| + \sum_{i=0}^{n} D_{2i}\right)\xi + \varepsilon\left(\left|B_2(\beta^{-1})\right| + \sum_{i=0}^{n} D_{2i}\right) \cdot |G_{nf}(\beta^{-1})|\right. \\ & \left. \cdot \left[I - \varepsilon\left(\sum_{i=0}^{n} D_{4i} \cdot |\beta^{h_i}|\right) \cdot |G_{nf}(\beta^{-1})|\right]^{-1}\right\} \\ & \cdot \left[|B_3(\beta^{-1})| + \sum_{i=0}^{n}(D_{3i} \cdot |\beta^{h_i}|)\right].\end{aligned}$$

Proof. This result follows immediately from Theorem 1 by setting $\alpha = 0$, $r = 1$. ■

COROLLARY 2 *Suppose that the nominal slow subsystem (4) and the nominal fast subsystem (8) are $D(\alpha, r)$-stable. If (29) and (30) hold for all $\beta = e^{j\theta}$, $\theta \in [0, 2\pi]$, then the uncertain discrete singularly perturbed system (1) is $D(\alpha, r)$-stable if (31a) holds for $\beta = e^{j\theta}$, $\theta \in [0, 2\pi]$.*

Proof. If $|\beta| \le 1$, then we get

$$|\alpha + r\beta^{-1}| \ \ge\ |r\beta^{-1}| - |\alpha| = r|\beta^{-1}| - |\alpha| \ge r - |\alpha| > 0.$$

Hence, the multiple poles of $(\alpha + r\beta^{-1})^{-h_i}$, i.e., $\beta = -r/\alpha$, are outside the unit disk $|\beta| \le 1$. Equivalently, $(\alpha + r\beta^{-1})^{-h_i}$ is analytic in $|\beta| \le 1$. Moreover, we obtain that $G_{ns}(\alpha + r\beta^{-1})$, $[I - \Xi(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})|]^{-1}$, and $\Gamma(\beta)$ are analytic and continuous for all $|\beta| \le 1$, respectively. By Lemma 4, the evaluation of inequalities in Theorem 1 for all $|\beta| \le 1$ is equivalent to their evaluation on the boundary of $|\beta| \le 1$. This completes the proof. ■

5. Numerical example

Consider the uncertain discrete system (1) with

$$A_{10} = \begin{bmatrix} -0.1 & 0.2 \\ -0.1 & -0.3 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.1 & -0.05 \\ -0.2 & -0.1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.03 & -0.1 \\ -0.1 & 0.03 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 0.2 & -0.3 \\ -1.3 & -0.6 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & -0.5 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0.1 & -0.3 \\ 0.3 & -0.2 \end{bmatrix},$$

$$A_{30} = \begin{bmatrix} -0.3 & 3.4 \\ -2.9 & 1.2 \end{bmatrix}, \quad A_{31} = \begin{bmatrix} 0.1 & 1.8 \\ -2.3 & 2.1 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} -0.3 & -0.2 \\ -1.6 & 0.8 \end{bmatrix},$$

$$A_{40} = \begin{bmatrix} -2.2 & -1.1 \\ 2.7 & 3.6 \end{bmatrix}, \quad A_{41} = \begin{bmatrix} 2.8 & 3.8 \\ -2.1 & 1.6 \end{bmatrix}, \quad A_{42} = \begin{bmatrix} -2.8 & -3.1 \\ 3.2 & -3.1 \end{bmatrix},$$

$h_0 = 0$, $h_1 = 1$, $h_2 = 2$, $\varepsilon = 0.01$. Moreover, the upper bounds of the structured perturbations are given in the following:

$$D_{10} = \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, D_{11} = \begin{bmatrix} 0.01 & 0.002 \\ 0.01 & 0.01 \end{bmatrix}, D_{12} = \begin{bmatrix} 0.003 & 0.01 \\ 0.002 & 0.03 \end{bmatrix},$$

$$D_{20} = \begin{bmatrix} 0.03 & 0.01 \\ 0.01 & 0.02 \end{bmatrix}, D_{21} = \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.02 \end{bmatrix}, D_{22} = \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.03 \end{bmatrix},$$

$$D_{30} = \begin{bmatrix} 0.01 & 0.03 \\ 0.01 & 0.01 \end{bmatrix}, D_{31} = \begin{bmatrix} 0.01 & 0.03 \\ 0.01 & 0.02 \end{bmatrix}, D_{32} = \begin{bmatrix} 0.02 & 0.03 \\ 0.01 & 0.01 \end{bmatrix},$$

$$D_{40} = \begin{bmatrix} 0.02 & 0.01 \\ 0.01 & 0.03 \end{bmatrix}, D_{41} = \begin{bmatrix} 0.03 & 0.01 \\ 0.01 & 0.02 \end{bmatrix}, D_{42} = \begin{bmatrix} 0.01 & 0.01 \\ 0.02 & 0.02 \end{bmatrix}.$$

In what follows, the $D(\alpha, r)$-stability problem with $\alpha = 0.1$ and $r = 0.7$ is investigated according to Theorem 1. Since $\max_{|\beta|\le 1} \rho[(\alpha+r\beta^{-1})^{-1}K(\alpha+r\beta^{-1})] = 0.6418$ and $\max_{|\beta|\le 1} \rho[(\alpha + r\beta^{-1})^{-1}\varepsilon B_4(\alpha + r\beta^{-1})] = 0.3185$, the nominal slow subsystem (4) and fast subsystem (8) are $D(\alpha, r)$-stable in view of (6c), (9b), and Lemma 1. Moreover, since

$$\max_{|\beta|\le 1} \| \Xi(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})| \,\| = 0.3030$$

$$\max_{|\beta|\le 1} \left\| \left[\varepsilon \sum_{i=0}^{n} D_{4i} \cdot |(\alpha + r\beta^{-1})^{-h_i}| \right] \cdot |G_{ns}(\alpha + r\beta^{-1})| \right\| = 0.0044,$$

and

$$\max_{|\beta|\le 1} \rho\{\Gamma(\beta) \cdot |G_{ns}(\alpha + r\beta)| \cdot [\, I - \Xi(\beta) \cdot |G_{ns}(\alpha + r\beta^{-1})|\,]^{-1}\} = 0.8982,$$

the inequalities (29), (30), and (31a) are satisfied. By Theorem 1, the uncertain discrete system under consideration is $D(0.1, 0.7)$-stable. For convenience of simulation, let

$$\Delta A_{10} = \begin{bmatrix} -0.02 & 0.01 \\ 0.02 & 0.01 \end{bmatrix}, \quad \Delta A_{11} = \begin{bmatrix} -0.01 & 0.002 \\ -0.01 & 0.01 \end{bmatrix}, \ \Delta A_{12} = \begin{bmatrix} -0.003 & -0.01 \\ 0.002 & -0.002 \end{bmatrix},$$

$$\Delta A_{20} = \begin{bmatrix} -0.03 & -0.01 \\ -0.01 & -0.02 \end{bmatrix}, \Delta A_{21} = \begin{bmatrix} -0.02 & 0.01 \\ 0.02 & -0.02 \end{bmatrix}, \Delta A_{22} = \begin{bmatrix} 0.02 & -0.01 \\ 0.02 & -0.03 \end{bmatrix},$$

$$\Delta A_{30} = \begin{bmatrix} -0.01 & -0.03 \\ -0.01 & -0.01 \end{bmatrix}, \Delta A_{31} = \begin{bmatrix} -0.01 & 0.03 \\ 0.01 & -0.02 \end{bmatrix}, \Delta A_{32} = \begin{bmatrix} -0.02 & 0.03 \\ -0.01 & -0.01 \end{bmatrix},$$

$$\Delta A_{40} = \begin{bmatrix} -0.02 & 0.01 \\ 0.01 & -0.03 \end{bmatrix}, \Delta A_{41} = \begin{bmatrix} -0.03 & 0.01 \\ -0.01 & -0.02 \end{bmatrix}, \Delta A_{42} = \begin{bmatrix} -0.01 & -0.01 \\ 0.02 & -0.02 \end{bmatrix}.$$

Table 1 shows the simulation results of the distance between the poles of the uncertain discrete system and the center $(0.1, 0.7)$ of disk $D(0.1, 0.7)$ for various ε . This table shows that the uncertain discrete system is $D(0.1, 0.7)$-stable for $\varepsilon \le 0.032$. However, if $\varepsilon \ge 0.033$, the poles of the uncertain discrete system may lie outside the disk $D(0.1, 0.7)$, i.e., the $D(0.1, 0.7)$-stability of the uncertain discrete system cannot be guaranteed.

	$\varepsilon =$ 0.008	$\varepsilon =$ 0.01	$\varepsilon =$ 0.02	$\varepsilon =$ 0.03	$\varepsilon =$ 0.032	$\varepsilon =$ 0.033	$\varepsilon =$ 0.035
Pole 1	0.6360	0.5303	0.5764	0.6105	0.6163	0.6191	0.6245
Pole 2	0.5195	0.5303	0.5764	0.6105	0.6163	0.6191	0.6245
Pole 3	0.5195	0.6443	0.4542	0.5262	0.5385	0.5445	0.5562
Pole 4	0.5310	0.5268	0.4542	0.5262	0.5385	0.5445	0.5562
Pole 5	0.5310	0.5268	0.3726	0.4066	0.6965	**0.7208**	**0.7151**
Pole 6	0.3187	0.3488	0.6251	0.6835	0.6965	**0.7208**	**0.7151**
Pole 7	0.3187	0.3488	0.5948	0.6835	0.4131	0.4162	0.4224
Pole 8	0.3369	0.3407	0.4976	0.4654	0.4599	0.4573	0.4524
Pole 9	0.4084	0.4332	0.4976	0.4654	0.4599	0.4573	0.4524
Pole 10	0.2438	0.2549	0.2936	0.3196	0.3241	0.3262	0.3303
Pole 11	0.2438	0.2549	0.2936	0.3196	0.3241	0.3262	0.3303
Pole 12	0.0569	0.0630	0.0778	0.0835	0.0842	0.0846	0.0852

Table 1. Distance between the poles of the uncertain discrete system and the center $(0.1, 0)$ of disk $D(0.1, 0.7)$ for various ε

6. Conclusions

The robust D-stability of uncertain discrete two-time-scale systems with multiple time delays has been considered in this paper. The reduced-order models, i.e. the slow and fast subsystems of the system, are first derived. Under the condition that the slow and fast subsystems of the nominal system are both $D(\alpha, r)$-stable, we have proposed a frequency-domain robust D-stability criterion such that the slow and fast subsystems of the original uncertain system are $D(\alpha, r)$-stable. A delay-dependent criterion has also been proposed to guarantee the robust D-stability of the original uncertain system subject to structured perturbations. A numerical example has been provided to illustrate our main results.

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