Control and Cybernetics

vol. **32** (2003) No. 4

The eigenvalue derivatives of linear damped systems

by

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Abstract: In this note, the derivatives of eigenvalues with respect to the model parameters for linear damped systems is proposed by means of kronecker algebra and matrix calculus. A numerical example is also provided to illustrate the use of the main result.

Keywords: derivatives of eigenvalues, linear damped systems.

1. Introduction

In recent years, the stability and performance of linear damped systems has been widely investigated. In the past, there have been a number of interesting developments in responses, amplitude bounds, or eigenvalue bounds for linear damped systems. For instance, the amplitude bounds of linear damped systems have been proposed in Hu and Schiehlen (1996) and Schiehlen and Hu (1995). Furthermore, the response bounds of linear damped systems have been investigated in Hu and Eberhard (1999), Nicholson (1987a), and Yae and Inman (1987). Besides, the eigenvalue bounds of nonclassically damped systems have been derived in Nicholson (1987b).

There exist various applications demonstrating that an appropriate choice of the eigenvalues of a closed-loop system does not necessarily give a reliable indication as to the sufficiency of the stability margin since the derivatives of eigenvalues with respect to the model parameters may be very large. Consequently, the derivatives of eigenvalues with respect to the model parameters for linear damped systems are of importance. Furthermore, when exploring the dynamic behavior and its derivatives of eigenvalues with respect to the model parameters, kronecker algebra and matrix calculus frequently furnish shorter and more concise calculations. It is the purpose of this note to estimate the derivatives of eigenvalues with respect to the model parameters using the kronecker algebra approach and the matrix calculus approach.

2. Problem formulation and main results

For convenience, we define some notations that will be used throughout this note as follows:

$\Re^{m\times n}$:= the set of all real m by n matrices,
$C^{m\times n}$:= the set of all complex m by n matrices,
I_r := the unit matrix with r rows and r columns,
$Q>0$:= the matrix Q is a symmetric positive definite matrix,
$\frac{\partial\lambda}{\partial M} := \left[\frac{\partial\lambda}{\partial m_{ij}}\right]$ a real-valued function λ differentiated with respect to a matrix M with, $M=[m_{ij}]$
$A\otimes B$:= the kronecker product of A and B,
E_{ij} := the elementary matrix E_{ij} is defined as the matrix (of order $n\times n$) which has a unity in the (i,j)th position and all other elements zero,
$A^{\#L}$:= the left inverse matrix of the matrix A,
$\|A\|$:= the induced Euclidean norm of the matrix A.

Before presenting the problem formulation, let us introduce some lemmas, which will be used in the proof of the main theorem.

LEMMA 2.1 *(Weinmann, 1991) If $A\in C^{n\times m}$ and $z\in C^{s\times 1}$, then $A\otimes z=(I_n\otimes z)A$.*

LEMMA 2.2 *(Weinmann, 1991) If $A\in C^{n\times n}$ and $c\in C$, then $\frac{\partial det(A)}{\partial c}=tr\left[\frac{\partial A}{\partial c}\cdot adj(A)\right]$.*

Consider the following uncertain linear damped system

$$J(M)\ddot{y}(t)+B(M)\dot{y}(t)+K(M)y(t)=\tau(t), \tag{1}$$

where $y(t)\in\mathfrak{R}^{n}$ is the displacement vector, $\tau(t)\in\mathfrak{R}^{n}$ is the external excitation, $J\in\mathfrak{R}^{n\times n}$ is the matrix of inertial moment, $B\in\mathfrak{R}^{n\times n}$ is the damping matrix, $K\in\mathfrak{R}^{n\times n}$ is the stiffness matrix, and $M\in\mathfrak{R}^{r\times s}$ is the parameter's matrix. Clearly, the characteristic equation of system (1) is given by

$$\det[\lambda^2 J+\lambda B+K]=0. \tag{2}$$

The objective of this note is to estimate the derivatives of eigenvalues with respect to the parameter's matrix without the use of eigenvectors.

Now we present the main result.

THEOREM 2.1 *Assume that λ_i is a simple eigenvalue of system (1). Then the derivatives of eigenvalues with respect to the parameter's matrix $M:=[m_{jk}]\in\Re^{r\times s}$ of the system (1) are given by*

$$\frac{\partial\lambda_i}{\partial m_{jk}} = -\left\{tr\left[\left(\lambda_i^2\frac{\partial J}{\partial m_{jk}} + \lambda_i\frac{\partial B}{\partial m_{jk}} + \frac{\partial K}{\partial m_{jk}}\right)\cdot adj(\lambda_i^2 J + \lambda_i B + K)\right]\right\}$$
$$\cdot\left\{tr[(2\lambda_i J + B)\cdot adj(\lambda_i^2 J + \lambda_i B + K)]\right\}^{-1}, \quad \forall j \in \underline{r}, k \in \underline{s}.$$

Proof. At first, the existence of solution to $\frac{\partial\lambda_i}{\partial M}$ is proved as follows. For an eigenvalue λ_i, there exists $a_i \in C^{n\times 1}$ with $a_i \neq 0$ satisfying $(\lambda_i^2 J + \lambda_i B + K)a_i = 0$. Taking partial derivatives of both sides of the above equation with respect to the matrix $M \in \Re^{r\times s}$ yields

$$\frac{\partial(\lambda_i^2 J + \lambda_i B + K)a_i}{\partial M} = \frac{\partial(\lambda_i^2 J a_i)}{\partial M} + \frac{\partial(\lambda_i B a_i)}{\partial M} + \frac{\partial(K a_i)}{\partial M} = 0. \tag{3}$$

Furthermore, one has

$$\begin{aligned}\frac{\partial(\lambda_i^2 J a_i)}{\partial M} &= \frac{\partial\lambda_i^2}{\partial M}\otimes(Ja_i) + \lambda_i^2\frac{\partial(Ja_i)}{\partial M}\\ &= \left[2\lambda_i\frac{\partial\lambda_i}{\partial M}\right]\otimes(Ja_i) + \lambda_i^2\left[\frac{\partial J}{\partial M}(I_s\otimes a_i) + (I_r\otimes J)\frac{\partial a_i}{\partial M}\right],\end{aligned} \tag{4a}$$

$$\begin{aligned}\frac{\partial(\lambda_i B a_i)}{\partial M} &= \frac{\partial\lambda_i}{\partial M}\otimes(Ba_i) + \lambda_i\frac{\partial(Ba_i)}{\partial M}\\ &= \frac{\partial\lambda_i}{\partial M}\otimes(Ba_i) + \lambda_i\left[\frac{\partial B}{\partial M}(I_s\otimes a_i) + (I_r\otimes B)\frac{\partial a_i}{\partial M}\right]\end{aligned} \tag{4b}$$

and

$$\frac{\partial(Ka_i)}{\partial M} = \frac{\partial K}{\partial M}(I_s\otimes a_i) + (I_r\otimes K)\frac{\partial a_i}{\partial M}. \tag{4c}$$

Combining (3) and (4) gives

$$\left[2\lambda_i\frac{\partial\lambda_i}{\partial M}\right]\otimes(Ja_i) + \lambda_i^2\frac{\partial J}{\partial M}(I_s\otimes a_i) + \frac{\partial\lambda_i}{\partial M}\otimes(Ba_i) + \lambda_i\frac{\partial B}{\partial M}(I_s\otimes a_i)$$
$$+\frac{\partial K}{\partial M}\otimes(I_s\otimes a_i) + \lambda_i^2(I_r\otimes J)\frac{\partial a_i}{\partial M} + \lambda_i(I_r\otimes B)\frac{\partial a_i}{\partial M} + (I_r\otimes K)\frac{\partial a_i}{\partial M} = 0.$$

It can be readily obtained that

$$\left(\frac{\partial\lambda_i}{\partial M}\right)\otimes[2\lambda_i Ja_i + Ba_i] = -\lambda_i^2\frac{\partial J}{\partial M}(I_s\otimes a_i) - \lambda_i\frac{\partial B}{\partial M}(I_s\otimes a_i)$$
$$-\frac{\partial K}{\partial M}\otimes(I_s\otimes a_i) - \left[I_r\otimes(\lambda_i^2 J + \lambda_i B + K)\right]\frac{\partial a_i}{\partial M}.$$

Applying Lemma 1 to above equation, we obtain

$$\left[I_r \otimes (2\lambda_i J a_i + B a_i)\right] \frac{\partial \lambda_i}{\partial M} = -\lambda_i^2 \frac{\partial J}{\partial M}(I_s \otimes a_i) - \lambda_i \frac{\partial B}{\partial M}(I_s \otimes a_i)$$
$$-\frac{\partial K}{\partial M} \otimes (I_s \otimes a_i) - \left[I_r \otimes (\lambda_i^2 J + \lambda_i B + K)\right] \frac{\partial a_i}{\partial M} \,. \tag{5}$$

Furthermore one has $\det(2\lambda_i J + B) \neq 0$, which implies $(2\lambda_i J + B)a_i = (2\lambda_i J a_i + B a_i) \neq 0, \forall a_i \neq 0$. Hence we conclude that the matrix $[I_r \otimes (2\lambda_i J a_i + B a_i)] \in C^{rn\times r}$ has full column rank, which implies that the left inverse of matrix $[I_r \otimes (2\lambda_i J a_i + B a_i)]$ exists. Postmultiplying (5) by $[I_r \otimes (2\lambda_i J a_i + B a_i)]^{\#L}$ gives

$$\frac{\partial \lambda_i}{\partial M} = -[I_r \otimes (2\lambda_i J a_i + B a_i)]^{\#L}$$
$$\cdot \left\{ \lambda_i^2 \frac{\partial K}{\partial M}(I_s \otimes a_i) + \lambda_i^2 \frac{\partial B}{\partial M}(I_s \otimes a_i) + \frac{\partial K}{\partial M} \otimes (I_s \otimes a_i) \right.$$
$$\left. + [I_r \otimes (\lambda_i^2 J + \lambda_i B + K)] \frac{\partial a_i}{\partial M} \right\},$$

so that the existence of $\frac{\partial \lambda_i}{\partial M}$ is guaranteed. By Lemma 2 with (2), one has

$$\frac{\partial \det(\lambda_i^2 J + \lambda_i B + K)}{\partial m_{jk}} = tr\left[\frac{\partial(\lambda_i^2 J + \lambda_i B + K)}{\partial m_{jk}} \cdot adj(\lambda_i^2 J + \lambda_i B + K)\right] = 0. \tag{6}$$

Furthermore, it can be shown that

$$\frac{\partial(\lambda_i^2 J + \lambda_i B + K)}{\partial m_{jk}} = 2\lambda_i \frac{\partial \lambda_i}{\partial m_{jk}} J + \lambda_i^2 \frac{\partial J}{\partial m_{jk}} + \frac{\partial \lambda_i}{\partial m_{jk}} B + \lambda_i \frac{\partial B}{\partial m_{jk}} + \frac{\partial K}{\partial m_{jk}}$$
$$= \frac{\partial \lambda_i}{\partial m_{jk}}(2\lambda_i J + B) + \lambda_i^2 \frac{\partial J}{\partial m_{jk}} + \lambda_i \frac{\partial B}{\partial m_{jk}} + \lambda_i \frac{\partial K}{\partial m_{jk}}. \tag{7}$$

Substituting (7) into (6), yields

$$tr\left[\frac{\partial \lambda_i}{\partial m_{jk}}(2\lambda_i J + B) \cdot adj(\lambda_i^2 J + \lambda_i B + K)\right]$$
$$= tr\left[-\left(\lambda_i^2 \frac{\partial J}{\partial m_{jk}} + \lambda_i \frac{\partial B}{\partial m_{jk}} + \frac{\partial K}{\partial m_{jk}}\right) \cdot adj(\lambda_i^2 J + \lambda_i B + K)\right],$$

which implies that

$$\frac{\partial \lambda_i}{\partial m_{jk}} = -\left\{ tr\left[\left(\lambda_i^2 \frac{\partial J}{\partial m_{jk}} + \lambda_i \frac{\partial B}{\partial m_{jk}} + \frac{\partial K}{\partial m_{jk}}\right) \cdot adj(\lambda_i^2 J + \lambda_i B + K)\right]\right\}$$
$$\cdot \left\{ tr\left[(2\lambda_i J + B) \cdot adj(\lambda_i^2 J + \lambda_i B + K)\right]\right\}^{-1}.$$

This completes our proof. ■

REMARK 2.1 *The characteristic equation of det*$[\lambda B + K] = 0$ *with the assumptions that*

(A1) B *is nonsingular;*

(A2) Both matrices B *and* K *can be simultaneously diagonalized has been considered in Prells and Friswell (2000). This is the special case of (3.) with* $J = 0$. *It is noted that the methodology used in our main result is different from that used in Prells and Friswell (2000). Furthermore, even if both (A1) and (A2) are not satisfied, our main result may be applied to the system (1).*

3. Illustrative example

Consider the linear damped system

$$J(M)\ddot{y}(t) + B(M)\dot{y}(t) + K(M)y(t) = \tau(t), \tag{8}$$

where $y(t) \in \Re^2$ is the displacement vector, $\tau(t) \in \Re^2$ is the external excitation,

$$J = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ K = \begin{bmatrix} 2 & 1 \\ 1 & \beta \end{bmatrix},$$

and $M := [\alpha \quad \beta]$ is parameter's matrix with $1 \leq \alpha \leq 3$ and $0.5 \leq \beta \geq 1.5$. We assume that the nominal values are $\alpha = 2$ and $\beta = 1$. In this case, from (3.), the eigenvalues of the system (8) are

$$\lambda_{1,normal} = -0.2459 + 1.1484j,\ \lambda_{2,normal} = -0.2459 - 1.1484j,$$
$$\lambda_{3,normal} = 0.2459 + 0.5496j,\ \lambda_{4,normal} = 0.2459 - 0.5496j.$$

By Theorem 1, the derivatives of eigenvalues of system (8) can be estimated by

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial M} &= \left[\, 0.0762 - 0.1128j \ \ 0.1524 - 0.2256j \,\right], \\
\frac{\partial \lambda_2}{\partial M} &= \left[\, 0.0762 + 0.1128j \ \ 0.1524 + 0.2256j \,\right], \\
\frac{\partial \lambda_3}{\partial M} &= \left[\, -0.0762 - 0.0599j \ \ -0.1524 - 0.1198j \,\right], \\
\frac{\partial \lambda_4}{\partial M} &= \left[\, -0.0762 + 0.0599j \ \ -0.1524 + 0.1198j \,\right].
\end{aligned}$$

In the case of $\alpha = 2.05$ and $\beta = 0.95$ (i.e., $\Delta\alpha = 0.05$ and $\Delta\beta = -0.05$), the eigenvalues can be approximately calculated as

$$\begin{aligned}
\lambda_1 &= -0.2482 + 1.144j = \lambda_{1,normal} + \Delta\lambda_1 \\
&\cong -0.2459 + 1.1484j + \frac{\partial \lambda_1}{\partial M}\begin{bmatrix}\Delta\alpha \\ \Delta\beta\end{bmatrix} = -0.24971 + 1.15404j, \\
\lambda_2 &= -0.2482 - 1.144j = \lambda_{2,normal} + \Delta\lambda_2 \\
&\cong -0.2459 - 1.1484j + \frac{\partial \lambda_2}{\partial M}\begin{bmatrix}\Delta\alpha \\ \Delta\beta\end{bmatrix} = -0.24971 - 1.15404j, \\
\lambda_3 &= 0.2482 + 0.525j = \lambda_{3,normal} + \Delta\lambda_3 \\
&\cong 0.2459 + 0.5496j + \frac{\partial \lambda_3}{\partial M}\begin{bmatrix}\Delta\alpha \\ \Delta\beta\end{bmatrix} = 0.24971 + 0.5526j, \\
\lambda_4 &= 0.2482 - 0.525j = \lambda_{4,normal} + \Delta\lambda_4 \\
&\cong 0.2459 - 0.5496j + \frac{\partial \lambda_4}{\partial M}\begin{bmatrix}\Delta\alpha \\ \Delta\beta\end{bmatrix} = 0.24971 - 0.5526j,
\end{aligned}$$

4. Conclusions

In this note, the calculation derivatives of eigenvalues with respect to the model parameters for linear damped systems has been proposed by means of kronecker algebra and matrix calculus. A numerical example has also been provided to illustrate the use of the main result.

Acknowledgment The author thanks the National Science Council of ROC, for supporting this work under Grant 91-2213-E-214-008.

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