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Book review:

Handbook of Brownian Motion - Facts and Formulae (Second Edition)

by

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Brownian motion as well as other diffusion processes play a meaningful role in stochastic analysis. They are very important from theoretical point of view and very useful in applications. Diffusions and their local times are applied in financial mathematics, economics, engineering and stochastic optimization.

Brownian motion as a stochastic process was considered by Bachelier and Einstein at the beginning of the 20^{th} century. However, Wiener was the first to introduce its solid mathematical foundations. Lévy made a significant contribution to modern analysis of the process. Brownian motion is an important example of a linear diffusion, i.e. a continuous strong Markov process taking values on an interval.

The aim of the book is to give an exposition of facts and formulae associated to Brownian motion in a style of a handbook. It is divided into two parts, which are preceded by the prefaces to the first and to the second edition. Part I is devoted to theory, while Part II contains tables with numbered formulae associated to Brownian motion and other diffusions. There are appendices at the end of each part.

Part I entitled "Theory" consists of six chapters and one appendix. Chapter I is devoted to general theory of stochastic processes. Apart from basic definitions, Markov processes are considered. Particular attention is paid to the Feller and Feller-Dynkin processes. In the last part of the chapter the notion of martingale is introduced and some facts concerning theory of martingales are presented. One may find here, in particular, the Doob-Meyer decomposition for submartingales and some known martingale inequalities, such as the Burkholder-Davis-Gundy inequality.

Description of linear diffusions is the subject of Chapter II. The authors recall their characteristics, i.e. speed measure, scale function and killing measure. Moreover, basic properties of diffusions and their local times are presented. Brief review of ergodic results connected with local times of recurrent diffusions completes the chapter.

Chapter III is dedicated to readers who are especially interested in stochastic calculus. This theory has an essential meaning for applications, especially for financial mathematics. Integration with respect to Brownian motion, the Ito and Tanaka formulae as well as the Cameron-Martin-Girsanov transformations of probability measure are basic tools in the theory of option pricing and many other fields. Apart from them, stochastic differential equations and their solutions are discussed in the chapter.

Chapter IV contains definition and basic distributional properties of Brownian motion. Besides such commonly known properties as spatial homogeneity, symmetry or time reversibility, especially important is Lévy's martingale characterization, Hölder continuity, formula for quadratic variation on intervals and property of infinite variation. Subsequently, some facts connected with Brownian local times and excursion processes are presented. Chapter IV is completed by an exposition of properties and formulae concerning processes related to Brownian motion, like Brownian bridge, Brownian motion with drift, Bessel processes and geometric Brownian motion. The last process is used for description of financial and insurance instruments. Therefore, theoretical considerations presented by the authors are useful in the analysis of various types of options. One should also emphasize a possibility of applications of geometric Brownian motion in mathematical physics.

Chapter V is dedicated to Ray-Knight theorems. Discussed results concern the Markovian character of local times. The laws of local times of Brownian motion, Brownian motion with drift and Bessel process are characterized. The last part of the chapter contains tables, which summarize the results presented.

In Chapter VI differential systems associated to Brownian motion are presented. The basic tool for finding distributions of functionals is the Feynman-Kac formula. Functionals stopped at exponential time, the first exit time, inverse additive functional and the first range time are considered. The last class of functionals is especially important, since the first range time is the stopping time with many applications.

Appendix 1 is a collection of basic characteristics and facts associated with some commonly used diffusions. Besides various types of processes mentioned before, the Ornstein-Uhlenbeck as well as the radial and squared radial Ornstein-Uhlenbeck processes are characterized.

Part II contains tables of distributions of functionals of Brownian motion and other related diffusions. The tables are divided into nine sections with respect to the type of diffusion, which occurs in the formulae. The authors consider functionals of diffusions with up to five types of stopping times: exponentially distributed stopping time independent of diffusion, the first hitting time of a point, the first exit time from an interval, inverse local time at a point and the first range time at a level. The structure of tables is displayed via their triple numbering system. The numbers refer to the stopping time, a functional, and the Laplace transform or to the distribution itself. The numbering system causes that the structure of tables is clear. Introduction to Part II is very useful for the readers, because it contains lists of stopping times, diffusions and functionals connected with the numbering system.

Appendix 2 is a brief exposition of special functions and their properties. It is valuable, because many special functions appear the in formulae. Appendix 3 is dedicated to inverse Laplace transforms. Appendix 4 on differential equations is needful, when the Feynman-Kac formula is used. Readers who need to calculate moments of functionals of stochastic processes may use theory from Appendix 5, which is devoted to n-fold differentiation. In the final part of the book there is a rich bibliography as well as the subject index.

This edition of the book is an extension of the previous one. It contains more than 1000 new formulae in comparison with the first edition. There are new facts and formulae connected with geometric Brownian motion, the Feynman-Kac formula and the radial Ornstein-Uhlenbeck process. There are three new appendices. Some formulae from the first edition are excluded. The exclusion concerns formulae, which can be easily obtained from the other ones. The authors also wanted to avoid an additional extension of the number of pages.

Apart from the presentation of many formulae concerning functionals associated to Brownian motion and other diffusions, the advantage of this book is its theoretical part. It is very important to understand the theory related to the applied tools. Sometimes formulae need some modifications in order to adopt them to real applications. In this case it is very useful to know how they were derived. The second advantage is the presence of references to monographs for almost all generally formulated equations in Part I. Nevertheless, not every formula has a sufficient commentary concerning its specific applications. This may constitute a difficulty for a reader not acquainted with the subject. However, one can easy forget this inconvenience, because the book is written very carefully and there are no observable misprints.

From the mathematical point of view, the theory is presented very precisely, although it is not written in the classical form of definitions and theorems. The presentation is made without proofs. I agree with the opinion of the authors that such an approach would be welcome by the readers. The theoretical part of the book is also valuable as a brief review of the material about diffusions and their functionals, which should be known by students, engineers or economists using them in practice.

I think the book is very useful for readers applying Brownian motion and other diffusions.

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