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Second order optimality conditions for bang–bang control problems [1]

by

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Abstract: Second order necessary and sufficient optimality conditions for bang–bang control problems have been studied in Milyutin, Osmolovskii (1998). These conditions amount to testing the positive (semi–)definiteness of a quadratic form on a critical cone. The assumptions are appropriate for numerical verification only in some special cases. In this paper, we study various transformations of the quadratic form and the critical cone which will be tailored to different types of control problems in practice. In particular, by means of a solution to a linear matrix differential equation, the quadratic form can be converted to perfect squares. We demonstrate by three practical examples that the conditions obtained can be verified numerically.

Keywords: bang–bang control, second order necessary and sufficient conditions, critical cone, transformation of quadratic forms, numerical verification of second order conditions, van der Pol oscillator.

1. Introduction

There exists an extensive literature on second order sufficient conditions (SSC) for optimal control problems with control appearing nonlinearly, see Dunn (1995, 1996), Levitin, Milyutin, Osmolovskii (1978), Maurer (1981), Maurer, Pickenhain (1995), Maurer, Oberle (2002), Milyutin, Osmolovskii (1998), Osmolovskii

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(1988, 1988a, 1988b, 1995), Osmolovskii, Lempio (2000, 2002), Zeidan (1994) and further literature cited in these papers. SSC require that a certain quadratic form be positive definite on the so called critical cone. In practice, the test for SSC can be performed by checking whether an associated matrix Riccati equation has a bounded solution under appropriate boundary conditions. The Riccati approach has been extended to discontinuous controls (broken extremals) by Osmolovskii, Lempio (2002). The importance of SSC is due to its crucial role in the sensitivity analysis of parametric optimal control problems, see Malanowski (1992, 1993, 1994, 2001), Malanowski, Maurer (1996, 1998, 2001), Dontchev et al. (1995), Augustin, Maurer (2001a,2001b).

Optimal control problems with *control appearing linearly* lead either to bang–bang controls or to singular controls. First and higher order necessary optimality conditions have been studied, e.g., by Bressan (1985), Schättler (1988) and Sussmann (1979, 1987a, 1987b) for the generic properties of bang–bang controls. General second order necessary and sufficient conditions for an extremal with a discontinuous control (see Osmolovskii, 1995) can be derived from the theory of higher order conditions in Levitin, Milyutin and Osmolovskii (1978). The main results for bang-bang controls which follow from these general conditions are given in Milyutin and Osmolovskii (1998). Some proofs missing in that book will appear in Osmolovskii (2003). The literature on SSC for bang–bang controls is rather scarce both in theory and numerics. Only very recently, one may observe a revived interest in bang–bang controls and several approaches to SSC have been developed almost in parallel.

Sarychev (1997) has obtained first and second order optimality conditions for time–optimal bang–bang controls. It is not clear from the article mentioned, though, how one might apply the obtained conditions to practical examples. Noble, Schättler (2001) develop sufficient conditions for broken extremals which, however, are only applicable under the assumption that the reference trajectory can locally be embedded into a sufficiently smooth field of extremals. Felgenhauer (2003) discusses bang–bang controls where the dynamics are linear in control and state. Agrachev, Stefani, Zezza (2002) treat bang–bang control problems with fixed final time and are able to reduce the control problem to a finite–dimensional optimization problem with respect to the switching times as optimization variables. We are not aware of any practical bang–bang control problem in the literature except the one given in Ledzewicz, Schättler (2002) where SSC have been tested numerically.

Our aim is to develop SSC for bang–bang controls under verifiable assumptions. This goal will be achieved by deriving several representations of the quadratic form and the critical cone in Milyutin, Osmolovskii (1998), which are more convenient for numerical computations. For time–optimal bang–bang controls with fixed initial and terminal conditions, this program was already carried out in Maurer, Osmolovskii (2001). In the present article, we extend the analysis therein to bang–bang controls with very general state boundary conditions. Pontryagin's minimum principle and the bang–bang property are discussed in

Section 2. In Section 3, the critical cone is introduced and its properties are studied. Second order necessary and sufficient optimality conditions are given in terms of the positive (semi–)definiteness of a quadratic form on the critical cone. A control problem from economics illustrates the use of SSC. Section 4 presents the $Q-$transformation whereby the quadratic form is rewritten into a more convenient form using a solution Q of a linear differential equation. The general form of the boundary conditions for Q is developed. Positive definiteness conditions are given under which the quadratic form is transformed into perfect squares. In Section 5, we discuss two numerical examples that illustrate the numerical procedure for testing the positive definiteness of the corresponding quadratic forms.

2. Bang–bang control problems on nonfixed time intervals

2.1. Optimal control problems with control appearing linearly

We consider optimal control problems with control appearing linearly. Let $x(t) \in \mathbb{R}^{d(x)}$ denote the state variable and $u(t) \in \mathbb{R}^{d(u)}$ the control variable in the time interval $t \in \Delta = [t_0, t_1]$ with a non–fixed initial time t_0 and final time t_1.

$$\text{Minimize} \qquad \mathcal{J}(t_0, t_1, x, u) = J(t_0, x(t_0), t_1, x(t_1)) \tag{1}$$

subject to the constraints

$$\dot{x}(t) = f(t, x(t), u(t)), \quad u(t) \in U, \quad (t, x(t)) \in \mathcal{Q}, \quad t_0 \le t \le t_1, \tag{2}$$

$$\begin{aligned} &F(t_0, x(t_0), t_1, x(t_1)) \le 0, \quad K(t_0, x(t_0), t_1, x(t_1)) = 0, \\ &(t_0, x(t_0), t_1, x(t_1)) \in \mathcal{P}, \end{aligned} \tag{3}$$

where the control variable appears linearly in the system dynamics,

$$f(t, x, u) = a(t, x) + B(t, x)u\,. \tag{4}$$

Here, F, K, a are vector functions, B is a $d(x) \times d(u)$ matrix function, $\mathcal{P} \subset \mathbb{R}^{2+2d(x)}$, $\mathcal{Q} \subset \mathbb{R}^{1+d(x)}$ are open sets and $U \subset \mathbb{R}^{d(u)}$ is a convex polyhedron. The functions J, F, K are assumed to be twice continuously differentiable on $\mathcal{P}$ and the functions a, B are twice continuously differentiable on $\mathcal{Q}$. The dimensions of F, K are denoted by $d(F), d(K)$. We shall use the abbreviations

$$x_0 = x(t_0),\ x_1 = x(t_1),\ p = (t_0, x_0, t_1, x_1).$$

A trajectory

$$\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_1])$$

is said to be *admissible*, if $x(\cdot)$ is absolutely continuous, $u(\cdot)$ is measurable bounded and the pair of functions $(x(t), u(t))$ on the interval $\Delta = [t_0, t_1]$ with the end-points $p = (t_0, x(t_0), t_1, x(t_1))$ satisfies the constraints (2), (3).

Definition 2.1 *The trajectory $\mathcal{T}$ affords a Pontryagin minimum, if there is no sequence of admissible trajectories*

$$\mathcal{T}^n = (x^n(t), u^n(t) \mid t \in [t_0^n, t_1^n]), \quad n = 1, 2, \ldots,$$

such that the following properties hold with $\Delta^n = [t_0^n, t_1^n]$:

(a) $\mathcal{J}(\mathcal{T}^n) < \mathcal{J}(\mathcal{T}) \quad \forall\, n$ *and* $t_0^n \to t_0,\ t_1^n \to t_1$ *for* $n \to \infty$;

(b) $\max\limits_{\Delta^n \cap \Delta} |x^n(t) - x(t)| \to 0$ *for* $n \to \infty$*;*

(c) $\int\limits_{\Delta^n \cap \Delta} |u^n(t) - u(t)|\, dt \to 0$ *for* $n \to \infty$*.*

Note that for a fixed time interval Δ, a Pontryagin minimum corresponds to an L_1-local minimum with respect to the control variable.

2.2. First order necessary optimality conditions

Let

$$\mathcal{T} = (x(t), u(t) \mid t \in [t_0, t_1])$$

be a fixed admissible trajectory such that the control $u(\cdot)$ is a piecewise constant function on the interval $\Delta = [t_0, t_1]$. In order to make the notations simpler we do not use such symbols and indices as zero, hat or asterisk to distinguish this trajectory from others.

Denote by

$$\theta = \{\tau_1, \ldots, \tau_s\}, \quad t_0 < \tau_1 < \ldots < \tau_s < t_1$$

the finite set of all discontinuity points (jump points) of the control $u(t)$. Then $\dot{x}(t)$ is a piecewise continuous function whose discontinuity points belong to θ, and hence $x(t)$ is a piecewise smooth function on Δ. Henceforth we shall use the notation

$$[u]^k = u^{k+} - u^{k-}$$

are the left hand and the right hand values of the control $u(t)$ at τ_k, respectively. Similarly, we denote by $[\dot{x}]^k$ the jump of the function $\dot{x}(t)$ at the same point.

Let us formulate a first-order necessary condition for optimality of the trajectory $\mathcal{T}$ – the Pontryagin minimum principle. To this end we introduce the Pontryagin function

$$H(t, x, \psi, u) = \psi f(t, x, u) = \psi a(t, x) + \psi B(t, x) u, \tag{5}$$

where ψ is a row-vector of dimension $d(\psi) = d(x)$ while x, u, f, F and K are column-vectors. The factor of the control u in the Pontryagin function is called the *switching function*

$$\sigma(t, x, \psi) = D_u H(t, x, \psi, u) = \psi B(t, x) \tag{6}$$

which is a row vector of dimension $d(u)$. Denote by l the end-point Lagrange function

$$l(\alpha_0, \alpha, \beta, p) = \alpha_0 J(p) + \alpha F(p) + \beta K(p),$$

where α and β are row-vectors with $d(\alpha) = d(F)$, $d(\beta) = d(K)$, and α_0 is a number. We introduce a collection of Lagrange multipliers

$$\lambda = (\alpha_0, \alpha, \beta, \psi(\cdot), \psi_0(\cdot))$$

such that

$$\psi(\cdot) : \Delta \to \mathbb{R}^{d(x)}, \quad \psi_0(\cdot) : \Delta \to \mathbb{R}^1$$

are continuous on Δ and continuously differentiable on each interval of the set $\Delta \setminus \theta$. In the sequel, we shall denote first or second order partial derivatives by the subscripts referring to the variables.

Denote by M_0 the set of the normed collections λ satisfying the minimum principle conditions for the trajectory $\mathcal{T}$:

$$\alpha_0 \geq 0, \quad \alpha \geq 0, \quad \alpha F(p) = 0, \quad \alpha_0 + \sum \alpha_i + \sum |\beta_j| = 1, \tag{7}$$

$$\dot{\psi} = -H_x, \quad \dot{\psi}_0 = -H_t \quad \forall t \in \Delta \setminus \theta, \tag{8}$$

$$\psi(t_0) = -l_{x_0}, \quad \psi(t_1) = l_{x_1}, \quad \psi_0(t_0) = -l_{t_0}, \quad \psi_0(t_1) = l_{t_1}, \tag{9}$$

$$\min_{u \in U} H(t, x(t), \psi(t), u) = H(t, x(t), \psi(t), u(t)) \quad \forall t \in \Delta \setminus \theta, \tag{10}$$

$$H(t, x(t), \psi(t), u(t)) + \psi_0(t) = 0 \quad \forall t \in \Delta \setminus \theta. \tag{11}$$

The derivatives l_{x_0} and l_{x_1} are taken at the point $(\alpha_0, \alpha, \beta, p)$, where $p = (t_0, x(t_0), t_1, x(t_1))$, and the derivatives H_x, H_t are evaluated at the point $(t, x(t), u(t), \psi(t))$, $t \in \Delta \setminus \theta$. The condition $M_0 \neq \emptyset$ constitutes the first order necessary condition for a Pontryagin minimum of the trajectory $\mathcal{T}$, which is the so called Pontryagin minimum principle, see Pontryagin et al. (1961), Milyutin, Osmolovskii (1998).

Theorem 2.1 *If the trajectory $\mathcal{T}$ affords a Pontryagin minimum, then the set M_0 is nonempty. The set M_0 is a finite-dimensional compact set and the projector $\lambda \mapsto (\alpha_0, \alpha, \beta)$ is injective on M_0.*

In the sequel, it will be convenient to use the simple abbreviation (t) for indicating all arguments $(t, x(t), u(t), \psi(t))$, e.g., $H(t) = H(t, x(t), u(t), \psi(t))$, $\sigma(t) = \sigma(t, x(t), \psi(t))$. The continuity of the pair of functions $(\psi_0(t), \psi(t))$ at the points $t_k \in \theta$ constitutes the Weierstrass–Erdmann necessary conditions for nonsmooth extremals. We formulate one more condition of this type which is important for the statement of the second-order conditions for extremal with jumps in the control. Namely, for $\lambda \in M_0$, $\tau_k \in \theta$ consider the function

$$(\Delta_k H)(t) = H(t, x(t), \psi(t), u^{k+}) - H(t, x(t), \psi(t), u^{k-}) = \sigma(t)\,[u]^k. \tag{12}$$

Proposition 2.1 *For each $\lambda \in M_0$ the following equalities hold*

$$\frac{d}{dt}(\Delta_k H)\big|_{t=\tau_k-0} = \frac{d}{dt}(\Delta_k H)\big|_{t=\tau_k+0}, \quad k=1,\ldots,s.$$

Consequently, for each $\lambda \in M_0$ the function $(\Delta_k H)(t)$ has a derivative at the point $\tau_k \in \theta$. Define the quantity

$$D^k(H) = -\frac{d}{dt}(\Delta_k H)(\tau_k).$$

Then, the minimum condition (8) implies the following property:

Proposition 2.2 *For each $\lambda \in M_0$ the following conditions hold:*

$$D^k(H) \geq 0, \quad k=1,\ldots,s. \tag{13}$$

The value $D^k(H)$ also can be written in the form

$$\begin{aligned} D^k(H) &= -H_x^{k+}H_\psi^{k-} + H_x^{k-}H_\psi^{k+} - [H_t]^k \\ &= \dot\psi^{k+}\dot x^{k-} - \dot\psi^{k-}\dot x^{k+} + [\psi_0]^k, \end{aligned}$$

where H_x^{k-} and H_x^{k+} are the left hand and the right hand values of the function $H_x(t,x(t),u(t),\psi(t))$ at τ_k, respectively, $[H_t]^k$ is a jump of the function $H_t(t)$ at τ_k, etc. It also follows from the above representation that we have

$$D^k(H) = -\dot\sigma(\tau_k^{\pm})[u]^k \tag{14}$$

where the values on the right hand side agree for the derivative $\dot\sigma(\tau_k^+)$ from the right and the derivative $\dot\sigma(\tau_k^-)$ from the left. In the case of a *scalar* control u, the total derivative $\sigma_t + \sigma_x\dot x + \sigma_\psi\dot\psi$ does not contain the control variable explicitly and hence the derivative $\dot\sigma(t)$ is *continuous* at τ_k.

Proposition 2.3 *For any $\lambda \in M_0$ we have*

$$l_{x_0}\dot x(t_0) + l_{t_0} = 0, \quad l_{x_1}\dot x(t_1) + l_{t_1} = 0. \tag{15}$$

Proof. The equalities (15) follow from the equality $\psi(t)\dot x(t)+\psi_0(t)=0$ evaluated for $t=t_0$ and $t=t_1$ together with the transversality conditions

$$\psi(t_0) = -l_{x_0}, \quad \psi_0(t_0) = -l_{t_0}, \quad \psi(t_1) = l_{x_1}, \quad \psi_0(t_1) = l_{t_1}.$$

2.3. Integral cost function, unessential variables, strong minimum

It is well known that any control problem with a cost functional in integral form

$$\mathcal{J} = \int_{t_0}^{t_1} f_0(t,x(t),u(t))\,dt \tag{16}$$

can be brought to the canonical form (1) by introducing a new state variable y defined by the state equation

$$\dot{y} = f_0(t, x, u), \quad y(t_0) = 0\,. \tag{17}$$

This yields the cost function $\mathcal{J} = y(t_1)$. The control variable is assumed to appear linearly in the function f_0,

$$f_0(t, x, u) = a_0(t, x) + B_0(t, x)u\,. \tag{18}$$

It follows from equations (8) and (9) that the adjoint variable ψ^y associated with the new state variable y is given by $\psi^y(t) \equiv \alpha_0$, which yields the Pontryagin function (5) in the form

$$\begin{aligned} H(t, x, \psi, u) &= \alpha_0 f_0(t, x, u) + \psi f(t, x, u) \\ &= \alpha_0 a_0(t, x) + \psi a(t, x) + (\alpha_0 B_0(t, x) + \psi B(t, x))u. \end{aligned} \tag{19}$$

Hence, the switching function is given by

$$\sigma(t, x, \psi) = \alpha_0 B_0(t, x) + \psi B(t, x), \quad \sigma(t) = \sigma(t, x(t), \psi(t)). \tag{20}$$

The component y is called an *unessential* component in the augmented problem. The general definition of an unessential component is as follows:

Definition 2.2 *The state variable x_i, i.e., the i–th component of the state vector x is called* unessential *if the function f does not depend on x_i and if the functions F, J, K are affine in $x_{i0} = x_i(t_0)$ and $x_{i1} = x_i(t_1)$.*

Unessential components should not be taken into consideration in the definition of a minimum. This leads to the definition of a *strong minimum* which is a stronger type than the Pontryagin minimum in Definition 1. The strong minimum refers to the proximity of the state components in the trajectory only. In the following, let $\underline{x}$ denote the vector of all essential components of state vector x.

Definition 2.3 *We say that the trajectory $\mathcal{T}$ affords a strong minimum if there is no sequence of admissible trajectories*

$$\mathcal{T}^n = (x^n(t), u^n(t) \mid t \in [t_0^n, t_1^n]), \quad n = 1, 2, \ldots$$

such that

(a) $\mathcal{J}(\mathcal{T}^n) < \mathcal{J}(\mathcal{T})$,

(b) $t_0^n \to t_0, \quad t_1^n \to t_1, \quad x^n(t_0) \to x(t_0)$ *($n \to \infty$),*

(c) $\max_{\Delta^n \cap \Delta} |\underline{x}^n(t) - \underline{x}(t)| \to 0$ *($n \to \infty$), where* $\Delta^n = [t_0^n, t_1^n]$.

The *strict* strong minimum is defined in a similar way, with the strict inequality (a) replaced by the non-strict one and the trajectory $\mathcal{T}^n$ required to be different from $\mathcal{T}$ for each n.

2.4. Bang-bang control

The intuitive definition of a bang–bang control is that of a control which assumes values only in the vertex set of the admissible polyhedron U in (2). We shall need a slightly more restrictive definition of a bang–bang control to obtain the sufficient conditions in Theorem 3.2. Let

$$\text{Arg}\min_{v\in U}\ \sigma(t)v$$

be the set of points $v \in U$ where the minimum of the linear function $\sigma(t)v$ is attained. For a given extremal trajectory $\mathcal{T} = \{\,(x(t), u(t)) \mid t \in \Delta\,\}$ with a piecewise constant control $u(t)$ we say that $u(t)$ is a *bang-bang control* if there exists $(\psi_0, \psi) \in M_0$ such that

$$\text{Arg}\min_{v\in U} \sigma(t)v = [u(t-0), u(t+0)]\,, \tag{21}$$

where $[u(t-0), u(t+0)]$ denotes the line segment spanned by the vectors $u(t-0)$ and $u(t+0)$ in $\mathbb{R}^{d(u)}$. Note that $[u(t-0), u(t+0)]$ is a singleton $\{u(t)\}$ at each continuity point of the control $u(t)$ with $u(t)$ being a vertex of the polyhedron U. Only at the points $t_k \in \theta$ does the line segment $[u^{k-}, u^{k+}]$ coincide with an edge of the polyhedron.

If the control is *scalar*, $d(u) = 1$ and $U = [u_{\min}, u_{\max}]$, then the bang–bang property is equivalent to

$$\sigma(t) \neq 0 \quad \forall\, t \in \Delta \setminus \theta$$

which yields the control law

$$u(t) = \left\{ \begin{array}{ll} u_{\min}, & \text{if} \quad \sigma(t) > 0 \\ u_{\max}, & \text{if} \quad \sigma(t) < 0 \end{array} \right\} \quad \forall\, t \in \Delta \setminus \theta. \tag{22}$$

For vector–valued control inputs, condition (21) imposes further restrictions. For example, if U is the unit cube in $\mathbb{R}^{d(u)}$, condition (21) precludes simultaneous switching of the control components. This property holds in many examples. Condition (21) will be indispensable in the sensitivity analysis of optimal bang–bang controls.

3. Quadratic necessary and sufficient optimality conditions

In this section, we shall formulate a quadratic necessary optimality condition of a Pontryagin minimum (Definition 2.1) for given bang–bang control. A strengthening of this quadratic condition yields a quadratic sufficient condition for a strong minimum (Definition 2.3). These quadratic conditions are based on the properties of a quadratic form on the so called critical cone, whose elements are first order variations along a given trajectory $\mathcal{T}$. The main results of this section (Theorems 3.1 and 3.2) are due to Osmolovskii, see Milyutin and Osmolovskii (1998), Part 2, Chapter 3. Proofs missing in this book will appear in Osmolovskii (2003).

3.1. Critical cone

For a given trajectory $\mathcal{T}$ we introduce the space $\mathcal{Z}(\theta)$ and the *critical cone* $\mathcal{K} \subset \mathcal{Z}(\theta)$. Denote by $P_\theta C^1(\Delta, \mathbb{R}^{d(x)})$ the space of piecewise continuous functions

$$\bar{x}(\cdot) : \Delta \to \mathbb{R}^{d(x)},$$

continuously differentiable on each interval of the set $\Delta \setminus \theta$. For each $\bar{x} \in P_\theta C^1(\Delta, \mathbb{R}^{d(x)})$ and for $\tau_k \in \theta$ we set

$$\bar{x}^{k-} = \bar{x}(\tau_k - 0), \quad \bar{x}^{k+} = \bar{x}(\tau_k + 0), \quad [\bar{x}]^k = \bar{x}^{k+} - \bar{x}^{k-}.$$

Set $\bar{z} = (\bar{t}_0, \bar{t}_1, \xi, \bar{x})$, where $\bar{t}_0, \bar{t}_1 \in \mathbb{R}^1$, $\xi \in \mathbb{R}^s$, $\bar{x} \in P_\theta C^1(\Delta, \mathbb{R}^{d(x)})$. Thus,

$$\bar{z} \in \mathcal{Z}(\theta) := \mathbb{R}^2 \times \mathbb{R}^s \times P_\theta C^1(\Delta, \mathbb{R}^{d(x)}).$$

For each $\bar{z}$ we set

$$\tilde{x}_0 = \bar{x}(t_0) + \bar{t}_0 \dot{x}(t_0), \quad \tilde{x}_1 = \bar{x}(t_1) + \bar{t}_1 \dot{x}(t_1), \quad \tilde{p} = (\bar{t}_0, \tilde{x}_0, \bar{t}_1, \tilde{x}_1). \tag{23}$$

The vector $\tilde{p}$ is considered as a column vector. Note that $\bar{t}_0 = 0$, respectively, $\bar{t}_1 = 0$ for *fixed* initial time t_0, respectively, final time t_1. Denote by $I_F(p) = \{i \in \{1, \dots, d(F)\} \mid F_i(p) = 0\}$ the set of indices of all active endpoint inequalities $F_i(p) \le 0$ at the point $p = (t_0, x(t_0), t_1, x(t_1))$. Denote by $\mathcal{K}$ the set of all $\bar{z} \in \mathcal{Z}(\theta)$ satisfying the following conditions:

$$J'(p)\tilde{p} \le 0, \quad F_i'(p)\tilde{p} \le 0 \; \forall\, i \in I_F(p), \quad K'(p)\tilde{p} = 0, \tag{24}$$

$$\dot{\bar{x}}(t) = f_x'(t, x(t), u(t))\bar{x}(t), \quad [\bar{x}]^k = [\dot{x}]^k \xi_k, \quad k = 1, \dots, s, \tag{25}$$

where $p = (x(t_0), t_0, x(t_1), t_1)$.

It is obvious that $\mathcal{K}$ is a convex finite-dimensional and finite-faced cone in the space $\mathcal{Z}(\theta)$. We call it *the critical cone*. Each element $\bar{z} \in \mathcal{K}$ is uniquely defined by numbers $\bar{t}_0$, $\bar{t}_1$, a vector ξ and the initial value $\bar{x}(t_0)$ of the function $\bar{x}(t)$.

Proposition 3.1 *For any $\lambda \in M_0$ and $\bar{z} \in \mathcal{K}$ we have*

$$l_{x_0}\bar{x}(t_0) + l_{x_1}\bar{x}(t_1) = 0. \tag{26}$$

Proof. Integrating the equality $\psi(\dot{\bar{x}} - f_x \bar{x}) = 0$ on $[t_0, t_1]$ and using the adjoint equation $\dot{\psi} = -\psi f_x$ we obtain $\int\limits_{t_0}^{t_1} \frac{d}{dt}(\psi \bar{x})\, dt = 0$, whence

$$(\psi\bar{x})|_{t_0}^{t_1} - \sum_{k=1}^{s} [\psi\bar{x}]^k = 0.$$

From the jump conditions $[\bar{x}]^k = [\dot{x}]^k \xi_k$ and the equality $\psi(t)\dot{x}(t) + \psi_0(t) = 0$ it follows that

$$[\psi\bar{x}]^k = \psi(\tau_k)[\dot{x}]^k \xi_k = [\psi\dot{x}]^k \xi_k = -[\psi_0]^k \xi_k = 0 \quad \forall\, k.$$

Then the equation $(\psi\bar{x})|_{t_0}^{t_1} = 0$ together with the transversality conditions $\psi(t_0) = -l_{x_0}$ and $\psi(t_1) = l_{x_1}$ imply (26). ■

PROPOSITION 3.2 *For any $\lambda \in M_0$ and $\bar{z} \in \mathcal{K}$ we have*

$$\sum_{i=1}^{s} \alpha_i F_{ip}\tilde{p} + \beta K_p \tilde{p} = 0. \tag{27}$$

Proof. For $\lambda \in M_0$ and $\bar{z} \in \mathcal{K}$, we have by Propositions 2.3 and 3.1

$$\bar{t}_0(l_{x_0}\dot{x}(t_0) + l_{t_0}) + \bar{t}_1(l_{x_1}\dot{x}(t_1) + l_{t_1}) + l_{x_0}\bar{x}(t_0) + l_{x_1}\bar{x}(t_1) = 0\,.$$

Now using the equalities $\tilde{x}_0 = \bar{x}(t_0) + \bar{t}_0\dot{x}(t_0)$, $\tilde{x}_1 = \bar{x}(t_1) + \bar{t}_1\dot{x}(t_1)$, and $\tilde{p} = (\bar{t}_0, \tilde{x}_0, \bar{t}_1, \tilde{x}_1)$ we get $l_p\tilde{p} = 0$, which is equivalent to condition (27). ■

Two important properties of the critical cone follow from Proposition 3.2.

PROPOSITION 3.3 *For any $\lambda \in M_0$ and $\bar{z} \in \mathcal{K}$, we have*

$$\alpha_0 J'(p)\tilde{p} = 0, \quad \alpha_i F_i'(p)\tilde{p} = 0 \quad \forall i \in I_F(p).$$

PROPOSITION 3.4 *Suppose that there exist $\lambda \in M_0$ with $\alpha_0 > 0$. Then adding the equalities*

$$\alpha_i F_i'(p)\tilde{p} = 0 \; \forall i \in I_F(p),$$

to the system (24), (25) defining $\mathcal{K}$, one can omit the inequality

$$J'(p)\tilde{p} \le 0,$$

in that system without affecting $\mathcal{K}$.

Thus, $\mathcal{K}$ is defined by condition (25) and by the condition $\tilde{p} \in \mathcal{K}_0$, where $\mathcal{K}_0$ is the cone in $\mathbb{R}^{2d(x)+2}$ given by (24). But if there exists $\lambda \in M_0$ with $\alpha_0 > 0$, then we can put

$$\mathcal{K}_0 = \{\tilde{p} \in \mathbb{R}^{d(x)+2} \mid F_i'(p)\tilde{p} \le 0,\; \alpha_i F_i'(p)\tilde{p} = 0 \;\forall\, i \in I_F(p),\; K'(p)\tilde{p} = 0\}. \tag{28}$$

If, in addition, $\alpha_i > 0$ holds for all $i \in I_F(p)$, then $\mathcal{K}_0$ is a subspace in $\mathbb{R}^{d(x)+2}$.

An explicit representation of the variations $\bar{x}(t)$ in (25) is obtained as follows. For each $k = 1, ..., s$, define the vector functions $y^k(t)$ as the solutions to the system

$$\dot{y} = f_x(t)y, \quad y(\tau_k) = [\dot{x}]^k, \quad t \in [\tau_k, t_1].$$

For $t < \tau_k$ we put $y^k(t) = 0$ which yields the jump $[y^k]^k = [\dot{x}]^k$. Moreover, define $y^0(t)$ as the solution to the system

$$\dot{y} = f_x(t)y, \quad y(t_0) = \bar{x}(t_0) =: \bar{x}_0.$$

By the superposition principle for linear ODEs it is obvious that we have

$$\bar{x}(t) = \sum_{k=1}^{s} y^k(t)\xi_k \; + y^0(t)$$

from which we obtain the representation

$$\tilde{x}_1 = \sum_{k=1}^{s} y^k(t_1)\xi_k \; + y^0(t_1) + \dot{x}(t_1)\bar{t}_1 \,. \tag{29}$$

Furthermore, denote by $x(t;\tau_1,...,\tau_s)$ the solution of the state equation (2) using the values of the optimal bang–bang control with switching points $\tau_1, ...\tau_s$. It easily follows from elementary properties of ODEs that the partial derivatives of state trajectories w.r.t. to the switching points is given by

$$\frac{\partial x}{\partial \tau_k}(t;\tau_1,...,\tau_s) = -y^k(t) \quad \text{for} \;\; t \geq \tau_k, \;\; k = 1,...,s. \tag{30}$$

This gives the following expression for $\bar{x}(t)$:

$$\bar{x}(t) = -\sum_{k=1}^{s} \frac{\partial x}{\partial \tau_k}(t)\xi_k \; + y^0(t). \tag{31}$$

In a special case that frequently arises in practice, we can use these formulas to show that $\mathcal{K} = \{0\}$. This property then yields a first order sufficient condition in view of Theorem 3.2. Namely, consider the problem with an integral cost functional (16) where the initial time $t_0 = \hat{t}_0$ is fixed, while the final time t_1 is free, and where the initial and final values of the state variables are given: minimize

$$\mathcal{J} = \int_{t_0}^{t_1} f_0(t,x,u)dt \tag{32}$$

subject to

$$\dot{x} = f(t,x,u), \quad x(t_0) = \hat{x}_0, \; x(t_1) = \hat{x}_1, \quad u(t) \in U. \tag{33}$$

In the definition of $\mathcal{K}$ we then have $\bar{t}_0 = 0$, $\bar{x}(t_0) = 0$, $\tilde{x}(t_1) = 0$. The condition $\bar{x}(t_0) = 0$ implies that $y^0(t) \equiv 0$ whereas the condition $\tilde{x}(t_1) = 0$ yields in view of the representation (29)

$$\sum_{k=1}^{s} y^k(t_1)\xi_k \; + \dot{x}(t_1)\bar{t}_1 = 0.$$

This equation leads to the following statement:

Proposition 3.5 *In problem (32), (33), assume that the $s+1$ vectors*

$$y^k(t_1) = -\frac{\partial x}{\partial \tau_k}(t_1) \ \ (k=1,...,s), \ \ \dot{x}(t_1)$$

are linearly independent. Then the critical cone is $\mathcal{K} = \{0\}$.

We conclude this subsection with a special property of the critical cone for time–optimal control problems with fixed initial time and state,

$$t_1 \ \to \ \min, \ \ \dot{x} = f(t,x,u), \ u \in U, \ \ t_0 = \hat{t}_0, \ \ x(t_0) = \hat{x}_0, \ \ K(x(t_1)) = 0, \qquad (34)$$

where f is defined by (4). The following result generalizes Proposition 3.1 from Maurer, Osmolovskii (2001) and will be used in Example 5.2 to simplify the critical cone.

Proposition 3.6 *Suppose that there exists $(\psi_o, \psi) \in M_0$ such that $\alpha_0 > 0$. Then $\bar{t}_1 = 0$ holds for each $\bar{z} = (\bar{t}_1, \xi, \bar{x}) \in \mathcal{K}$.*

Proof. For arbitrary $(\psi_0, \psi) \in M_0$ and $\bar{z} = (\bar{t}_1, \xi, \bar{x}) \in \mathcal{K}$ we infer from the proof of Proposition 3.1 that $\psi(t)\bar{x}(t)$ is a constant function on $[t_0, t_1]$. In view of the relations $\psi(t_1) = \beta K_{x_1}(x(t_1))$, $K_{x_1}(x(t_1))\tilde{x}_1 = 0$ and $\tilde{x}_1 = \bar{x}(t_1) + \dot{x}(t_1)\bar{t}_1$ we get

$$0 = (\psi\bar{x})(t_0) = (\psi\bar{x})(t_1) = \psi(t_1)(\tilde{x}_1 - \dot{x}(t_1)\bar{t}_1) = -\psi(t_1)\dot{x}(t_1)\bar{t}_1 = \psi_0(t_1)\bar{t}_1\,.$$

Since $\psi_0(t_1) = \alpha_0 > 0$, this relation yields $\bar{t}_1 = 0$. ■

In the case of $\alpha_0 > 0$ we note as a consequence that the critical cone is a subspace defined by the conditions

$$\begin{array}{l} \dot{\bar{x}} = f_x(t)\bar{x}, \quad [\bar{x}]^k = [\dot{x}]^k\xi_k \quad (k=1,...,s), \\ \bar{t}_0 = \bar{t}_1 = 0, \quad \bar{x}(t_0) = 0, \quad K_{x_1}(x(t_1))\bar{x}(t_1) = 0. \end{array} \qquad (35)$$

3.2. Quadratic necessary optimality conditions

Let us introduce a quadratic form on the critical cone $\mathcal{K}$ defined by the conditions (24), (25). For each $\lambda \in M_0$ and $\bar{z} \in \mathcal{K}$ we set

$$\Omega(\lambda, \bar{z}) = \langle A\tilde{p}, \tilde{p}\rangle + \sum_{k=1}^{s}\left(D^k(H)\xi_k^2 + 2[H_x]^k\bar{x}^k_{\rm av}\xi_k\right) + \int_{\Delta}\langle H_{xx}\bar{x}(t), \bar{x}(t)\rangle\, dt, \quad (36)$$

where

$$\begin{array}{rcl} \langle A\tilde{p}, \tilde{p}\rangle & = & \langle l_{pp}\tilde{p}, \tilde{p}\rangle + 2\dot{\psi}(t_0)\tilde{x}_0\bar{t}_0 + (\dot{\psi}_0(t_0) - \dot{\psi}(t_0)\dot{x}(t_0))\bar{t}_0^2 \\ & & -2\dot{\psi}(t_1)\tilde{x}_1\bar{t}_1 - (\dot{\psi}_0(t_1) - \dot{\psi}(t_1)\dot{x}(t_1))\bar{t}_1^2, \end{array} \qquad (37)$$

$$l_{pp} = l_{pp}(\alpha_0, \alpha, \beta, p), \ \ p = (t_0, x(t_0), t_1, x(t_1)),$$

$$H_{xx} = H_{xx}(t, x(t), u(t), \psi(t)), \quad \bar{x}^k_{\mathrm{av}} = \frac{1}{2}(\bar{x}^{k-} + \bar{x}^{k+}).$$

Note that the functional $\Omega(\lambda, \bar{z})$ is linear in λ and quadratic in $\bar{z}$. Also note that for a problem on a fixed time interval $[t_0, t_1]$ we have $\bar{t}_0 = \bar{t}_1 = 0$ and, hence, the quadratic form (37) reduces to $\langle A\tilde{p}, \tilde{p}\rangle = \langle l_{pp}\tilde{p}, \tilde{p}\rangle$. The following theorem gives the main second order necessary condition of optimality.

THEOREM 3.1 *If the trajectory $\mathcal{T}$ affords a Pontryagin minimum, then the following Condition $\mathcal{A}$ holds: the set M_0 is nonempty and*

$$\max_{\lambda \in M_0} \Omega(\lambda, \bar{z}) \geq 0 \quad \textit{for all} \;\; \bar{z} \in \mathcal{K}.$$

We call Condition $\mathcal{A}$ the necessary quadratic condition, although it is truly quadratic only if M_0 is a singleton. In the last case we have an *accessory problem*: minimize the quadratic form Ω on the critical cone $\mathcal{K}$.

3.3. Quadratic sufficient optimality conditions

A natural strengthening of the necessary Condition $\mathcal{A}$ turns out to be a sufficient optimality condition not only for a Pontryagin minimum, but also for a strong minimum, see Definition 2.3. The following result has been obtained in Milyutin, Osmolovskii (1998), Part 2, Chapter 3, section 12.4, and Osmolovskii (2003).

THEOREM 3.2 *Let the following Condition $\mathcal{B}$ be fulfilled for the trajectory $\mathcal{T}$:*
- *(a) $u(t)$ is a bang-bang control for which condition (21) holds,*
- *(b) there exists $\lambda \in M_0$ such that $D^k(H) > 0, k = 1, \ldots, s$,*
- *(c) $\max_{\lambda \in M_0} \Omega(\lambda, \bar{z}) > 0$ for all $\bar{z} \in \mathcal{K} \setminus \{0\}$.*

Then $\mathcal{T}$ is a strict strong minimum.

Note that the condition (c) is automatically fulfilled, if $\mathcal{K} = \{0\}$, which gives a first order sufficient condition for a strong minimum in the problem. A specific situation where $\mathcal{K} = \{0\}$ holds was described in Proposition 3.5. Also note that the condition (c) is automatically fulfilled if there exists $\lambda \in M_0$ such that

$$\Omega(\lambda, \bar{z}) > 0 \text{ for all } \bar{z} \in \mathcal{K} \setminus \{0\}. \tag{38}$$

Example: Resource allocation problem. Let $x(t)$ be the stock of a resource and let the control $u(t)$ be the investment rate. The control problem is to *maximize* the overall consumption

$$\int_0^{t_1} x(t)(1 - u(t))\, dt$$

on a fixed time interval $[0, t_1]$ subject to

$$\dot{x}(t) = x(t)\, u(t), \quad x(0) = x_0 > 0, \quad 0 \leq u(t) \leq 1\,.$$

The Pontryagin function (5) for the equivalent minimization problem is

$$H = \alpha_0 x(u-1) + \psi\, xu = -\alpha_0 x + \sigma u, \quad \sigma(x,\psi) = x(\alpha_0 + \psi).$$

A straightforward discussion of the minimum principle shows that the optimal solution has one switching point $\tau_1 = t_1 - 1$ for $t_1 > 1$. Moreover, we can take $\alpha_0 = 1$ and find

$$u(t) = \left\{ \begin{array}{lll} 1 & , & 0 \le t \le \tau_1 \\ 0 & , & \tau_1 \le t \le t_1 \end{array} \right\},$$

$$(x(t),\psi(t)) = \left\{ \begin{array}{lll} (x_0 e^t, -e^{-(t-\tau_1)}) & , & 0 \le t \le \tau_1 \\ (x_0 e^{\tau_1}, t - t_1) & , & \tau_1 \le t \le t_1 \end{array} \right\}$$

The switching function is $\sigma(t) = x(t)(1-\psi(t))$ for which we compute $\dot\sigma(\tau_1) = x_0 e^{\tau_1} \ne 0$. Here we have $k = 1$, $[u]^1 = -1$ and thus obtain $D^1(H) = -\dot\sigma(\tau_1)[u]^1 = \dot\sigma(\tau_1) > 0$ in view of (12) and (14). Hence, conditions (a) and (b) of Theorem 3.2 hold. The check of condition (c) is rather simple since the quadratic form (36) reduces here to $\Omega(\lambda,\bar z) = D^1(H)\xi_1^2$. This relation follows from $H_{xx} \equiv 0$ and $[H_x]^1 = (1+\psi(\tau_1))[u]^1 = 0$ and the fact that the quadratic form (37) vanishes. Note that the above control problem can not be handled in the class of *convex* optimization problems. This means that the necessary conditions do not automatically imply optimality of the computed solution.

We conclude this subsection with the case of a *time–optimal* control problem (34) with a *single switching point*, i.e., $s = 1$. Assume that $\alpha_0 > 0$ for a given $\lambda \in M_0$. Then, by Proposition 3.6 we have $\bar t_1 = 0$ and thus the critical cone is the subspace defined by (35). In this case, the quadratic form Ω can be computed explicitly as follows. Denote by $y(t)$, $t \in [\tau_1, t_1]$, the solution to the Cauchy problem

$$\dot y = f_x y, \quad y(\tau_1) = [\dot x]^1.$$

The following assertion is obvious: if $(\xi,\bar x) \in \mathcal{K}$, then $\bar x(t) = 0$ for $t \in [t_0,\tau_1)$ and $\bar x(t) = y(t)\xi$ for $t \in (\tau_1, t_1]$. Therefore, the inequality $K_{x_1}(x(t_1))y(t_1) \ne 0$ would imply $\mathcal{K} = \{0\}$. Consider now the case $K_{x_1}(x(t_1))y(t_1) = 0$. This condition always holds for time–optimal problems with a *scalar* function K and $\alpha_0 > 0$. Indeed, the condition $\frac{d}{dt}(\psi y) = 0$ implies $(\psi y)(t) = const.$ in $[\tau_1, t_1]$, whence

$$(\psi y)(t_1) = (\psi y)(\tau_1) = \psi(\tau_1)[\dot x]^1 = \sigma(\tau_1)[u]^1 = 0.$$

Using the transversality condition $\psi(t_1) = \beta K_{x_1}(x(t_1))$ and the inequality $\beta \ne 0$ (if $\beta = 0$, then $\psi(t_1) = 0$ and hence $\psi(t) = 0$ and $\psi_0(t) = 0$ in $[t_0, t_1]$) we see that the equality $(\psi y)(t_1) = 0$ implies the equality $K_{x_1}(x(t_1)y(t_1) = 0$.

Observe now that the cone $\mathcal{K}$ is a one–dimensional subspace on which the quadratic form has the representation $\Omega = \rho\xi^2$, where

$$\rho := D^1(H) - [\dot\psi]^1[\dot x]^1 + \int_{\tau_1}^{t_1} (y(t))^* H_{xx}(t) y(t)\, dt + (y(t_1))^*(\beta K)_{x_1x_1} y(t_1). \quad (39)$$

This gives the following result.

Proposition 3.7 *Suppose that we have found an extremal for the time–optimal control problem (35) that has one switching point and satisfies* $\alpha_0 > 0$ *and* $K_{x_1}(x(t_1)y(t_1) = 0$. *Then the inequality* $\rho > 0$ *with* ρ *defined in (39) is equivalent to the positive definiteness of* Ω *on* $\mathcal{K}$.

4. Sufficient conditions for positive definiteness of the quadratic form Ω on the critical cone $\mathcal{K}$

Assume that the following conditions are fulfilled for the trajectory $\mathcal{T}$:

(i) $u(t)$ is a bang-bang control with $s \geq 1$ switching points;

(ii) there exists $\lambda \in M_0$ such that $D^k(H) > 0,\ k = 1, \dots, s$.

Let $\lambda \in M_0$ be a fixed element (possibly, different from that in the assumption (ii)) and let $\Omega = \Omega(\lambda, \cdot)$ be the quadratic form (36) for this element. According to Theorem 3.2, *the positive definiteness of* Ω *on the critical cone* $\mathcal{K}$ *is a sufficient condition for a strict strong minimum of the trajectory.* Recall that $\mathcal{K}$ is defined by (25) and the condition $\tilde{p} \in \mathcal{K}_0$ where $\tilde{p} = (\bar{t}_0, \tilde{x}_0, \bar{t}_1, \tilde{x}_1)$, $\tilde{x}_0 = \bar{x}(t_0) + \bar{t}_0\dot{x}(t_0)$, $\tilde{x}_1 = \bar{x}(t_1) + \bar{t}_1\dot{x}(t_1)$. The cone $\mathcal{K}_0$ is defined by (28) in the case $\alpha_0 > 0$ and by (24) in the general case.

Now our aim is to find sufficient conditions for the positive definiteness of the quadratic form Ω on the cone $\mathcal{K}$. In what follows we shall use some ideas and results presented in Maurer, Osmolovskii (2001) and in Osmolovskii, Lempio (2002), who have extended the Riccati approach from Maurer, Pickenhain (1995), Zeidan (1994) to broken extremals.

4.1. Q-transformation of Ω on $\mathcal{K}$

Let $Q(t)$ be a symmetric matrix on $[t_0, t_1]$ with piecewise continuous entries which are absolutely continuous on each interval of the set $[t_0, t_1] \setminus \theta$. Therefore, Q may have a jump at each point $\tau_k \in \theta$. For $\bar{z} \in \mathcal{K}$ we obviously have

$$\int_{t_0}^{t_1} \frac{d}{dt}\langle Q\bar{x}, \bar{x}\rangle\, dt = \langle Q\bar{x}, \bar{x}\rangle\Big|_{t_0}^{t_1} - \sum_{k=1}^{s} [\langle Q\bar{x}, \bar{x}\rangle]^k\,,$$

where $[\langle Q\bar{x}, \bar{x}\rangle]^k$ is the jump of the function $\langle Q\bar{x}, \bar{x}\rangle$ at the point $\tau_k \in \theta$. Using the equation $\dot{\bar{x}} = f_x\bar{x}$ with $f_x = f_x(t, x(t), u(t))$, we obtain

$$\sum_{k=1}^{s} [\langle Q\bar{x}, \bar{x}\rangle]^k + \int_{t_0}^{t_1} \langle(\dot{Q} + f_x^* Q + Q f_x)\bar{x}, \bar{x}\rangle\, dt - \langle Q\bar{x}, \bar{x}\rangle(t_1) + \langle Q\bar{x}, \bar{x}\rangle(t_0) = 0,$$

where the asterisk denotes transposition. Adding this zero-form to Ω and using the equality $[H_x]^k = -[\dot\psi]^k$ we get

$$\begin{aligned}\Omega \;=\; & \langle A\tilde p, \tilde p\rangle - \langle Q\bar x, \bar x\rangle(t_1) + \langle Q\bar x, \bar x\rangle(t_0)\\ &+ \sum_{k=1}^{s}\Big(D^k(H)\xi_k^2 - 2[\dot\psi]^k \bar x_{\rm av}^k \xi_k + [\langle Q\bar x, \bar x\rangle]^k\Big)\\ &+ \int_{t_0}^{t_1} \langle (H_{xx} + \dot Q + f_x^* Q + Q f_x)\bar x, \bar x\rangle\, dt.\end{aligned} \tag{40}$$

We shall call this formula the *Q-transformation of Ω on $\mathcal{K}$.*

In order to eliminate the integral term in Ω we assume that $Q(t)$ satisfies the following linear matrix differential equation,

$$\dot Q + f_x^* Q + Q f_x + H_{xx} = 0 \quad \text{on } [t_0, t_1] \setminus \theta. \tag{41}$$

It is interesting to note that the same equation is obtained from the modified Riccati equation in Maurer, Pickenhain (1995), equation (47), when all control variables are on the boundary of the control constraints. Using (41) the quadratic form (40) reduces to

$$\Omega = \omega_0 + \sum_{k=1}^{s} \omega_k\,, \tag{42}$$

$$\omega_k := D^k(H)\xi_k^2 - 2[\dot\psi]^k \bar x_{\rm av}^k \xi_k + [\langle Q\bar x, \bar x\rangle]^k\,, \quad k = 1, \ldots, s, \tag{43}$$

$$\omega_0 := \langle A\tilde p, \tilde p\rangle - \langle Q\bar x, \bar x\rangle(t_1) + \langle Q\bar x, \bar x\rangle(t_0). \tag{44}$$

Thus, we have proved the following statement:

PROPOSITION 4.1 *Let $Q(t)$ satisfy the linear differential equation (41) on $[t_0, t_1] \setminus \theta$. Then for each $\bar z \in \mathcal{K}$ the representation (42) holds.*

Now our goal is to derive conditions such that $\omega_k > 0$, $k = 0, \ldots, s$, holds on $\mathcal{K} \setminus \{0\}$. To this end we shall express ω_k via the vector $(\xi_k, \bar x^{k-})$. We use the formula

$$\bar x^{k+} = \bar x^{k-} + [\dot x]^k \xi_k, \tag{45}$$

which implies

$$\langle Q^{k+}\bar x^{k+}, \bar x^{k+}\rangle = \langle Q^{k+}\bar x^{k-}, \bar x^{k-}\rangle + 2\langle Q^{k+}[\dot x]^k, \bar x^{k-}\rangle \xi_k + \langle Q^{k+}[\dot x]^k, [\dot x]^k\rangle \xi_k^2.$$

Consequently,

$$[\langle Q\bar x, \bar x\rangle]^k = \langle [Q]^k \bar x^{k-}, \bar x^{k-}\rangle + 2\langle Q^{k+}[\dot x]^k, \bar x^{k-}\rangle \xi_k + \langle Q^{k+}[\dot x]^k, [\dot x]^k\rangle \xi_k^2.$$

Using this relation together with

$$\bar x_{\rm av}^k = \bar x^{k-} + \frac{1}{2}[\dot x]^k \xi_k$$

in the definition (43) of ω_k, we obtain

$$\begin{aligned}\omega_k &= \{D^k(H) + \Big(([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k\Big)[\dot{x}]^k\}\,\xi_k^2 \\ &\quad +2\Big(([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k\Big)\bar{x}^{k-}\xi_k + (\bar{x}^{k-})^*[Q]^k\bar{x}^{k-}.\end{aligned} \tag{46}$$

Here $[\dot{x}]^k$ and $\bar{x}^{k-}$ are column vectors, while $([\dot{x}]^k)^*$, $(\bar{x}^{k-})^*$ and $[\dot{\psi}]^k$ are row vectors. By putting

$$q_{k+} = ([\dot{x}]^k)^* Q^{k+} - [\dot{\psi}]^k, \quad b_{k+} = D^k(H) + (q_{k+})[\dot{x}]^k \tag{47}$$

we get

$$\omega_k = (b_{k+})\xi_k^2 + 2(q_{k+})\bar{x}^{k-}\xi_k + (\bar{x}^{k-})^*[Q]^k\bar{x}^{k-}. \tag{48}$$

Note that ω_k is a quadratic form in the variables $(\xi_k, \bar{x}^{k-})$ with the matrix

$$M_{k+} = \begin{pmatrix} b_{k+} & q_{k+} \\ (q_{k+})^* & [Q]^k \end{pmatrix}, \tag{49}$$

where q_{k+} is a row vector and $(q_{k+})^*$ is a column vector.

Similarly, using the relation

$$\bar{x}^{k-} = \bar{x}^{k+} - [\dot{x}]^k\xi_k,$$

we obtain

$$[\langle Q\bar{x}, \bar{x}\rangle]^k = \langle [Q]^k\bar{x}^{k+}, \bar{x}^{k+}\rangle + 2\langle Q^{k-}[\dot{x}]^k, \bar{x}^{k+}\rangle\xi^k - \langle Q^{k-}[\dot{x}]^k, [\dot{x}]^k\rangle\xi_k^2.$$

This formula, together with the relation

$$\bar{x}^k_{\text{av}} = \bar{x}^{k+} - \frac{1}{2}[\dot{x}]^k\xi_k,$$

leads to the representation

$$\omega_k = (b_{k-})\xi_k^2 + 2(q_{k-})\bar{x}^{k+}\xi_k + (\bar{x}^{k+})^*[Q]^k\bar{x}^{k+}, \tag{50}$$

where

$$q_{k-} = ([\dot{x}]^k)^* Q^{k-} - [\dot{\psi}]^k, \quad b_{k-} = D^k(H) - (q_{k-})[\dot{x}]^k. \tag{51}$$

We consider (50) as a quadratic form in the variables $(\xi_k, \bar{x}^{k+})$ with the matrix

$$M_{k-} = \begin{pmatrix} b_{k-} & q_{k-} \\ (q_{k-})^* & [Q]^k \end{pmatrix}. \tag{52}$$

Since the right hand sides of equalities (48) and (50) are connected by the relation (45), the following statement obviously holds.

PROPOSITION 4.2 *For each $k = 1, \ldots, s$, the positive (semi)definiteness of the matrix M_{k-} is equivalent to the positive (semi)definiteness of the matrix M_{k+}.*

Now we can prove two theorems.

THEOREM 4.1 *Assume that $s = 1$. Let $Q(t)$ be a solution of the linear differential equation (41) on $[t_0, t_1] \setminus \theta$ which satisfies two conditions:*
(i) the matrix M_{1+} is positive semidefinite and
(ii) the quadratic form ω_0 is positive on the cone $\mathcal{K}_0 \setminus \{0\}$.
Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

Proof. Take an arbitrary element $\bar{z} \in \mathcal{K}$. Conditions (i) and (ii) imply that $\omega_k \geq 0$ for $k = 0, 1$, and hence $\Omega = \omega_0 + \omega_1 \geq 0$ for this element. Assume now that $\Omega = 0$. Then, $\omega_k = 0$ for $k = 0, 1$. In virtue of (ii) the equality $\omega_0 = 0$ implies that $\bar{t}_0 = \bar{t}_1 = 0$ and $\bar{x}(t_0) = \bar{x}(t_1) = 0$. The last two equalities together with equation $\dot{\bar{x}} = f_x \bar{x}$ show that $\bar{x}(t) = 0$ in $[t_0, \tau_1) \cup (\tau_1, t_1]$. Now using formula (43) for $\omega_1 = 0$, as well as the conditions $D^1(H) > 0$ and $\bar{x}^{1-} = \bar{x}^{1+} = 0$ we obtain that $\xi_1 = 0$. Consequently, we have $\bar{z} = 0$ which means that Ω is positive on $\mathcal{K} \setminus \{0\}$. ■

THEOREM 4.2 *Assume that $s \geq 2$. Let $Q(t)$ be a solution of the linear differential equation (41) on $[t_0, t_1] \setminus \theta$, which satisfies the following conditions:*
(a) the matrix M_{k+} is positive semidefinite for each $k = 1, \ldots, s$;
(b) $b_{k+} := D^k(H) + (q_{k+})[\dot{x}]^k > 0$ for each $k = 1, \ldots, s-1$;
(c) the quadratic form ω_0 is positive on the cone $\mathcal{K}_0 \setminus \{0\}$.
Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

Proof. Take an arbitrary element $\bar{z} \in \mathcal{K}$. Conditions (a) and (c) imply that $\omega_k \geq 0$ for $k = 0, 1, \ldots, s$ and hence $\Omega \geq 0$ for this element.

Assume that $\Omega = 0$. Then $\omega_k = 0$ for $k = 0, 1, \ldots, s$. In virtue of (c) the equality $\omega_0 = 0$ implies that $\bar{t}_0 = \bar{t}_1 = 0$ and $\bar{x}(t_0) = \bar{x}(t_1) = 0$. The last two equalities together with equation $\dot{\bar{x}} = f_x \bar{x}$ show that $\bar{x}(t) = 0$ in $[t_0, \tau_1) \cup (\tau_s, t_1]$ and hence $\bar{x}^{1-} = \bar{x}^{s+} = 0$. The conditions $\omega_1 = 0$, $\bar{x}^{1-} = 0$ and $b_{1+} > 0$ by formula (48) (with $k = 1$) yield $\xi_1 = 0$. Then $[\bar{x}]^1 = 0$ and hence $\bar{x}^{1+} = 0$. The last equality together with equation $\dot{\bar{x}} = f_x \bar{x}$ show that $\bar{x}(t) = 0$ in (t_1, t_2) and hence $\bar{x}^{2-} = 0$. Similarly, the conditions $\omega_2 = 0$, $\bar{x}^{2-} = 0$ and $b_{2+} > 0$ by formula (48) (with $k = 2$) imply that $\xi_2 = 0$ and $\bar{x}(t) = 0$ in (t_2, t_3). Therefore, $\bar{x}^{3-} = 0$, etc. Continuing this process we get that $\bar{x} \equiv 0$ and $\xi_k = 0$ for $k = 1, \ldots, s-1$. Now, using formula (43) for $\omega_s = 0$, as well as the conditions $D^s(H) > 0$ and $\bar{x} \equiv 0$ we obtain that $\xi_s = 0$. Consequently, we have $\bar{z} = 0$ which means that Ω is positive on $\mathcal{K} \setminus \{0\}$. ■

Similarly, using representation (50) for ω_k we can prove the following statement:

THEOREM 4.3 *Let $Q(t)$ be a solution of the linear differential equation (41) on $[t_0, t_1] \setminus \theta$ which satisfies the following conditions:*
(a') the matrix M_{k-} is positive semidefinite for each $k = 1, \dots, s$;
(b') $b_{k-} := D^k(H) - (q_{k-})[\dot{x}]^k > 0$ for each $k = 2, \dots, s$ (if $s = 1$, then this condition is not required);
(c) the quadratic form ω_0 is positive on the cone $\mathcal{K}_0 \setminus \{0\}$.
Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

4.2. The case of fixed initial values t_0 and $x(t_0)$

Consider the problem (1)-(3) with additional constraints $t_0 = \hat{t}_0$ and $x(t_0) = \hat{x}_0$. In this case we have additional equalities in the definition of the critical cone $\mathcal{K}$: $\bar{t}_0 = 0$ and $\tilde{x}_0 := \bar{x}(t_0)+\bar{t}_0\dot{x}(t_0) = 0$ whence $\bar{x}(t_0) = 0$. The last equality together with the equation $\dot{\bar{x}} = f_x\bar{x}$ shows that $\bar{x}(t) = 0$ in $[t_0, \tau_1)$ whence $\bar{x}^{1-} = 0$. From definitions (44) and (37) of ω_0 and $\langle A\tilde{p}, \tilde{p}\rangle$, respectively, it follows that for each $\bar{z} \in \mathcal{K}$ we have

$$\omega_0 = \langle A_1\tilde{p}, \tilde{p}\rangle - \langle Q(t_1)(\tilde{x}_1 - \bar{t}_1\dot{x}(t_1)), (\tilde{x}_1 - \bar{t}_1\dot{x}(t_1))\rangle, \tag{53}$$

where

$$\begin{aligned}\langle A_1\tilde{p}, \tilde{p}\rangle \;\; = \;\; & l_{t_1t_1}\bar{t}_1^2 + 2l_{t_1x_1}\tilde{x}_1\bar{t}_1 + \langle l_{x_1x_1}\tilde{x}_1, \tilde{x}_1\rangle \\ & -2\dot{\psi}(t_1)\tilde{x}_1\bar{t}_1 - (\dot{\psi}_0(t_1) - \dot{\psi}(t_1)\dot{x}(t_1))\bar{t}_1^2.\end{aligned} \tag{54}$$

The equalities $\bar{t}_0 = 0$ and $\tilde{x}_0 = 0$ hold also for each element $\tilde{p}$ of the finite dimensional and finite–faced cone $\mathcal{K}_0$ given by (28) for $\alpha_0 > 0$ and by (24) in the general case. Rewriting the terms ω_0 we get a quadratic form in the variables $(\bar{t}_1, \tilde{x}_1)$ generated by the matrix

$$B := \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix},$$

where

$$\begin{aligned} B_{11} &= l_{t_1t_1} + \dot{\psi}(t_1)\dot{x}(t_1) - \dot{\psi}_0(t_1) - \dot{x}(t_1)^*Q(t_1)\dot{x}(t_1), \\ B_{12} &= l_{t_1x_1} - \dot{\psi}(t_1) + \dot{x}(t_1)^*Q(t_1), \\ B_{22} &= l_{x_1x_1} - Q(t_1). \end{aligned} \tag{55}$$

The property $\bar{x}(t) = 0$ in $[t_0, \tau_1)$ for $\bar{z} \in \mathcal{K}$ allows to refine Theorems 4.1 and 4.2.

THEOREM 4.4 *Assume that the initial values $t_0 = \hat{t}_0$ and $x(t_0) = \hat{x}_0$ are fixed in the problem (1)-(3), and let s=1. Let $Q(t)$ be a continuous solution of the linear differential equation (41) on $[\tau_1, t_1]$ which satisfies two conditions:*
(i) $b_1 := D^1(H) + \Big(([\dot{x}]^1)^*Q(\tau_1) - [\dot{\psi}]^1\Big)[\dot{x}]^1 \geq 0$;
(ii) the quadratic form ω_0 is positive on the cone $\mathcal{K}_0 \setminus \{0\}$.
Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

Proof. Continue $Q(t)$ arbitrarily as a solution of differential equation (41) to the whole interval $[t_0, t_1]$ with possible jump at the point τ_1. Note that the value b_1 in condition (i) is the same as the value b_{1+} for the continued solution, and hence $b_{1+} \geq 0$. Let $\bar{z} \in \mathcal{K}$ and hence $\bar{x}^{1-} = 0$. Then by (48) with $k = 1$ the condition $b_{1+} \geq 0$ implies the inequality $\omega_1 \geq 0$. Condition (ii) implies the inequality $\omega_0 \geq 0$. Consequently $\Omega = \omega_0 + \omega_1 \geq 0$. Further arguments are the same as in the proof of Theorem 4.1. ■

THEOREM 4.5 *Assume that the initial values $t_0 = \hat{t}_0$ and $x(t_0) = \hat{x}_0$ are fixed in the problem (1)-(3) and $s \geq 2$. Let $Q(t)$ be a solution of the linear differential equation (41) on $(\tau_1, t_1] \setminus \theta$ which satisfies the following conditions:*
(a) the matrix M_{k+} is positive semidefinite for each $k = 2, \ldots, s$;
(b) $b_{k+} := D^k(H) + (q_{k+})[\dot{x}]^k > 0$ for each $k = 1, \ldots, s-1$;
(c) the quadratic form ω_0 is positive on the cone $\mathcal{K}_0 \setminus \{0\}$.
Then Ω is positive on $\mathcal{K} \setminus \{0\}$.

Proof. Again, without loss of generality we can consider $Q(t)$ as a discontinuous solution of equation (41) on the whole interval $[t_0, t_1]$. Let $\bar{z} \in \mathcal{K}$. Then by (48) with $k = 1$ the conditions $b_{1+} > 0$ and $\bar{x}^{1-} = 0$ imply the inequality $\omega_1 \geq 0$. Further arguments are the same as in the proof of Theorem 4.2. ■

4.3. Q-transformation of Ω to perfect squares

We shall formulate special jump conditions for the matrix Q at each point $\tau_k \in \theta$. This will make it possible to transform Ω to perfect squares and thus to prove its positive definiteness on $\mathcal{K}$.

PROPOSITION 4.3 (OSMOLOVSKII, LEMPIO, 2002) *Suppose that*

$$b_{k+} := D^k(H) + (q_{k+})[\dot{x}]^k > 0 \tag{56}$$

and that Q satisfies the jump condition at τ_k

$$b_{k+}[Q]^k = (q_{k+})^*(q_{k+}), \tag{57}$$

where $(q_{k+})^*$ is a column vector while q_{k+} is a row vector. Then ω_k can be written as the perfect square

$$\begin{aligned} \omega_k &= (b_{k+})^{-1}\left((b_{k+})\xi_k + (q_{k+})(\bar{x}^{k-})\right)^2 \\ &= (b_{k+})^{-1}\left(D^k(H)\xi_k + (q_{k+})(\bar{x}^{k+})\right)^2. \end{aligned} \tag{58}$$

Proof. Using (48), (56), and (57), we obtain

$$\begin{aligned} \omega_k &= (b_{k+})\xi_k^2 + 2(q_{k+})\bar{x}^{k-}\xi_k + (\bar{x}^{k-})^*[Q]^k\bar{x}^{k-} \\ &= (b_{k+})^{-1}\left((b_{k+})^2\xi_k^2 + 2(q_{k+})\bar{x}^{k-}(b_{k+})\xi_k + \left((q_{k+})\bar{x}^{k-}\right)^2\right) \\ &= (b_{k+})^{-1}\left((b_{k+})\xi_k + (q_{k+})(\bar{x}^{k-})\right)^2. \end{aligned}$$

Since

$$(b_{k+})\xi_k + (q_{k+})\bar{x}^{k-} = \left(D^k(H) + (q_{k+})[\dot{x}]^k\right)\xi_k + (q_{k+})\bar{x}^{k-}$$

$$= D^k(H)\xi_k + (q_{k+})[\bar{x}]^k + (q_{k+})\bar{x}^{k-} = D^k(H)\xi_k + (q_{k+})\bar{x}^{k+},$$

we see that equality (58) holds. ■

THEOREM 4.6 *Let $Q(t)$ satisfy the linear differential equation (41) on $[t_0, t_1] \setminus \theta$ and let conditions (56) and (57) hold for each $k = 1, \ldots, s$. Let ω_0 be positive on $\mathcal{K}_0 \setminus \{0\}$. Then Ω is positive on $\mathcal{K} \setminus \{0\}$.*

Proof. By Proposition 4.3 and formulae (48), (49) the matrix M_{k+} is positive semidefinite for each $k = 1, \ldots, n$. Now using Theorem 4.1 for $s = 1$ and Theorem 4.2 for $s \geq 2$ we obtain that Ω is positive on $\mathcal{K} \setminus \{0\}$.

Similar assertions hold for the jump conditions that use left hand values of Q at each point $\tau_k \in \theta$. ■

PROPOSITION 4.4 (OSMOLOVSKII, LEMPIO, 2002) *Suppose that*

$$b_{k-} := D^k(H) - (q_{k-})[\dot{x}]^k > 0 \tag{59}$$

and that Q satisfies the jump condition at τ_k

$$b_{k-}[Q]^k = (q_{k-})^*(q_{k-}). \tag{60}$$

Then

$$\begin{aligned} \omega_k &= (b_{k-})^{-1}\left((b_{k-})\xi_k + (q_{k-})(\bar{x}^{k+})\right)^2 \\ &= (b_{k-})^{-1}\left(D^k(H)\xi_k + (q_{k-})(\bar{x}^{k-})\right)^2. \end{aligned} \tag{61}$$

THEOREM 4.7 *Let $Q(t)$ satisfy the linear differential equation (41) on $[t_0, t_1] \setminus \theta$, and let conditions (59) and (60) hold for each $k = 1, \ldots, s$. Let ω_0 be positive on $\mathcal{K}_0 \setminus \{0\}$. Then Ω is positive on $\mathcal{K} \setminus \{0\}$.*

5. Numerical examples

5.1. Minimal fuel consumption of a car

The following optimal control problem has been treated by Oberle, Pesch (2000) as an exercise of applying the minimum principle. Consider a car whose dynamics (position x_1 and velocity x_2) are subject to friction and gravitational forces. The acceleration $u(t)$ is proportional to the fuel consumption. Thus the control problem is to minimize the total fuel consumption

$$\mathcal{J} = \int_0^{t_1} u(t)\,dt \tag{62}$$

in a time interval $[0, t_1]$ subject to the dynamic constraints, boundary conditions and the control constraints

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{u}{mx_2} - \alpha g - \frac{c}{m}x_2^2, \tag{63}$$

$$x_1(0) = 0,\ x_2(0) = 1, \quad x_1(t_1) = 10,\ x_2(t_1) = 3, \tag{64}$$

$$u_{\min} \le u(t) \le u_{\max}, \quad 0 \le t \le t_1. \tag{65}$$

The final time t_1 is unspecified. The following values of the constants will be used in computations below:

$$m = 4, \quad \alpha = 1, \quad g = 10, \quad c = 0.4, \quad u_{\min} = 100, \quad u_{\max} = 140.$$

In view of the integral cost criterion (62) we consider the Pontryagin function (Hamiltonian) (19) where we can put $\alpha_0 = 1$,

$$H(x_1, x_2, \psi_1, \psi_2, u) = u + \psi_1 x_2 + \psi_2 \left(\frac{u}{mx_2} - \alpha g - \frac{c}{m}x_2^2 \right). \tag{66}$$

The adjoint equations (8) are

$$\dot{\psi}_1 = 0, \quad \dot{\psi}_2 = -\psi_1 + \psi_2 \left(\frac{u}{mx_2^2} + \frac{2c}{m}\, x_2 \right). \tag{67}$$

The condition (11) evaluated for the free final time t_1 yields the additional boundary condition

$$u(t_1) + 3\psi_1(t_1) + \psi_2(t_1) \left(\frac{u(t_1)}{3m} - \alpha g - \frac{9c}{m} \right) = 0. \tag{68}$$

The switching function

$$\sigma(x, \psi) = D_u H = 1 + \frac{\psi_2}{mx_2}, \quad \sigma(t) = \sigma(x(t), \psi(t)),$$

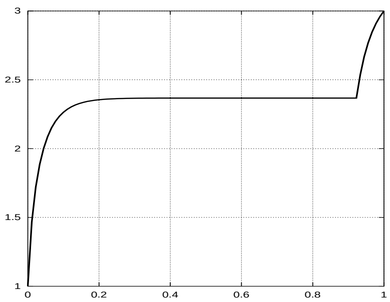
determines the control law as

$$u(t) = \left\{ \begin{array}{ll} u_{\min}, & \text{if} \quad \sigma(t) > 0 \\ u_{\max}, & \text{if} \quad \sigma(t) < 0 \end{array} \right\}.$$

Computations give evidence to the fact that the optimal control is bang–bang with one switching point τ_1,

$$u(t) = \left\{ \begin{array}{ll} u_{\min}, & 0 \le t \le \tau_1 \\ u_{\max}, & \tau_1 \le t \le t_1 \end{array} \right\}.$$

We have used both the code BNDSCO of Oberle, Grimm (1989) and the package NUDOCCCS of Büskens (1998) to compute the switching point τ_1, the final



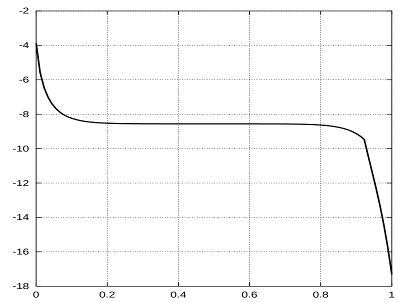


Figure 1. Optimal fuel consumption for a car: optimal state x_2 (left) and adjoint variable ψ_2 (right) on the normalized time interval $[0, 1]$

time t_1 and the adjoint variables $\psi(t)$. The following numerical results allow to reconstruct the complete solution that is displayed in Fig. 1:

$$\begin{array}{rclcrcll}
\tau_1 & = & 3.924283925 & , & t_1 & = & 4.25407390 & , \\
\psi_1(0) & = & -42.24169870 & , & \psi_2(0) & = & -3.87639606 & , \\
x_1(\tau_1) & = & 9.08646352 & , & x_2(\tau_1) & = & 2.36732904 & , \\
\psi_1(t_1) & = & -42.24169870 & , & \psi_2(t_1) & = & -17.31509202 & .
\end{array}$$

We will show that this trajectory satisfies the assumptions of Proposition 3.5 which yields the critical cone $\mathcal{K} = \{0\}$. It can be verified immediately that the computed vectors

$$\frac{\partial x}{\partial \tau_1}(t_1) = (-0.6326710\,,\,-0.7666666)^*, \quad \dot{x}(t_1) = (3.0\,,\,0.7666666)^*$$

are linearly independent. Moreover, we find, in view of (14),

$$D^1(H) = -\dot{\sigma}(\tau_1)\,[u]^1 = 0.472397 \cdot 40 > 0\,,$$

Theorem 3.2 shows that the computed bang–bang control is indeed a strong minimum.

5.2. Time–optimal control of the van der Pol oscillator with a nonlinear boundary condition

In Maurer, Osmolovskii (2001), the time–optimal control of a van der Pol oscillator with a fixed initial and terminal state was studied. Here, we consider the same problem but replace the two terminal conditions by one nonlinear terminal condition. This allows us to demonstrate the evaluation of the quadratic boundary conditions (53)–(55) for the matrix $Q(t_1)$. The control problem is to minimize the endtime t_1 subject to the constraints

$$\dot{x}_1(t) = x_2(t)\,,\ \dot{x}_2(t) = -x_1(t) + x_2(t)(1 - x_1^2(t)) + u(t)\,, \tag{69}$$

$$x_1(0) = 1.0,\ x_2(0) = 1.0\,, \quad x_1(t_1)^2 + x_2(t_1)^2 = r^2,\ r = 0.2\,, \tag{70}$$

$$|\,u(t)\,| \le 1 \quad \text{for}\ \ t \in [0, t_1\,]\,. \tag{71}$$

The Pontryagin (or Hamiltonian) function (5) is given by

$$H(x,u,\psi) = \psi_1 x_2 + \psi_2\,(-x_1 + x_2(1-x_1^2) + u)\,. \tag{72}$$

The adjoint equations (8) and boundary conditions (9) are

$$\begin{array}{ll} \dot\psi_1 = \psi_2(1+2x_1x_2), & \psi_1(t_1) = 2\beta x_1(t_1), \\ \dot\psi_2 = -\psi_1 - \psi_2(1-x_1^2), & \psi_2(t_1) = 2\beta x_2(t_1). \end{array} \tag{73}$$

The boundary condition (11) associated with the free final time t_1 leads to

$$1 + \psi_1(t_1)x_2(t_1) + \psi_2(t_1)\,(-x_1(t_1) + x_2(t_1)(1-x_1(t_1)^2) + u(t_1))\; = 0\,, \tag{74}$$

where we have taken $\alpha_0 = 1$. The switching function is $\sigma(t) = H_u(t) = \psi_2(t)$. The structure of the optimal solution in Maurer, Osmolovskii (2001) for fixed terminal conditions $x_1(t_1) = x_2(t_1) = 0$ suggests that the optimal control for the boundary condition $x_1(t_1)^2 + x_2(t_1)^2 = r^2$, $r = 0.2$, is bang–bang with one switching point τ_1,

$$u(t) = \left\{ \begin{array}{rll} -1 & \text{for} & 0 \le t \le \tau_1 \\ 1 & \text{for} & \tau_1 \le t \le t_1 \end{array} \right\}. \tag{75}$$

In particular, we get the switching condition

$$\sigma(\tau_1) = \psi_2(\tau_1) = 0. \tag{76}$$

Using either the boundary value solver BNDSCO of Oberle, Grimm (1989) or the direct optimization routine NUDOCCCS of Büskens (1998) we obtain the following set of selected values for the switching point, final time and state and adjoint variables:

$$\begin{array}{rclcrcll} \tau_1 & = & 0.7139356 & , & t_1 & = & 2.864192 & , \\ \psi_1(0) & = & 0.9890682 & , & \psi_2(0) & = & 0.9945782 & , \\ x_1(\tau_1) & = & 1.143759 & , & x_2(\tau_1) & = & -0.5687884 & , \\ \psi_1(\tau_1) & = & 1.758128 & , & \psi_2(\tau_1) & = & 0.0 & , \\ x_1(t_1) & = & 0.06985245 & , & x_2(t_1) & = & -0.1874050 & , \\ \psi_1(t_1) & = & 0.4581826 & , & \psi_2(t_1) & = & -1.229244 & , \\ \beta & = & 3.279646 & . & & & & \end{array} \tag{77}$$

We have two alternatives to check sufficient conditions. One way is to use Theorem 4.4 by solving the linear equation (41). Another possibility is offered by the direct evaluation of the quadratic form as given in Proposition 3.7. Let us begin with testing the assumptions of Theorem 4.4 and consider the symmetric 2×2–matrix

$$Q(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}(t) & Q_{22}(t) \end{pmatrix}.$$

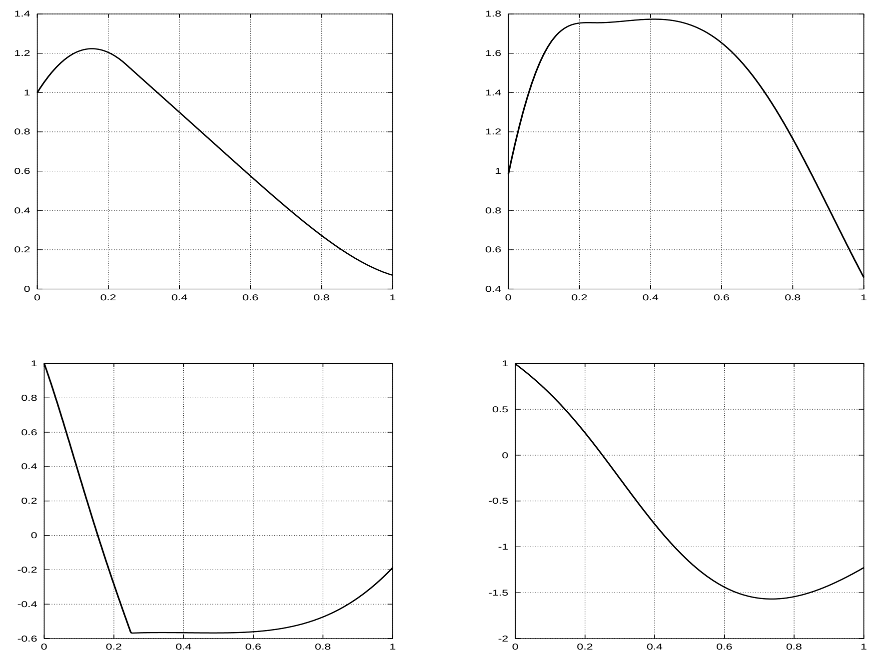


Figure 2. Time–optimal control of the van der Pol oscillator: state variables (left column) and adjoint variables (right column) on the normalized time interval $[0, 1]$.

The linear equations $\dot{Q} = -Qf_x - f_x^* Q - H_{xx}$ in (41) yield the following ODEs:

$$\begin{aligned} \dot{Q}_{11} &= 2\,Q_{12}(1 + 2x_1x_2) + 2\psi_2 x_2, \\ \dot{Q}_{12} &= -(Q_{11} + Q_{12}(1 - x_1^2) + Q_{22}(1 + 2x_1x_2) + 2\psi_2 x_1, \\ \dot{Q}_{22} &= -2\,(Q_{12} + Q_{22}(1 - x_1^2)). \end{aligned} \tag{78}$$

In view of Theorem 4.4 we need to determine a solution $Q(t)$ only in the interval $[\tau_1, t_1]$ such that

$$D^1(H) + (q_{k+})[\dot{x}]^1 > 0, \quad q_{k+} = ([\dot{x}]^1)^* Q(\tau_1) - [\dot{\psi}]^1,$$

holds and the quadratic form ω_0 in (53)–(55) is positive definite on the cone $\mathcal{K}_0$ defined in (28). Since $\psi_2(\tau_1) = 0$ we get from (14)

$$D^1(H) = -\dot{\sigma}(\tau_1)[u]^1 = 2 \cdot \psi_1(\tau_1) = 2 \cdot 1.758128 > 0\,.$$

Furthermore, in view of $[\dot{\psi}]^1 = 0$ we obtain the condition

$$D^1(H) + ([\dot{x}]^1)^* Q(\tau_1)[\dot{x}]^1 = 2 \cdot 1.758128 + 4Q_{22}(\tau_1) > 0,$$

i.e., we have to choose an initial value $Q_{22}(\tau_1) > -0.879064$. By Proposition 3.6, we have $\bar{t}_1 = 0$ for every element $\bar{z} = (\bar{t}_1, \xi, \bar{x}) \in \mathcal{K}$. Therefore, by (55) we must check that the matrix $B_{22} = 2\beta\, I_2 - Q(t_1)$ is positive definite on the critical cone $\mathcal{K}_0$ defined in (28), i.e., on the cone

$$\mathcal{K}_0 = \{(\bar{t}_1, v_1, v_2) \mid \bar{t}_1 = 0,\ x_1(t_1)v_1 + x_2(t_1)v_2 = 0\}\,.$$

Thus, the variations (v_1, v_2) are related by $v_2 = -v_1\, x_1(t_1)/x_2(t_1)$. Evaluating the quadratic form $\langle (2\beta\, I_2 - Q(t_1))(v_1, v_2), (v_1, v_2)\rangle$ with $v_2 = -v_1\, x_1(t_1)/x_2(t_1)$, we arrive at the test

$$c = \left(2\beta\left(1 + \left(\frac{x_1}{x_2}\right)^2\right) - \left(Q_{11} - 2\frac{x_1}{x_2}Q_{12} + (\frac{x_1}{x_2})^2 Q_{22}\right)\right)(t_1)\ > 0\,.$$

A straightforward integration of the ODEs (78) using the solution data (77) and the initial values $Q_{11}(\tau_1) = Q_{12}(\tau_1) = Q_{22}(\tau_1) = 0$ gives the numerical results

$$Q_{11}(t_1) = 0.241897,\ \ Q_{12}(t_1) = -0.706142,\ \ Q_{22}(t_1) = 1.163448,$$

which yield the positive value $c = 7.593456 > 0$. Thus, Theorem 4.4 asserts that the bang–bang control characterized by (77) provides a strict strong minimum.

The alternative test for SSC is based on Proposition 3.7. The variational system $\dot{y}(t) = f_x(t)y(t),\ y(\tau_1) = [\dot{x}]^1$, for the variation $y = (y_1, y_2)$ leads to the variational system

$$\begin{aligned} &\dot{y}_1 = y_2, && y_1(\tau_1) = 0,\\ &\dot{y}_2 = -(1 + 2x_1 2x_2)y_1 + (1 - x_1^2)y_2, && y_2(\tau_1) = 2, \end{aligned}$$

for which we compute

$$y_1(t_1) = 4.929925\,, \quad y_2(t_1) = 1.837486\,.$$

Note that the relation $K_{x_1}(x(t_1))y(t_1) = 2(x_1(t_1)y_1(t_1) + x_2(t_1)y_2(t_1) = 0$ holds. By Proposition 3.7 we have to show that the quantity ρ in (39) is positive,

$$\rho = D^1(H) - [\dot{\psi}]^1[\dot{x}]^1 + \int_{\tau_1}^{t_1} (y(t))^* H_{xx}(t)y(t)\, dt + (y(t_1))^*(\beta K)_{x_1x_1} y(t_1)\ > 0\,.$$

Using $[\dot{\psi}]^1 = 0$ and $(y(t_1))^*(\beta K)_{x_1x_1} y(t_1) = 2\beta(y_1(t_1)^2 + y_2(t_1)^2)$, we finally obtain

$$\rho = D^1(H) + 184.550\ > 0.$$

6. Conclusion

We have studied second order sufficient conditions for optimal bang–bang controls. The original form of these conditions as given in Osmolovskii (1995) and Milyutin, Osmolovskii (1998) required that an associated quadratic form be positive definite on the critical cone. A direct numerical verification of this test can be carried out only in rather special cases. Therefore, the main objective of this paper was to study several representations of the critical cone and transformations of the quadratic form such as to obtain a more practical second order test. In particular, it was useful to compute elements of the critical cone as variations of the state trajectory with respect to the switching points and initial conditions. Moreover, by means of the solution to a linear matrix ODE, the quadratic form could be converted to perfect squares. The second order test has been successfully applied to three numerical examples representing different types of control problems. More examples with applications of bang–bang controls to nonlinear optics may be found in Kim (2002) and Kim et al. (2003).

After finishing this paper, we became interested in exploring the relations between the SSC in Theorem 3.2 and in Agrachev, Stefani, Zezza (2001, 2002). A careful study of the second order variations of state trajectories w.r.t. switching points reveals that the conditions in Theorem 3.2 and in Agrachev et al. (2002) are indeed equivalent under the assumption $\alpha_0 > 0$. These results will be reported in a future paper. The theoretical studies also showed that the SSC in Agrachev et al. (2002) can be checked numerically by a suitable implementation of the program NUDOCCCS of Büskens (1998). Together with the methods in this article we have thus found several possibilities of testing SSC. Another promising aspect is that the results of this study can be used in the development of a theoretical and numerical sensitivity analysis for optimal bang–bang controls.

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