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On robustness of the regularity property of maps

by

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Abstract: The problem considered in the paper can be described as follows. We are given a continuous mapping from one metric space into another which is regular (in the sense of metric regularity or, equivalently, controllability at a linear rate) near a certain point. How small may be an additive perturbation of the mapping which destroys regularity? The paper contains a new proof of a recent theorem of Dontchev-Lewis-Rockafellar for linear perturbations of maps between finite-dimensional Banach spaces and an exact estimate for Lipschitz perturbations of maps between complete metric spaces.

Keywords: metric regularity, openness at a linear rate, pseudo-Lipschitz (Aubin) property, slope, regularity criterion.

1. Introduction

We start with a brief informal description of the problems to be dealt with in the paper. Suppose we have an equation $F(x) = y$ (or an inclusion $y \in F(x)$ with F being a set-valued operator). Let $S(y)$ be the collection of solutions of the equation depending on the right-hand side parameter y. One of the important and often asked questions is how to check that the set of solutions does not change sharply under a minor change of the parameter. A Lipschitz-type behavior of the solution set is often considered satisfactory. Properly formulated, it leads to the concept of the *pseudo-Lipschitz*, or *Aubin property* (a precise definition will be given later).

An elementary but very important fact is that a reformulation of this property in terms of the mapping F leads to the concept of *metric regularity*, one of the most fundamental in nonsmooth and, generally, nonlinear analysis.

Let X and Y be complete metric spaces, and let $F : X \rightrightarrows Y$ be a set-valued mapping of which we shall always assume that Gr $F = \{(x, y) : \ y \in \text{Gr } F\}$, the graph of F, is a closed set. We shall denote the distance in either space by the same symbol $d(\cdot, \cdot)$; this should not cause any confusion as the content of the parenthesis always explains to which space this applies. Given $(\bar{x}, \bar{y})$, it is said that F is *metrically regular near* $(\bar{x}, \bar{y})$ if there are $K > 0$, $\varepsilon > 0$ such that

$$d(x, F^{-1}(y)) \leq K d(y, F(x)),$$

provided $d(x, \bar{x}) < \varepsilon$, $d(y, \bar{y}) < \varepsilon$. The lower bound of all such K is called the *rate* (or the *norm*) of metric regularity of F near $(\bar{x}, \bar{y})$ and is denoted $\mathrm{Reg}F(\bar{x}, \bar{y})$.

In case of a single-valued mapping we slightly change the terminology and notation and say that F is metrically regular near $\bar{x}$ and write $\mathrm{Reg}F(\bar{x})$.

The concept of metric regularity acquired this final form after decades of developments whose starting points were the Banach-Schauder open mapping theorem and the Ljusternik-Graves theorem on local openness of smooth maps with surjective derivatives. Each of these theorem can be interpreted as a theorem on metric regularity of corresponding maps. We refer to Ioffe (2002) for a detailed historical discussion.

The relationship of metric regularity to local solvability and Lipschitz stability of solutions of equations or inclusions can be stated in an equally general setting. Namely, it is said that F *covers* (or is *open*) at a linear rate near $(\bar{x}, \bar{y})$ if there are $r > 0$, $\varepsilon > 0$ such that whenever $d(x, \bar{x}) < \varepsilon$, $d(y, \bar{y}) < \varepsilon$, the inclusion

$$B(y, rt) \subset F(B(x, t))$$

holds for all sufficiently small t.

The upper bound of such r is called the *rate* of covering of F near $(\bar{x}, \bar{y})$ and is denoted $\mathrm{Sur}F(\bar{x}, \bar{y})$.

Furthermore, F is said to be *pseudo-Lipschitz* (or to have the *Aubin property*) near $(\bar{x}, \bar{y})$ if there are $K > 0$ and $\varepsilon > 0$ such that for any y of a neighborhood of $\bar{y}$ the function $d(y, F(\cdot))$ satisfies the Lipschitz condition with constant not exceeding K in the ε-ball around $\bar{x}$.[1] The lower bound of such K is the *Aubin constant* of F near $(\bar{x}, \bar{y})$.

The fact of a principal importance is that all three concepts are equivalent in the following sense: F is metrically regular near $(\bar{x}, \bar{y})$ at the rate K if and only if it is open near $(\bar{x}, \bar{y})$ with the rate K^{-1} and if and only if the inverse mapping F^{-1} is pseudo-Lipschitz near $(\bar{y}, \bar{x})$ with constant K. The proof of this equivalence, which is rather a simple reformulation of the definitions can be found in Ioffe (2002).

In view of this fact, it seems to be convenient to have an umbrella word to refer to the three properties when there is no need to specify one of them. The term *first order regular* seems to be a suitable candidate. However, to simplify the terminology, we adopt in this paper the following convention, a

[1] This amounts to saying that

$$F(u) \bigcap B(\bar{y}, \varepsilon) \subset \{y : \ d(y, F(x)) \leq K d(x, u)\}$$

provided $d(x, \bar{x}) < \varepsilon$, $d(u, \bar{x}) < \varepsilon$.

sort of return to the terminology used in the 1970s, which will allow us to use the most convenient of the three equivalent properties whenever necessary:

Convention: A set-valued mapping will be called *regular* near $(\bar{x}, \bar{y})$ if it has the three equivalent properties.

Now let us return to the main question we are going to discuss: how the regularity property can be affected by perturbations of the mapping. Suppose now that Y is a Banach space and we perturb F by adding to it a certain single-valued mapping H. What can be said about regularity of the perturbed mapping $F + H$? The following fundamental theorem was proved by Miljutin in 1980 (see Dimitruk, Miljutin, Osmolovskii, 1980).

THEOREM 1.1. *Let X be a complete metric space, let Y be a Banach space, let $F : X \to Y$ be a (single-valued mapping) defined in a neighborhood of $\bar{x} \in X$ which is regular near $\bar{x}$ with* $\mathrm{Sur}F(\bar{x}) = r$. *Let $H : X \to Y$ be another mapping also defined in a neighborhood of $\bar{x}$ and satisfying there the Lipschitz condition with constant $L < r$. Then $F + H$ is regular near $\bar{x}$ and*

$$\mathrm{Sur}(F + H)(\bar{x}) \geq r - L.$$

This result remains valid (with only a slight modification of the proof) for the case when F is a set-valued mapping and, with a somewhat greater effort, when H is also a set-valued mapping and the Lipschitz constant relates to the Hausdorff metric in the space of closed subsets of Y (see Ioffe, 2000). [2]

Surprisingly, the natural question whether this lower estimate is precise was not discussed till very recently when Dontchev, Lewis, Rockafellar (2002) showed that at least in two cases, when both X and Y are finite dimensional Banach spaces and when X is also a Banach space and the mapping F is positively homogeneous of degree one (that is, when $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$), (a) *the lower bound of Lipschitz constants of operators H such that $F + H$ is not regular is precisely* $\mathrm{Sur}F(\bar{x}, \bar{y})$ and (b) *moreover, in either case the lower bound is realized by linear maps of rank one.*

These results provide a partial answer to the question and, in turn, in an equally natural way, lead to further questions of whether it is possible to extend the results to broader classes of set-valued mappings. As far as part (b) is concerned, the answer is generally negative (although recently Mordukhovich, 2003, described a class of set-valued mappings from a Banach space into $I\!R^n$ for which (b) holds). Already in the case of $X = Y = H$, a Hilbert space, and F single-valued the lower bound of $\mathrm{Sur}(F + A)(\bar{x})$ over all linear mappings A with norms equal or smaller that $L \leq \mathrm{Sur}F(\bar{x})$ can be strictly greater than $\mathrm{Sur}F(\bar{x}) - L$.

[2] To be precise: the results quoted, as stated, deal with global rather than local regularity. However the reduction to local results is straightforward in each case. For an independent proof of the local version of Miljutin's theorem see e.g. Ioffe (1987).

An example of such F, even having reasonably good differentiability properties, was given in Ioffe (2002).

The final result of this paper (Theorem 4.1) shows that (a) is valid for arbitrary single-valued mappings from metric into normed spaces, that is to say, that the lower estimate in the theorem of Miljutin is exact on the class of locally Lipschitz perturbations. The question of whether or not a similar fact is valid for set-valued mappings remains open.

We also give a new proof of (b) for set-valued mappings between finite dimensional spaces based not on the powerful machinery of subdifferential calculus (see e.g. Ioffe, 2002, for connections between subdifferential calculus and the three regularity properties) but rather on simpler calculations involving properties of the so-called *slope* which is the simplest and the most precise tool to characterize the regularity property. We start by discussing necessary properties of slope in the next section.

2. Slope and the regularity criterion

In this section we formulate the principal local regularity criteria for set-valued mappings. This criterion is based on the concept of *slope* introduced in 1980 by De Giorgi-Marino and Tosques (1980).

Definition 2.1. *Let X be a metric space, and let f be a function on X with values in $(-\infty, \infty]$ which is finite at x. The quantity*

$$|\nabla f|(x) = \limsup_{\substack{u \to x \\ u \neq x}} \frac{(f(x) - f(u))^+}{d(x,u)}$$

is called the slope of f at x.

The meaning of this concept is very simple: this is just the highest speed of decrease of the function from the given point. Slope also can be considered the quantitative measure of the qualitative concept of *calmness* due to Clarke and Rockafellar.

If X is a Banach space and f is Fréchet differentiable at x, then $|\nabla f|(x) = \|f'(x)\|$ (that is, the slope coincides with the norm of the derivative). More generally, if f is uniformly directionally differentiable, that is — if the directional derivative $f'(x;h)$ exists for any h and $t^{-1}(f(x+th) - f(x) - f'(x;h))$ goes to zero uniformly on the unit ball as $t \to 0$, then

$$|\nabla f|(x) = [\inf_{\|h\|=1} f'(x;h)]^- = \sup_{\|h\|=1} [f'(x;h)]^-,$$

where $\alpha^- = \max\{0, -\alpha\}$. For a lower semicontinuous function on a finite dimensional space slopes are completely defined by the lower Hadamard directional derivative:

$$f^-(x;h) = \liminf_{\substack{h' \to h \\ t \searrow 0}} t^{-1}(f(x+th') - f(x)).$$

For such functions the value of the slope is defined as in the formula above with f' replaced by f^-.

We turn now to the general regularity criterion for set-valued maps, stated in terms of slopes. Let X, Y be metric spaces. With every $F : X \rightrightarrows Y$ we associate the following family of functions on $X \times Y$:

$$\varphi_y(x,v) = \begin{cases} d(y,v), & \text{if} \quad v \in F(x); \\ \infty, & \text{if} \quad v \notin F(x). \end{cases}$$

In other words,

$$\varphi_y(x,v) = d(y,v) + \chi_{\mathrm{Gr}\ F}(x,v),$$

the second term in the sum being the *indicator* of Gr F, that is the function equal to zero on Gr F and infinity outside.

We shall also consider the following family of α-distances in $X \times Y$:

$$d_\alpha((x,v),(u,w)) = d(x,u) + \alpha d(v,w),$$

and by ∇_α we shall denote the slopes of functions on $X \times Y$ with respect to the α-distance, so that

$$|\nabla_\alpha f|(x,v) = \limsup_{\substack{(u,w)\to(x,v)\\(x,u)\neq(u,w)}} \frac{(f(x,v) - f(u,w))^+}{d_\alpha((x,v),(u,w))}.$$

Theorem 2.1. *(Ioffe, 2000). Let X and Y be complete metric spaces, let $F : X \rightrightarrows Y$ be a set-valued mapping with nonempty closed graph and let $(\bar{x},\bar{y}) \in$ Gr F. Let finally*

$$m(\alpha) = \liminf_{\substack{(x,y,v)\to(\bar{x},\bar{y},\bar{y})\\ y\neq v}} |\nabla_\alpha \varphi_y|(x,v).$$

Then

$$\mathrm{Sur}F(\bar{x},\bar{y}) \geq \frac{m(\alpha)}{1-\alpha m(\alpha)}.$$

Moreover, the equality actually holds if Y is a Banach space.

Implicit in this theorem is that, in case of a Banach Y, $\alpha m(\alpha) < 1$ for any α and $m(\alpha) \to \mathrm{Sur}F(\bar{x},\bar{y})$ as $\alpha \to 0$.

We refer to Ioffe (2000, 2001) for an explanation how all known subdifferential regularity criteria follow from the theorem.

The criterion assumes a nicer form if F has the property that the functions

$$\psi_y = d(y, F(\cdot))$$

are lower semicontinuous for all y (or at least for all y of a neighborhood of $\bar{y}$). This is always the case if both X and Y are finite dimensional Banach spaces.

THEOREM 2.2. *(Ioffe, 2000) If in addition to the assumptions of Theorem 1.1, the functions ψ_y are lower semicontinuous for all y of a neighborhood of $\bar{y}$, then*

$$\mathrm{Sur}F(\bar{x},\bar{y}) \geq \liminf_{\substack{(x,y)\to(\bar{x},\bar{y})\\ y\notin F(x)}} |\nabla\psi_y|(x).$$

Moreover, the equality holds if Y is a Banach space.

REMARK 2.1. *The assumption that Y is a Banach space is not necessary for the equality to hold in both theorems. This is also true if Y is a metric space with the the following approximate geodesic property: for any $y_1,\ y_2 \in Y$ and any $\varepsilon > 0$ there is a y such that $d(y,y_i) \leq (1/2)d(y_1,y_2)+\varepsilon$ (see Ioffe, 2001).*

REMARK 2.2. *Both functions φ_y and ψ_y, as functions of y, satisfy the Lipschitz condition with constant 2 (for the fixed values of arguments).*

REMARK 2.3. *The rates of surjection and metric regularity as well as the values and slopes of ψ_y of course depend on specific choice of distances in X and Y. A simple calculation shows however, that small change of a norm results in a small change of the rates and the slopes.*

3. The finite dimensional case.

In this section we shall apply Theorem 2.2 to prove the following result which is equivalent to the mentioned theorem of Dontchev, Lewis, Rockafellar (2002).

THEOREM 3.1. *Let X and Y be finite dimensional Banach spaces, let $F : X \rightrightarrows Y$ be a set-valued mapping from X into Y with closed graph which is regular near $(\bar{x},\bar{y}) \in \mathrm{Gr}\ F$ and $\mathrm{Sur}F(\bar{x},\bar{y}) = r > 0$. Then for any positive $\rho \leq r$ there is a linear operator $A : X \to Y$ of rank one with norm equal to ρ and such that $\mathrm{Sur}(F + A)(\bar{x},\bar{y} + A\bar{x}) = r - \rho$.*

The theorem proved in Dontchev, Lewis, Rockafellar (2002) corresponds to the case $\rho = r$. The proof given in Dontchev, Lewis, Rockafellar (2002) is based on the finite dimensional subdifferential regularity criterion for set-valued mappings, see Mordukhovich (1993), which reduces the problem to non-singularity of the coderivative at a given point (where "nonsingularity" of a homogeneous set-valued mapping means that the distance from zero to any value of the mapping is not smaller than the norm of the argument times a fixed constant). An obvious observation is that non-singularity of a homogeneous mapping can be destroyed by an addition of a rank one linear operator. The criterion is one of the most advanced results of finite dimensional nonsmooth analysis. We give below an alternative proof of the theorem which does not need any reference to subdifferential calculus.

We precede the proof with with a lemma which gives important information about slopes.

Lemma 3.1. *Let X be a finite dimensional Banach space with smooth norm, and let f be a lower semicontinuous function on X. Suppose that $|\nabla f|(x) = r > 0$. Then, for any $\varepsilon > 0$ there are $w \in X$, $r' > 0$ an $x^* \in X^*$ such that $\|x - w\| < \varepsilon$, $\|x^*\| = 1$, $r' \leq r + \varepsilon$ and*

$$f(u) + r'\langle x^*, u\rangle + \varepsilon\|u - w\| \geq f(w)$$

for all u in a neighborhood of w.

Proof. By the definition of slope for any $\delta > 0$ the function $f(u)+(r+\delta)\|u-x\|$ attains a local minimum at x.

Given an $\varepsilon > 0$, we choose a $\delta \in (0, \varepsilon)$, e.g. $\delta = \varepsilon/2$, and find a $\sigma < \delta$ to make sure that

$$f(u) + (r + \delta)\|u - x\| > f(x)$$

if $\|u - x\| \leq \sigma$, $x \neq u$.

Next, we choose $K > 0$ and $p > 1$ for which the following inequality holds:

$$r + \delta \leq K\sigma^{p-1} < \frac{r + \varepsilon}{p}.$$

This is clearly possible as $\delta < \varepsilon$.

Consider the function

$$g(u) = f(u) + K\|u - x\|^p,$$

Then

$$g(u) = f(u) + K\sigma^{p-1}\|u - x\| > g(x) = f(x)$$

if $\|u-x\| = \sigma$. It follows that $g(u)$ attains a local minimum at a certain w with $\|w - x\| < \sigma$. Clearly $w \neq x$. Indeed, as the norm is smooth and $p > 1$, the function $\|\cdot\|^p$ is smooth and its derivative at zero is zero. Therefore, $w = x$ would mean that $|\nabla f|(x) = |\nabla g|(x) = 0$, contrary to the assumption.

The derivative of $\|\cdot\|^p$ at w can thus be written in the form $p\|w - x\|^{p-1}x^*$, where $\|x^*\| = 1$ and $\langle x^*, w - x\rangle = \|w - x\|$. We have for u close to w:

$$g(w) \leq g(u) = f(u)+K\|u-x\|^p = f(u)+K\|w-x\|^{p-1}\langle x^*, u-w\rangle+o(\|u-w\|).$$

The proof is completed by setting $r' = Kp\|w - x\|^{p-1} \leq Kp\sigma^{p-1} < r + \varepsilon$.

Remark 3.1. *The lemma easily extends to spaces with Gâteaux differentiable renorms (or even to spaces with a Lipschitz Gâteaux differentiable bump) with the help of one of the variational principle of Borwein-Preiss.*

Proof of Theorem 3.1.

1. It follows from Remark 2.3 in the preceding section that we only need to prove the theorem under the additional assumption that the norm in Y is strictly convex and smooth. Indeed, let $\|\cdot\|_n$ be a sequence of smooth strictly convex norms in Y converging to $\|\cdot\|$. Let r_n be the corresponding rate of covering by F near $(\bar{x}, \bar{y})$ and let A_n be a rank one linear operator $X \to Y$ whose norm (corresponding to the n-th norm in Y) is $\rho_n \to \rho$. We can assume that A_n converge to a certain operator A which will obviously be of rank one and whose $\|\cdot\|$-norm is ρ. By Remark 2.3 the rates of covering of $F + A$ near $(\bar{x}, \bar{y} + A\bar{x})$ corresponding to the n-th norms converge to $\mathrm{Sur}(F+A)(\bar{x}, \bar{y} + A\bar{x})$. On the other hand, by Miljutin's theorem, the rates of covering of $F + A$ and $F + A_n$ (at the corresponding points) corresponding to the n-th norm differ by at most $\|A - A_n\|_n$.

2. For a finite dimensional space the functions $d(y, F(\cdot))$ are automatically lower semicontinuous if the graph of F is closed, so we can apply Theorem 2.2. It follows that $|\nabla_y \psi|(x) > 0$ for all $(x, y) \notin \mathrm{Gr}\, F$ of a neighborhood of $(\bar{x}, \bar{y})$, and in any neighborhood of $(\bar{x}, \bar{y})$ we can find an (x, y) such that $|\nabla \psi_y|(x)$ is arbitrarily close to r. By Lemma 3.1 the latter implies the existence of sequences $(x_n) \to \bar{x}$, $(y_n) \to \bar{y}$, $(r_n) \to r$, $(\varepsilon_n) \searrow 0$ and x_n^* such that $y_n \notin F(x_n)$, $\|x_n^*\| = 1$ and for any n

$$d(y_n, F(u)) + r_n \langle x_n^*, u - x_n \rangle + \varepsilon_n \|u - x_n\| \geq d(y_n, F(x_n)) \tag{1}$$

for all u of a neighborhood of x_n.

Let $v_n \in F(x_n)$ be the closest to y_n in $F(x_n)$. We can assume without loss of generality that v_n is the unique closest point to y_n in $F(x_n)$. Indeed, otherwise we can replace y_n by $\alpha_n y_n + (1 - \alpha_n) v_n$ with $\alpha_n \to 1$. As the norm in X is strictly convex, v_n becomes a unique closest point to y_n after the replacement.

For each n consider two complementary sets:

$$P_n = \{u : \ d(y_n, F(u)) \geq d(y_n, F(x_n)) + \rho \|u - x_n|\};$$

$$Q_n = \{u : \ d(y_n, F(u)) < d(y_n, F(x_n)) + \rho \|u - x_n\|\}.$$

The second set meets any neighborhood of x_n at infinitely many points as the slope of $d(y_n, F(\cdot))$ is positive. For any $u \in Q_n$, $u \neq x_n$ choose a $v(u) \in F(u)$ such that $\|y_n - v(u)\| = d(y_n, F(u))$. We claim that

$$u \in Q_n, \ u \to x_n, \ v \in F(u), \ d(y_n, F(u)) = \|v - y_n\| \quad \Rightarrow \quad v \to v_n. \tag{2}$$

Indeed, let w be any limiting point of such v. Clearly, $w \in F(x_n)$ as F is closed-graph. Therefore

$$d(y_n, F(x_n)) \leq \|y_n - w\| \leq \limsup_{\substack{u \to x_n \\ u \in Q_n}} d(y_n, F(u)) \leq d(y_n, F(x_n)) \tag{3}$$

which proves the claim as v_n is the unique closest point to y_n in $F(x_n)$.

3. Set $h_n = \|v_n - y_n\|^{-1}(v_n - y_n)$. We may assume that h_n converge to a certain h and x_n^* converge to a certain x^*, both being vectors of norm one. Set for $x \in X$

$$\begin{aligned} Ax &= \rho\langle x^*, x\rangle h, \\ \widehat{F}(x) &= F(x) + Ax, \\ \hat{y}_n &= y_n + Ax_n. \end{aligned}$$

Then A is a rank one linear operator of norm ρ.

We have

$$d(\hat{y}_n, \widehat{F}(x_n)) = d(y_n + Ax_n, F(x_n) + Ax_n) = d(y_n, F(x_n)). \tag{4}$$

If $u \in P_n$, then (as $\|A\| \le \rho$)

$$d(\hat{y}_n, \widehat{F}(u)) \ge d(y_n, F(u)) - \|A\|\|u - x_n\| \ge d(y_n, F(x_n)). \tag{5}$$

Let on the other hand $u \in Q_n$ be sufficiently close to x_n. Let $v(u) \in F(u)$ be such that $\|v(u)+A_n u-\hat{y}_n\| = d(\hat{y}_n, \widehat{F}(u))$. Let finally $y_n^*(u)$ satisfy $\|y_n^*(u)\| = 1$, $\langle y_n^*(u), v(u) - y_n\rangle = \|v(u) - y_n\|$. Then, by (1)

$$\begin{aligned} d(\hat{y}_n, \widehat{F}(u)) &= d(y_n, F(u) + A(u - x_n)) = \|v(u) + A(u - x_n) - y_n\| \\ &\ge \langle y_n^*(u), v(u) + A(u - x_n) - y_n)\rangle \\ &= \|v(u) - y_n\| + \langle y_n^*(u), A(u - x_n)\rangle \\ &\ge d(y_n, F(u)) + \rho\langle y_n^*(u), h\rangle\langle x_n^*, u - x_n\rangle \\ &\ge d(y_n, F(x_n)) - \varepsilon_n\|u - x_n\| - r_n\langle x_n^*, u - x_n\rangle \\ &\quad +\rho\langle y_n^*(u), h\rangle\langle x_n^*, u - x_n\rangle. \end{aligned}$$

Comparing this with (4), we get

$$d(\hat{y}_n, \hat{F}(u))+\varepsilon_n\|u-x_n\|+r_n\langle x_n^*, u-x_n\rangle-\rho\langle y_n^*(u), h\rangle\langle x^*, u-x_n\rangle \ge d(\hat{y}_n, \hat{F}(x_n)) \tag{6}$$

We have seen that $v(u) \to v_n$ when $u \to x_n$. As the norm in X is smooth, it follows that $y_n^*(u) \to y_n^*$ where $\|y_n^*\| = 1$ and $\langle y_n^*, v_n - y_n\rangle = \|v_n - y_n\|$. Hence by the definition of h, $\langle y_n^*, h\rangle \to 1$ as $n \to \infty$. Finally, $x_n^* \to x^*$. Therefore, (6) implies that there are $q_n \to r - \rho$ such that

$$d(\hat{y}_n, \widehat{F}(u)) + q_n\|u - x_n\| \ge d(\hat{y}_n, \widehat{F}(x_n))$$

for $u \in Q_n$ sufficiently close to x_n. Together with (5) this implies that the slope of $d(\hat{y}_n, \widehat{F}(\cdot))$ at x_n is not greater than q_n and therefore by Theorem 2.2 $\mathrm{Sur}\widehat{F}(\bar{x}) \le r - \rho$. But by Miljutin's theorem the opposite inclusion also holds.

4. A general robustness estimate

In this section we show that for single-valued continuous mappings from metric spaces into normed spaces the lower estimate given by Miljutin's theorem is precise in the class of locally Lipschitz perturbations.

Let us say that $F : X \to Y$ is a Lipschitz rank one mapping on an open set U if in a neighborhood of any point of U it can be represented (up to a diffeomorphism of the range space Y), in the form

$$F(x) = \xi(x)y,$$

where ξ is a real valued function satisfying the Lipschitz condition in the neighborhood.

THEOREM 4.1. *Let X be a metric space, let Y be a Banach space, and let $F : X \to Y$ be a continuous mapping which is regular near $\bar{x} \in X$ with* $\mathrm{Sur}F(\bar{x}) = \bar{r} > 0$. *Then, for any $0 < \rho \le \bar{r}$ there is a mapping $H : X \to Y$ which is a Lipschitz rank one mapping with Lipschitz constant ρ and such that* $\mathrm{Sur}(F + H)(\bar{x}) = \bar{r} - \rho$.

Proof. **Step** 1. Take a $z \ne F(\bar{x})$ and set $\varphi(x) = \|F(x) - z\|$. Suppose that for a certain x sufficiently close to $\bar{x}$, $|\nabla\varphi|(x) = r$. At the first step of the proof we shall show that, given a $\delta > 0$, there is a Lipschitz rank one mapping H_x with Lipschitz constant ρ, such that the slope of $\psi_x(\cdot) = \|F(\cdot) + H_x(\cdot) - z\|$ at x satisfies $|\nabla\psi_x|(x) = (r - \rho)^+$ and $H_x(u) = 0$ if $d(x, u) \ge \delta$.

To this end we first note that, as x is sufficiently close to $\bar{x}$, we can be sure that $r > 0$ (by the main regularity criterion) and $F(x) \ne z$. Set $\bar{y} = [\varphi(x)]^{-1}(F(x) - z)$ and define

$$a_\lambda = \inf\{\langle y^*, \bar{y}\rangle : \ \|y^*\| = 1, \ \langle y^*, v\rangle = \|v\| \text{ for some } v \text{ with } \|\bar{y} - v\| < \lambda\}.$$

Clearly, $a_\lambda \nearrow 1$ as $\lambda \to 0$.

As $|\nabla\varphi|(x) = r$,

$$\|F(u) - z\| \ge \|F(x) - z\| - rd(u, x) + o(d(u, x)). \tag{7}$$

and there is a sequence $(u_n) \to x$ such that

$$\|F(u_n) - z\| - \|F(x) - z\| = rd(u_n, x) + o(d(u_n, x)). \tag{8}$$

Take a small positive δ, set

$$\mu(t) = \max\{0, \min\{t, 2\delta - t\}\}$$

and define H_x as follows:

$$H_x(u) = \rho\mu(d(u, x))\bar{y}.$$

Then the Lipschitz constant of H_x is ρ and $H_x(u) = 0$ if $u = x$ or $d(u, x) \ge 2\delta$. We have for a y^* such that $\|y^*\| = 1$ and $\langle y^*, F(u) - z\rangle = \|F(u) - z\|$

$$\begin{aligned} \psi_x(u) - \psi_x(x) &= \|F(u) + H_x(u) - z\| - \|F(x) - z\| \\ &\ge \langle y^*, F(u) + H_x(u) - z\rangle - \|F(x) - z\| \\ &= \langle y^*, F(u) - z\rangle - \|F(x) - z\| + \rho\mu(d(u, x))\langle y^*, \bar{y}\rangle \\ &\ge \|F(u) - z\| - \|F(x) - z\| + \rho a_\lambda \mu(d(u, x)), \end{aligned}$$

where $\lambda = \|F(u) - F(x)\|$.

If $d(u,x) < \delta$, then $\mu(d(u,x)) = d(u,x)$ and therefore in view of (7) (and since $a_\lambda \to 1$ as $u \to x$)

$$\psi_x(u) - \psi_x(x) \ge -(r-\rho)d(u,x) + o(d(u,x) \ge -(r-\rho)^+ d(u,x) + o(d(u,x)).$$

This means that $|\nabla\psi_x|(x) \le (r-\rho)^+$.

On the other hand, for the u_n of (8) we have (as $a_\lambda \le 1$)

$$\begin{aligned}\psi_x(u_n) - \psi_x(x) &\le \|F(u_n) - z\| + \|H(u_n)\| - \|F(x) - z\| \\ &\le -rd(u_n,x) + \rho d(u_n,x) + o(d(u_n,x)),\end{aligned}$$

which shows that the slope of ψ_x is not smaller than $(r-\rho)^+$.

Step 2. We can now complete the proof of the theorem. As $\mathrm{Sur}F(\bar{x}) = \bar{r}$, there is are sequences $(x_n) \to \bar{x}$ and $(y_n) \to F(\bar{x})$ such that ($y_n \ne F(x_n)$ and) $|\nabla\varphi|(x_n) = r_n \to \bar{r}$. We shall consider two cases.

(A) $x_n = \bar{x}$ for infinitely many indices n. In this case we can assume that $x_n = \bar{x}$ for all x and $|\nabla\varphi(\bar{x})| = \bar{r}$. Then the mapping $H_{\bar{x}}$ gives the desired result.

(B) For all (sufficiently large) n, $x_n \ne \bar{x}$. In this case we may assume that all x_n are different. In other words,

$$\sigma_n = \min_{k \ne n} \|x_n - x_k\| > 0, \quad \forall\, n.$$

Clearly $\sigma_n \to 0$ as $n \to \infty$.

By Step 1 for any n there is a rank one Lipschitz mapping H_n with Lipschitz constant ρ such that

(a) $H_n(x_n) = 0$;
(b) $H_n(x) = 0$, if $\|x - x_n\| > \sigma_n/3$;
(c) $|\nabla\psi_n|(x_n) = (r_n - \rho)^+$, where $\psi_n(x) = \|F(x) + H_n(x) - y_n\|$.

It follows from (b) that the supports of H_n do not meet: if $H_n(x) \ne 0,\ H_m(u) \ne 0$, then $x \ne u$. Therefore the mapping

$$H(x) = \sum_1^\infty H_n(x)$$

is well defined, is rank one Lipschitz and its Lipschitz constant in a neighborhood of $\bar{x}$ is ρ. By Miljutin's theorem, $\mathrm{Sur}(F+H) \ge \bar{r} - \rho$. On the other hand, setting $\psi_y(x) = \|F(x) + H(x) - y\|$, we get from (b) and (c):

$$\lim_{n\to\infty} |\nabla\psi_{y_n}|(x_n) = |\nabla\psi_n|(x_n) \to \bar{r} - \rho,$$

and by the main regularity criterion, $\mathrm{Sur}(F+H)(\bar{x}) \le \bar{r} - \rho$.

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