Control and Cybernetics

vol. **32** (2003) No. 3

On stability analysis in shape optimisation : critical shapes for Neumann problem[1]

by

M. Dambrine[2], J. Sokołowski[3] and A. Żochowski[4]

[2]Laboratoire de Mathématiques Appliquées de Compiègne,
Université de Technologie de Compiègne,
former affiliation : Antenne de Bretagne de l'Ecole
Normale Supérieure de Cachan where this work was done;
e-mail:mdambrin@dma.utc.fr
[3]Institut Elie Cartan, Laboratoire de Mathématiques,
Université Henri Poincaré Nancy I,
B.P. 239, 54506 Vandoeuvre lès Nancy Cedex,
France; e-mail: sokolows@iecn.u-nancy.fr
[4]Systems Research Institute of the Polish Academy of Sciences,
ul. Newelska 6, 01-447 Warszawa, Poland;
e-mail: zochowsk@ibspan.waw.pl

Abstract: The stability issue of critical shapes for shape optimization problems with the state function given by a solution to the Neumann problem for the Laplace equation is considered. To this end, the properties of the shape Hessian evaluated at critical shapes are analysed. First, it is proved that the stability cannot be expected for the model problem. Then, the new estimates for the shape Hessian are derived in order to overcome the classical *two norms-discrepancy* well know in control problems, Malanowski (2001). In the context of shape optimization, the situation is similar compared to control problems, actually, the shape Hessian can be coercive only in the norm strictly weaker with respect to the norm of the second order differentiability of the shape functional. In addition, it is shown that an appropriate regularization makes possible the stability of critical shapes.

Keywords: shape optimisation, stability analysis, shape gradient, shape Hessian.

[1]Partially supported by the French–Polish research programme POLONIUM between INRIA–Lorraine and the Systems Research Institute of the Polish Academy of Sciences.

1. Introduction

The classical theory of calculus of variations connects the stability of a critical point of the functional E with the study of the second order derivative of E at this point. It turns out that in shape optimisation, one has to restrict the admissible family of domains to regular shapes, $\mathcal{C}^1$ or $\mathcal{C}^2$ for example, in order to easily define the shape derivatives. Since many shape functionals are known to attain the extremal values for smooth shapes, the use of the differential optimisation theory is rather natural. The existence of an optimal shape can be obtained either by solving the associated Euler equation, or by the compactness argument combined with an additional regularity of the optimal shapes. An additional question raised in connection with this approach is the following one, on which we focus our attention. Usually, if the second order shape derivative is considered at a critical point and, furthermore, the shape Hessian is non negative, the Hessian is only coercive in a strictly weaker topology compared with the topology for which the differentiability can be proved. This difficulty is a general feature of optimisation problems in the infinite dimensional function spaces setting, Malanowski (2001). In others words, the positivity of the Hessian turns out to be a necessary condition for any critical shape to be only a stable point i.e., to be a local strict minimum, but we can not know *a priori* if this condition is also sufficient or not. That definition of stability is natural and it is much stronger compared to any directional stability which would follow directly from the posivity of the Hessian at the critical point. This difficulty is pointed out by J. Descloux (1990) for the specific example i.e., for the minimization of the Dirichlet energy functional in two dimensions. In such a case the state function solves the Poisson equation with the Dirichlet boundary conditions.

This particular case has been studied by Dambrine and Pierre (2000), who have shown that, in this case, the positivity of the second derivative is sufficient to insure the stability of the critical point. Then, in Dambrine (2000), the method has been extended to a wider class of problems by relaxing the required assumptions on the spatial dimension, the differential operator and the functional. However, only the second order scalar elliptic equations with the Dirichlet boundary conditions were considered. In the papers of Belov, Fuji (1997) and Eppier (2000) the second order optimality conditions are also considered. In this work, we discuss the case of the Neumann boundary datum. This type of boundary conditions increases the difficulty of the study.

For the sake of simplicity, we consider the following model problem. Let f be a given function in $\mathcal{C}_0^\infty(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} f = 0. \tag{1}$$

Let α and v_0 denote two non-negative real numbers, and $d \geq 2$ be the space dimension. We define the class $\mathcal{O}_d$ as the family of all open bounded subsets Ω of $\mathbb{R}$ with the boundaries $\partial\Omega$ such that

(i) $\partial\Omega$ is a $\mathcal{C}^{3,\alpha}$ manifold of dimension $d-1$,

(ii) $supp(f) \subset \Omega$ and $\partial\Omega \cap supp(f) = \emptyset$,

(iii) $\mathfrak{V}(\Omega) = v_0$,

where $\mathfrak{V}$ is the restriction of the d-dimensional Lebesgue measure $\mathcal{L}^d$ to the class $\mathcal{O}_d$. The $\mathcal{C}^{3,\alpha}$ topology on $\mathcal{O}_d$ is considered in the paper. We explain later on why this regularity is needed for the stability analysis. The deformation fields $\mathbf{V}$, used in the framework of the speed method, Sokołowski and Zolésio (1992), are supposed to be sufficiently regular to preserve the admissible class $\mathcal{O}_d$, actually $\mathcal{V} = \mathcal{C}^{3,\alpha}(\mathbb{R}^d, \mathbb{R}^d)$ is the sufficient regularity assumption for our purposes. Under the assumption (1) the Neumann problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega, \\ \partial_n u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

admits the regular solution u_Ω, which is defined up to an additive constant. To make the solution of (2) unique, we use the following normalisation condition

$$\int_{\partial\Omega} u_\Omega = 0 \ . \tag{3}$$

We define the regularized energy shape functional E_σ on $\mathcal{O}_d$ by

$$\begin{aligned} E_\sigma(\Omega) &= \frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 - \int_\Omega f u_\Omega + \sigma\mathcal{H}^{d-1}(\partial\Omega), \\ &= -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 + \sigma\mathcal{H}^{d-1}(\partial\Omega). \end{aligned} \tag{4}$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)-$Hausdorff measure, so that $\mathcal{H}^{d-1}(\partial\Omega)$ is the perimeter of the smooth boundary $\partial\Omega$, and $\sigma \geq 0$ is the regularization parameter. Since the gradient of u_Ω is uniquely defined, the energy functional E_σ is well-defined on $\mathcal{O}_d$ even without any normalisation condition imposed on the state function u_Ω. We adress in this paper the following precise issue:

The identification of the conditions required for the specific shape optimization problem which assure for the functional E_σ the existence of a local strict maximum at the shape Ω^* in the topology of $\mathcal{O}_d$.

Such a shape Ω^* will be called a stable critical shape for the shape functional E_σ.

To simplify the presentation, we will focuss our attention on the energy shape functional $E = E_0$, that is — the shape functional without the perimeter term which is in fact a regularisation term. The results of this paper remain valid in cases of both $\sigma = 0$ and $\sigma > 0$. We mainly consider the question of stability of critical shapes and not the existence of optimal shapes. However, we do not know if any critical shape Ω^* exists for $\sigma = 0$. We can even present some non existence results which are given in Section 2.2. Let us point out that $\sigma > 0$ is sufficient for the existence of an optimal shape.

Section 2 is devoted to the computation of the shape gradient DE of E and to some remarks related to the topology of stable critical shapes. Actually, we prove that in some situations the critical shapes, if any, are unstable. In Section 3 the shape Hessian is evaluated at the critical shape and the question of its definiteness is adressed. In particular, we prove that the situation described above actually occurs. The second order shape differentiability of the functional E is obtained in the $\mathcal{C}^{2,\alpha}$ norm, however, the coercivity of the second derivative D^2E at any critical shape can only be expected, when it is the case, in the norm of the fractional Sobolev space $H^{1/2}$ on the boundary. Section 4 is the main part of this work, it includes the proof of the intermediate estimate

$$\left|e''_\Theta(t) - e''_\Theta(0)\right| \leq C\omega(\|\Theta - I_d\|_{2,\alpha})\|\langle \boldsymbol{X}_\Theta, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega_0)} \,. \tag{5}$$

Here, ω is a modulus of continuity; Θ is an arbitrary $\mathcal{C}^{2,\alpha}$ perturbation of the identity I_d in $\mathbb{R}^d$ and e_Θ denotes the restriction of the energy shape functional E to the specific path connecting Ω_0 with $\Theta(\Omega_0)$. Such a path is constructed using the flow of the given vector field $\boldsymbol{X}_\Theta$. In other words, our construction is based on the existence of the path, within the family of admissible shapes, connecting the given shape Ω and its image obtained by the diffeomorphism Θ, the image being denoted by $\Theta(\Omega)$. The particular path can be parametrized by the family of domains $(\Omega_t)_{t\in[0,1]}$ with $\Omega_0 = \Omega$ and $\Omega_1 = \Theta(\Omega)$. For such a parametrization, the function $e_\Theta(t) = E(\Omega_t)$ is defined over the interval $[0, 1]$. We keep in our notation the dependence on Θ to emphasize the following crucial fact that the main estimate (5) is uniform with respect to Θ, for sufficiently small appropriate norms of Θ. Section 5 is devoted to the study of E_σ, in particular, the existence of a stable critical shape is proved. In Section 6, some of the formulae needed in Section 4 are established. Now, we can state the main result of this work. We denote by $\boldsymbol{n}$ the exterior unit normal vector field to the boundary of $\Omega_0 \in \mathcal{O}_d$. $\mathcal{H}$ stands for the tangent hyperplane, in the space of deformation vector fields, to the volume shape constraints at the shape Ω_0.

Theorem 1.1 *Let $\Omega_0 \in \mathcal{O}_d$ be a critical shape of the energy functional E under the volume constraint $\mathfrak{V}(\Omega) = v_0$, and let Λ^* be the corresponding Lagrange multiplier. If the shape Hessian $D^2L_{\Lambda^*}(\Omega_0)$ of the Lagrangian $L_{\Lambda^*} = E + \Lambda^*\mathfrak{V}$ is negative definite, with the second order shape differential which satisfies the following inequality*

$$D^2L_{\Lambda^*}(\Omega_0)(\boldsymbol{V}, \boldsymbol{V}) \leq -C\|\langle \boldsymbol{V}, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega_0)} \tag{6}$$

for some $C > 0$ and for all fields $\boldsymbol{V} \in \mathcal{H}$, then Ω_0 is a stable critical shape for E in $\mathcal{O}_d$ i.e., a local strict maximum of E at Ω_0 occurs.

REMARK 1.1 *We make an intensive use of the shape calculus theory without recalling all the details, we refer e.g., to the book by J. Sokołowski and J.P. Zolésio (1992) and to the technical report by F. Murat and J. Simon (1977) where this theory is presented in the case of speed method and perturbation of identity method, respectively. We will use the speed method, but only with autonomous vector fields* $\boldsymbol{V}$*, for the shape sensitivity analysis. Since we deal only with regular shapes, this assumption is not restrictive. We use the notations from Sokołowski and Zolésio (1992). In particular,* $DE(\Omega;\boldsymbol{V})$ *denotes the directional derivative (Eulerian semi-derivative) of the energy functional* E *evaluated at the domain* Ω *in the direction of the field* $\boldsymbol{V}$*.*

The validity of estimate (6) is discussed in Section 3. Note that Theorem 1.1 is an easy consequence of estimate (5). Using the Taylor formula along the path Ω_t, we have

$$E(1) = E(0) + \int_0^1 (1-t)e''_\Theta(t)dt\ .$$

By the $H^{1/2}$-coercivity assumption combined with (5) we have

$$e''_\Theta(t) = \underbrace{e''_\Theta(0)}_{\leq -C\|\langle \boldsymbol{V},\boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega_0)}} + \underbrace{e''_\Theta(t) - e''_\Theta(0)}_{\leq C\omega(\eta)\|\langle \boldsymbol{V},\boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega_0)}},$$

Therefore, for η small enough, we get for all $t \in [0,1]$:

$$e''_\Theta(t) \leq -\frac{C}{2}\|\langle \boldsymbol{V},\boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega_0)} < 0 \ \Rightarrow\ E(1) < E(0).$$

2. Shape derivatives and the resulting Euler equation

2.1. Shape gradients and the Euler-Lagrange equation

We use tangential differential operators, in particular, the tangential gradient $\nabla_{\boldsymbol{\tau}}$ and the tangential divergence $\mathrm{div}_{\boldsymbol{\tau}}(\cdot)$. For the sake of simplicity, we omit in our notation any explicit dependence of tangential operators on the boundaries $\partial\Omega$ or $\partial\Omega_t$. Under the regularity assumption (i), it is known (see Sokołowski and Zolésio, 1992, section 2.29) that E is differentiable in $\mathcal{O}_d$ and that

$$\forall \boldsymbol{V} \in \mathcal{V}, \forall\Omega \in \mathcal{O}_d, \quad DE(\Omega;\boldsymbol{V}) = -\int_\Omega \langle \nabla u', \nabla u\rangle - \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \langle \boldsymbol{V},\boldsymbol{n}\rangle,$$

where u' denotes the shape derivative of the state function $u = u_\Omega$, solution of the boundary value problem (see Proposition 3-2 in Sokołowski and Zolésio, 1992)

$$\left\{\begin{array}{rcl} -\Delta v &=& 0 \text{ in } \Omega,\\ \partial_n v &=& \mathrm{div}_{\boldsymbol{\tau}}\left(\langle \boldsymbol{V},\boldsymbol{n}\rangle \nabla_{\boldsymbol{\tau}} u\right) \text{ on } \partial\Omega\ . \end{array}\right. \tag{7}$$

We introduce the notation $N := \operatorname{div}_{\boldsymbol{\tau}} (\langle \boldsymbol{V}, \boldsymbol{n}\rangle \nabla_{\boldsymbol{\tau}} u)$, used later on, especially in Section 4.

REMARK 2.1 *An integration by parts leads to*

$$\int_{\partial\Omega} \operatorname{div}_{\boldsymbol{\tau}} (\langle \boldsymbol{V}, \boldsymbol{n}\rangle \nabla_{\boldsymbol{\tau}} u) = -\int_{\partial\Omega} \langle \boldsymbol{V}, \boldsymbol{n}\rangle \langle \nabla_{\boldsymbol{\tau}} u, \nabla_{\boldsymbol{\tau}} 1\rangle.$$

The integral in the right hand side vanishes since $\nabla_{\boldsymbol{\tau}} 1 = 0$. It means that the Neumann problem (7) fulfills the compability condition required for the boundary datum, therefore, it is well posed.

Applying Green's formula and the boundary conditions in (7), we obtain

$$\begin{aligned}\int_{\Omega} \langle \nabla u', \nabla u\rangle &= \int_{\partial\Omega} u \partial_n u' - \int_{\Omega} u \Delta u' = \int_{\partial\Omega} u \operatorname{div}_{\boldsymbol{\tau}} (\langle \boldsymbol{V}, \boldsymbol{n}\rangle \nabla_{\boldsymbol{\tau}} u)\,,\\ &= -\int_{\partial\Omega} |\nabla_{\boldsymbol{\tau}} u|^2 \langle \boldsymbol{V}, \boldsymbol{n}\rangle = -\int_{\partial\Omega} |\nabla u|^2 \langle \boldsymbol{V}, \boldsymbol{n}\rangle.\end{aligned}$$

Therefore, the first order shape derivative of the energy functional is given by

$$\forall \boldsymbol{V} \in \mathcal{V}, \forall \Omega \in \mathcal{O}_d, \quad DE(\Omega; \boldsymbol{V}) = \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \langle \boldsymbol{V}, \boldsymbol{n}\rangle. \tag{8}$$

Note that we have obtained for the shape gradient an expression depending on the state function u_Ω, but independent of the shape derivative u' of the state function u_Ω. This property is specific for the energy functionals and it is no longer valid for any arbitrary elliptic shape functional, e.g., of the form

$$J(\Omega) = \int_{\Omega} g(x, u_\Omega, \nabla u_\Omega)\ ,$$

where g is a given smooth function. Anyway, we can expect that the conclusions of Theorem 1.1 can be extended for functionals of the above form, as it is the case of shape functionals for the Laplace equation with the Dirichlet boundary conditions (see Dambrine, Pierre, 2000, and Dambrine, 2000).

In the presence of the condition (iii) the shape optimisation problem under considerations is constrained. Hence, any critical point of shape functional solves the Euler equation for the Lagrangian L_Λ defined on the family $\mathcal{O}_d$ by

$$L_\Lambda(\Omega) = E(\Omega) + \Lambda\mathfrak{V}(\Omega) \quad \text{ with } \Lambda \in \mathbb{R}\setminus\{0\}. \tag{9}$$

Therefore, the existence of a critical shape defined by solutions of the Euler equation and denoted by Ω^* requires the existence of the nontrivial multiplier $\Lambda^* \neq 0$ such that the Euler equation

$$\forall \boldsymbol{V} \in \mathcal{V}, \quad DL_{\Lambda^*}(\Omega; \boldsymbol{V}) = 0 \tag{10}$$

admits at least one solution in $\mathcal{O}_d$. The critical shape Ω^* satisfies by definition

$$\forall \boldsymbol{V} \in \mathcal{V}, \quad DL_{\Lambda^*}(\Omega^*; \boldsymbol{V}) = \int_{\partial\Omega^*} \Big[\frac{1}{2}|\nabla u_{\Omega^*}|^2 + \Lambda^*\Big]\langle \boldsymbol{V}, \boldsymbol{n}\rangle = 0 \ .$$

Since the first order shape derivative vanishes (10) for all vector fields in $\mathcal{V}$, the critical shape Ω^* implies the additional boundary condition

$$\frac{1}{2}|\nabla u_{\Omega^*}|^2 + \Lambda^* = 0 \text{ on } \partial\Omega^*. \tag{11}$$

The function u_{Ω^*} solves the homogeneous Neumann problem, hence $|\nabla u_{\Omega^*}|^2 = |\nabla_{\boldsymbol{\tau}} u_{\Omega^*}|^2$ holds on $\partial\Omega^*$ and

$$\frac{1}{2}|\nabla_{\boldsymbol{\tau}} u_{\Omega^*}|^2 + \Lambda^* = 0 \text{ on } \partial\Omega^*. \tag{12}$$

The Euler equation provides a simple characterisation of the boundary of any critical shape Ω^* as a geometrical domain, which has the property that the overdetermined problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega^*, \\ \partial_n u &= 0 \text{ on } \partial\Omega^*, \\ \frac{1}{2}|\nabla_{\boldsymbol{\tau}} u_{\Omega^*}|^2 + \Lambda^* &= 0 \text{ on } \partial\Omega^*. \end{cases}$$

admits a solution.

2.2. Stability of domains determined by the Euler-Lagrange equation

We can deduce from the first order necessary optimality conditions further information on the topology of the optimal domain Ω^* which is assumed to be a stable critical shape.

Theorem 2.1 *Assume that $d = 2$ or $d = 3$. If Ω^* is a stable critical shape of E_σ in the family O_d, then any connected component of Ω^* cannot be diffeomorphic to the unit sphere S^{d-1}.*

Proof. We prove the theorem by contradiction, and distinguish the cases of $d = 2$ and $d = 3$. First, we fix the notation. Assume that the domain Ω^* has a finite number of connected components $(\Omega_i)_{i\in\{1,\dots,n\}}$. The boundary of Ω_i is denoted by $\partial\Omega_i$. We assume that the connected component denoted by Ω_1 is diffeomorphic to S^{d-1} and, in this way, we obtain a contradiction.

First, let us consider the bidimensional case. The boundary $\partial\Omega_1$ of Ω_1 is a single Jordan curve of the length $L > 0$. Hence, there exists a function γ of the class $\mathcal{C}^{3,\alpha}([0,2],\mathbb{R}^2)$ such that

(γi) $\gamma(0) = \gamma(L)$,

(γii) γ is one-to-one from $[0, L)$ onto $\partial\Omega_1$,

(γiii) $\forall s \in [0, L], \|\gamma'(s)\| = 1$.

The tangential gradient of the solution is given by the derivative $\frac{d}{ds} u_{\Omega^*}(\gamma(s))$. Equation (11) implies that

$$\left| \frac{d}{ds} u_{\Omega^*}(\gamma(s)) \right| = \sqrt{-2\Lambda^*}.$$

From the elliptic regularity theory it follows that the trace of the solution u_{Ω^*} on $\partial\Omega_1$ is continuous. Hence, there exists $\epsilon \in \{-1, 1\}$ such that

$$\forall s \in [0, L], \quad \frac{d}{ds} u_{\Omega^*}(\gamma(s)) = \epsilon\sqrt{-2\Lambda^*}.$$

Since $u_{\Omega^*}(\gamma(0)) = u_{\Omega^*}(\gamma(L))$, it follows that for any connected component

$$0 = u_{\Omega^*}(\gamma_i(0)) - u_{\Omega^*}(\gamma_i(|\partial\Omega_i|)) = \int_0^{|\partial\Omega_i|} \frac{d}{ds} u_{\Omega^*}(\gamma_i(s))\, ds = \epsilon_i \sqrt{-2\Lambda^*}|\partial\Omega_i|.$$

In other words, the Lagrange multiplier Λ^* is trivial. Therefore, the volume constraint can be ignored and the Euler-Lagrange equation (10) reduces to the Euler equation

$$\forall \boldsymbol{V} \in \mathcal{V}, \quad DE(\Omega^*; \boldsymbol{V}) = 0.$$

Moreover, we have on $\partial\Omega^*$

$$\nabla u_{\Omega^*} = 0 \Rightarrow u_{\Omega^*} \text{ is constant on } \partial\Omega^*.$$

By the normalisation condition $\int_{\Omega^*} u = 0$, this constant equals zero. Therefore, Ω^* is a $\mathcal{C}^{2,\alpha}$ domain such that for a function f there exists a solution of the following problem

$$\begin{cases} -\Delta u &= f \text{ in } \Omega^*, \\ u &= 0 \text{ on } \partial\Omega^*, \\ \partial_n u &= 0 \text{ on } \partial\Omega^*. \end{cases} \tag{13}$$

We show that the solution u_{Ω^*} vanishes outside of the support of the right hand side f. To this end, let us consider an open neighbourhood U of $\partial\Omega^*$. By the Holmgren unique continuation theorem there exists an open neighbourhood U of $\partial\Omega^*$ such that the local problem

$$\begin{cases} -\Delta u &= 0 \text{ in } U, \\ u &= 0 \text{ on } \partial\Omega^*, \\ \partial_n u &= 0 \text{ on } \partial\Omega^*. \end{cases} \tag{14}$$

admits a unique solution which is obviously the trivial solution $u = 0$. As U can always be restricted in such a way that $U \cap supp(f) = \emptyset$, the solution u_{Ω^*} vanishes outside of the support of the right hand side f. Let $\Omega \in \mathcal{O}_2$ be such that $\partial\Omega \subset U$. Then, the open set U is as well a neighbourhood of $\partial\Omega$ and u_{Ω^*} solves also

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \,, \\ \partial_n u &= 0 \text{ on } \partial\Omega \,. \end{cases}$$

The Holmgren Theorem (see for example Courand and Hilbert, 1989, page 238) provides the local uniqueness result for the state function since the C^2-boundary $\partial\Omega^*$ is not characteristic. In other words, u_{Ω^*} can be defined on the whole set U and u_{Ω^*} vanishes identically on U. Therefore, for all admissible domains Ω with $\partial\Omega \subset U$, and $\Omega \setminus U = \Omega^* \setminus U$, we have

$$E(\Omega) = -\frac{1}{2}\int_{\Omega} |\nabla u_{\Omega^*}|^2 = -\frac{1}{2}\int_{\Omega\setminus U} |\nabla u_{\Omega^*}|^2 = -\frac{1}{2}\int_{\Omega^*} |\nabla u_{\Omega^*}|^2 = E(\Omega^*).$$

Hence E is locally constant around Ω^*. Therefore, the critical shape Ω^* is unstable.

Let us now consider the tridimensionnal case. The same analysis remains valid except for the way to justify the triviality of the Lagrange multiplier and the regularity of $\partial\Omega^*$. The first order necessary optimality condition for $\partial\Omega^*$ takes the form

$$\frac{1}{2}|\nabla_{\boldsymbol{\tau}} u_{\Omega^*}|^2 + \Lambda^* = 0 \text{ on } \partial\Omega^*.$$

Therefore, $\nabla_{\boldsymbol{\tau}} u_{\Omega^*}$ is a continuous vector field by the elliptic regularity theory, which is tangent by definition. If we assume that Ω^* is the image of a ball by a given diffeomorphism, then the tangent vector field has to vanish at some point of $\partial\Omega^*$ as stated by the Hairy Sphere Theorem. On the other hand, the norm of the vector $\nabla_{\boldsymbol{\tau}} u_{\Omega^*}(x)$ equals $-2\Lambda^*$ for $x \in \partial\Omega^*$, and the norm $|\nabla_{\boldsymbol{\tau}} u_{\Omega^*}|$ is constant on $\partial\Omega^*$. Therefore, the norm equals zero everywhere on $\partial\Omega^*$. By the Holmgren theorem, $u_{\Omega^*} = 0$ in an open neighbourhood of $\partial\Omega^*$. We can conclude in the same way as in dimension two that the critical shape Ω^* is unstable. Note that this argument is still valid if $d = 2n + 1$, since the Hairy Sphere Theorem holds in such a case.

In conclusion, we claim that the only possible topologies for Ω^* in dimension 3 are domains with a smooth boundary of gender bigger or equal to one such as e.g., a torus. We restrict our attention to the case of critical shapes i.e., we assume that Ω^* is a critical shape and evaluate the shape Hessian $D^2 L_{\Lambda^*}(\Omega^*; \boldsymbol{V}, \boldsymbol{V})$.

3. The sign of shape Hessian at critical shapes

As it is the case in the classical theory of calculus of variations, the local extremes in constrained optimization problems imply the sign of the Hessian. We are going to verify the sign of the Hessian $D^2 L_\Lambda(\Omega^*; \boldsymbol{V}, \boldsymbol{V})$ on the kernel $\mathcal{H}$ of the constraints. Since the volume constraints are imposed, $\mathcal{H}$ is the kernel of $D\mathfrak{V}$ i.e.,

$$\mathcal{H} = \big\{ \boldsymbol{V} \in \mathcal{V}, \int_{\partial\Omega^*} \langle \boldsymbol{V}, \boldsymbol{n} \rangle = 0 \big\}. \tag{15}$$

Now, we evaluate the shape Hessian $D^2 L_\Lambda(\Omega^*; \boldsymbol{V}, \boldsymbol{V})$.

First, we recall the useful lemma for derivation of the shape Hessian. Note that the expression of the shape Hessian obtained via this lemma is not the same as the canonical form given by the structure theorem at the critical point. The form derived below is convenient to deal with for the stability analysis performed in Section 4.

LEMMA 3.1 *Let Ω be a domain in $\mathcal{O}$, $\boldsymbol{V}$ a deformation field in $\mathcal{V}$ and g be a smooth function in $\mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$. Let J be a shape functional defined on $\mathcal{O}$,*

$$\forall \Omega \in \mathcal{O}, \quad J(\Omega) = \int_{\partial\Omega} g \langle \boldsymbol{V}, \boldsymbol{n} \rangle.$$

Then J is differentiable on $\mathcal{O}$ in the direction $\boldsymbol{V}$ and, moreover, the first order shape derivative is given by

$$\forall \Omega \in \mathcal{O}, \quad DJ(\Omega; \boldsymbol{V}) = \int_{\partial\Omega} \big[g' + \operatorname{div}\big(g\boldsymbol{V}\big) \big] \langle \boldsymbol{V}, \boldsymbol{n} \rangle.$$

Proof. It is just a game between the Hadamard derivation Lemma and Green's formula. Actually, by Green's formula, we have

$$J(\Omega) = \int_{\partial\Omega} g \langle \boldsymbol{V}, \boldsymbol{n} \rangle = \int_{\Omega} \operatorname{div}\big(g\boldsymbol{V}\big);$$

so an application of the Hadamard formula to the domain integral leads to

$$DJ(\Omega; \boldsymbol{V}) = \int_{\Omega} \operatorname{div}\big(g'\boldsymbol{V}\big) + \operatorname{div}\big(\operatorname{div}\big(g\boldsymbol{V}\big)\boldsymbol{V}\big) \ .$$

This expression can be rewritten in the form of a boundary integral

$$DJ(\Omega; \boldsymbol{V}) = \int_{\partial\Omega} \big[g' + \operatorname{div}\big(g\boldsymbol{V}\big) \big] \langle \boldsymbol{V}, \boldsymbol{n} \rangle.$$

■

We recall the expression of the shape gradient for the volume constraints

$$D\mathfrak{V}(\Omega;\boldsymbol{V}) = \int_{\partial\Omega} \langle \boldsymbol{V}, \boldsymbol{n}\rangle.$$

By application of Lemma 3.1, we get

$$D^2\mathfrak{V}(\Omega;\boldsymbol{V},\boldsymbol{V}) = \int_{\partial\Omega} \operatorname{div}\big(\boldsymbol{V}\big)\langle \boldsymbol{V}, \boldsymbol{n}\rangle. \tag{16}$$

More delicate is application of Lemma 3.1 to the shape derivative of the energy functional $DE(\Omega,\boldsymbol{V})$ since the appropriate element g is defined only on Ω. Such a difficulty, which is common in the shape sensitivity analysis, does not imply any change of the related result of the lemma given in Dambrine (2000). We get

$$\begin{aligned} D^2E(\Omega;\boldsymbol{V},\boldsymbol{V}) = \int_{\partial\Omega} \langle \nabla u', \nabla u\rangle\langle \boldsymbol{V}, \boldsymbol{n}\rangle + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \operatorname{div}\big(\boldsymbol{V}\big)\langle \boldsymbol{V}, \boldsymbol{n}\rangle \\ + \frac{1}{2}\int_{\partial\Omega} \langle \nabla|\nabla u|^2, \boldsymbol{V}\rangle\langle \boldsymbol{V}, \boldsymbol{n}\rangle. \end{aligned} \tag{17}$$

At the critical shape Ω^*, (11) holds and a straightforward computation leads to

$$D^2L_\Lambda(\Omega^*;\boldsymbol{V},\boldsymbol{V}) = \int_{\partial\Omega^*} \langle \nabla u', \nabla u\rangle\langle \boldsymbol{V}, \boldsymbol{n}\rangle + \frac{1}{2}\int_{\partial\Omega^*} \langle \nabla|\nabla u|^2, \boldsymbol{V}\rangle\langle \boldsymbol{V}, \boldsymbol{n}\rangle. \tag{18}$$

Since $|\nabla u|^2$ is constant on $\partial\Omega^*$ by (11), we have $\langle \nabla|\nabla u|^2, \boldsymbol{V}\rangle = \partial_{\boldsymbol{n}}|\nabla u|^2\langle \boldsymbol{V}, \boldsymbol{n}\rangle$. As expected from the structure of the shape Hessian at a critical point, only the normal component of $\boldsymbol{V}$ is present in the second order shape derivative of the Lagrangian evaluated at Ω^*.

Now, the sign of Hessian is investigated on the hyperplane $\mathcal{H}$. Let us split the Hessian computed in (17) into two parts, which are analysed separately,

$$A(t) = \int_{\partial\Omega_t} \langle \nabla|\nabla u|^2, \boldsymbol{V}\rangle\langle \boldsymbol{V}, \boldsymbol{n}\rangle, \tag{19}$$

$$B(t) = \int_{\partial\Omega_t} \langle \nabla u', \nabla u\rangle\langle \boldsymbol{V}, \boldsymbol{n}\rangle. \tag{20}$$

First, we consider $A(0)$. The normal derivative $\partial_{\boldsymbol{n}}|\nabla u|^2$ is related to the geometry of $\partial\Omega^*$ as it is expressed in the following lemma.

LEMMA 3.2 *Let K denote the Gauss curvature on $\partial\Omega^*$. Then, we have*

$$\partial_{\boldsymbol{n}}|\nabla u|^2 = 2\ {}^t(\nabla_{\boldsymbol{\tau}} u)\ K\ \nabla_{\boldsymbol{\tau}} u,$$

furthermore, there exist two constants κ_{min} and κ_{max} such that

$$\kappa_{min}\ |\Lambda^*|\ \|\langle \boldsymbol{V}, \boldsymbol{n}\rangle\|_{L^2(\partial\Omega^*)} \le \frac{1}{2}A(0) \le \kappa_{max}\ |\Lambda^*|\ \|\langle \boldsymbol{V}, \boldsymbol{n}\rangle\|_{L^2(\partial\Omega^*)}.$$

Proof. The derivation of the above formula and of the estimates is based on computations in local coordinates. Let M be a given point on $\partial\Omega^*$. There exists an orthogonal coordinate system in $\mathbb{R}^3$ such that locally around $M = (0,0,0)$, $\partial\Omega^*$ admits the parametrisation $z = f(x,y)$ where f is chosen in such a way that $f(0,0) = f_x(0,0) = f_y(0,0) = 0$. On $\partial\Omega^*$, u is given by $u\big(x,y,f(x,y)\big)$ and the Neumann boundary condition leads to

$$\begin{aligned}\langle \nabla u, \boldsymbol{n}\rangle &= \frac{1}{\sqrt{1+f_x^2+f_y^2}}\begin{pmatrix}u_x\\ u_y\\ u_z\end{pmatrix}\cdot\begin{pmatrix}-f_x\\ -f_y\\ 1\end{pmatrix}\\ &= \frac{u_xf_x+u_yf_y-u_z}{\sqrt{1+f_x^2+f_y^2}} = 0 \text{ on } \partial\Omega^*. \qquad (21)\end{aligned}$$

Whence $u_xf_x+u_yf_y-u_z = 0$ on $\partial\Omega^*$, in particular, $u_z(M) = 0$. We differentiate (21) with respect to x and y. Taking the values at $M = (0,0,0)$ leads to

$$\begin{cases} f_{xx}u_x + f_{xy}u_y = u_{zx},\\ f_{xy}u_x + f_{yy}u_y = u_{zy}.\end{cases} \qquad (22)$$

We want to compute $\partial_{\boldsymbol{n}}|\nabla u|^2$, that is

$$\langle \nabla|\nabla u|^2, \boldsymbol{n}\rangle = \frac{2}{\sqrt{1+f_x^2+f_y^2}}\begin{pmatrix}u_xu_{xx}+u_yu_{yx}+u_zu_{zx}\\ u_xu_{xy}+u_yu_{yy}+u_zu_{zy}\\ u_xu_{xz}+u_yu_{yz}+u_zu_{zz}\end{pmatrix}\cdot\begin{pmatrix}-f_x\\ -f_y\\ 1\end{pmatrix}.$$

For the point M, we get from (22)

$$\begin{aligned}\langle \nabla|\nabla u|^2, \boldsymbol{n}\rangle =& \frac{2}{\sqrt{1+f_x^2+f_y^2}}\Big[u_x\big(f_{xx}u_x+f_{xy}u_y\big)+u_y\big(f_{xy}u_x+f_{yy}u_y\big)\Big],\\ =& \frac{2}{\sqrt{1+f_x^2+f_y^2}}\begin{pmatrix}u_x & u_y\end{pmatrix}\begin{pmatrix}f_{xx} & f_{xy}\\ f_{xy} & f_{yy}\end{pmatrix}\begin{pmatrix}u_x\\ u_y\end{pmatrix},\\ =& 2\ {}^t(\nabla_{\boldsymbol{\tau}}u)\ K\ \nabla_{\boldsymbol{\tau}}u.\end{aligned}$$

By compactness of $\partial\Omega^*$, the extremal eigenvalues of the continuous Gauss curvature K on $\partial\Omega^*$ attain the minimal and maximal values κ_{min} and κ_{max}, respectively. The constants κ_{min} and κ_{max} are determined by the minimal and the maximal values of the algebraic curvatures of arcs traced on $\partial\Omega^*$. Moreover, (11) relates the norm of $\nabla_{\boldsymbol{\tau}}u_{\Omega^*}$ to the Lagrange multiplier Λ^* so that

$$2|\Lambda^*|\kappa_{min} \leq^t (\nabla_{\boldsymbol{\tau}}u_{\Omega^*})\ K\ \nabla_{\boldsymbol{\tau}}u_{\Omega^*} \leq 2|\Lambda^*|\kappa_{max}.$$

■

REMARK 3.1 *We cannot completely determine the quantity* $\langle \nabla |\nabla u|^2, \boldsymbol{n}\rangle$ *as we know only two conditions for* ∇u *on* $\partial\Omega^*$ *so it is not sufficient to determine the three unknown coordinates. The situation is different for the Dirichlet boundary datum, we refer e.g., to Descloux (1990) for details.*

Now, we turn our attention to the term $B(0)$. The main characteristic of this term is that it is negative, so it has a sign.

LEMMA 3.3 *There exists a constant* $C > 0$ *depending only of* Ω^* *such that*

$$\forall \boldsymbol{V} \in \mathcal{H}, \quad B(0) \leq \ - \ C \ \|\langle \boldsymbol{V}, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega^*)} \ . \tag{23}$$

Proof. Since $\partial_{\boldsymbol{n}} u = 0$, then

$$B(0) = \int_{\partial\Omega^*} \langle \nabla_{\boldsymbol{\tau}} u', \nabla_{\boldsymbol{\tau}} u\rangle \langle \boldsymbol{V}, \boldsymbol{n}\rangle$$

and an integration by parts leads to

$$B(0) = -\int_{\partial\Omega^*} u' \operatorname{div}_{\boldsymbol{\tau}} (\nabla_{\boldsymbol{\tau}} u \langle \boldsymbol{V}, \boldsymbol{n}\rangle) = -\int_{\partial\Omega^*} u' \partial_{\boldsymbol{n}} u'.$$

Using the Green's formula we obtain

$$\int_{\partial\Omega^*} u' \partial_{\boldsymbol{n}} u' - \int_{\Omega^*} u' \Delta u' = \int_{\Omega^*} |\nabla u'|^2 \ ,$$

thus, $B(0) = -\|u'\|^2_{H^1_0(\Omega^*)}$. Since u' solves (7), the elliptic regularity for weak solution implies the first inequality below

$$\begin{aligned} \|u'\|_{H^1(\Omega^*)} &\leq C_1 \|\partial_n u'\|_{H^{-1/2}(\partial\Omega^*)} \leq C_2 \|\langle \boldsymbol{V}, \boldsymbol{n}\rangle \nabla_{\boldsymbol{\tau}} u\|_{H^{1/2}(\partial\Omega^*)} \\ &\leq C_3 \|\langle \boldsymbol{V}, \boldsymbol{n}\rangle\|_{H^{1/2}(\partial\Omega^*)} \end{aligned}$$

and we can justify the second and the third inequalities as follows. Since $\operatorname{div}_{\boldsymbol{\tau}}(\cdot)$ is a linear and continuous mapping from $H^{1/2}(\partial\Omega^*)$ into $H^{-1/2}(\partial\Omega^*)$, the second inequality holds. The third inequality can be deduced from the auxiliary multiplier lemma proved in Dambrine (2000) for the space $H^{1/2}(\partial\Omega^*) \times \mathcal{C}^1(\partial\Omega^*)$. The lemma can be used, since, from the elliptic regularity theory it follows that $\nabla_{\boldsymbol{\tau}} u$ is $\mathcal{C}^1$ on $\partial\Omega^*$. Obviously, the constants C_1, C_2 and C_3 depend on the geometrical attributes of the optimal shape Ω^*. ■

Conclusion. The sign of the Hessian $D^2 L_\Lambda(\Omega^*; \boldsymbol{V}, \boldsymbol{V})$ depends only of the geometry of Ω^*. It is clear that the geometrical constraints to be satisfied (see Theorem 2.1) are relatively strong, whence we are unable at the moment to present an explicit and simple example of the critical shape for which assumption (6) is verified. However, we have shown in this section that the principal term of shape Hessian, that is $B(0)$, is non-positive. Therefore, it seems reasonable to assume the existence of a critical point Ω^* for the energy functional such that the necessary condition of non-negativity of $D^2 L_\Lambda(\Omega^*; \boldsymbol{V}, \boldsymbol{V})$ is fulfilled.

The second remark is fundamental from the point of view of stability analysis Even if the Hessian $D^2 L_\Lambda(\Omega^*; \boldsymbol{V}, \boldsymbol{V})$ is anticoercive, the related inequality holds only for the $H^{1/2}$ norm and cannot be expected to be valid in the stronger norm of the differentiability, which is actually the $\mathcal{C}^2$ norm. We are in the situation indicated by J. Descloux and have to find a remedy for this difficulty in order to perform the stability analysis. Therefore, we are going to prove the key estimate (5) which allows for positive results in the stability analysis.

4. Proof of the stability estimate (5)

In this section, we fix the critical shape Ω_0 of E and a constant $\eta > 0$ which defines the maximal *size* in the norm $\mathcal{C}^{2,\alpha}$ of the perturbations which deforms the shape Ω_0. Let Θ denote an arbitrary diffeomorphism, an element of the ball with the centre $I_{\mathbb{R}^3}$ and of the radius η. We show in this section that the condition $D^2 L_\Lambda(\Omega_0; \boldsymbol{V}, \boldsymbol{V}) \geq c\|\boldsymbol{V}\|^2_{H^{1/2}(\partial\Omega_0)}$ implies the strict inequality for the energy functional $E(\Omega_0) < E(\Theta(\Omega_0))$.

The method proposed in Dambrine and Pierre (2000) and developed in Dambrine (2002) consists of the geometrical part with the construction of a path $\Omega(t)$ connecting the critical shape Ω_0 with the target shape $\Theta(\Omega_0)$, all elements of the path belonging to the family of domains with the fixed volume v_0. Such a construction is accomplished using the flow $\Phi_{\Theta,t}$ of the particular vector field $\boldsymbol{X}_\Theta$, with the intermediate shapes $\Omega(t)$ given by the image $\Phi_{\Theta,t}(\Omega_0)$ of the shape Ω_0. We consider the restriction of the shape functional to this path, that is

$$e_\Theta(t) = E\big(\Phi_t(\Omega_0)\big). \tag{24}$$

Then, the variations of the second derivative $e''_\Theta(t)$ of e_Θ with respect to the variable t can be estimated and the following result is established.

PROPOSITION 4.1 *There exists the modulus of continuity ω and the number η_0 such that for all $\eta \in (0, \eta_0)$ and each volume preserving diffeomorphism Θ with $\|\Theta - Id_{\mathbb{R}^d}\|_{3,\alpha} \leq \eta$,*

$$\Big| e''_\Theta(t) - e''_\Theta(0) \Big| \leq C\omega(\eta) \|\langle \boldsymbol{X}_\Theta, \boldsymbol{n}\rangle\|^2_{H^{1/2}(\partial\Omega_0)}\ , \tag{25}$$

for all $t \in [0, 1]$.

For the proof of Proposition 4.1, the arguments used for the Dirichlet problem in Dambrine and Pierre (2000) are adapted. We start with necessary preliminaries. The results of geometrical nature, which we use throughout the paper, are only reported. For the complete proofs we refer the reader to Dambrine (2000) in the bidimensional case and to Dambrine (2002) in the general case.

Preliminaries on the path construction within the family of admissible shapes. To simplify the notation, the dependence on Θ is not indicated whenever Θ is fixed, which is the case in the second part of the section. However, in the first part of the section we indicate the dependence on Θ, since we establish the uniform estimates with respect to Θ. As it is shown in Dambrine (2002), the sufficient conditions for the existence of a local strict minimum for energy functionals can be established by a construction of some specials paths within the admissible family of shapes. We use the same vector fields as those exploited in Dambrine (2002). We recall that, once an initial shape Ω_0 and an admissible perturbation Θ are chosen, then there exists the normal vector field $\boldsymbol{X}_\Theta$ such that

- the flow $\Phi_{\Theta,t}$ of the field $\boldsymbol{X}_\Theta$ at $t = 1$ maps Ω_0 onto $\Theta(\Omega_0)$
- the field $\boldsymbol{X}_\Theta$ is divergence-free,
- $\boldsymbol{X}_\Theta = m\check{\boldsymbol{n}}$, where $\check{\boldsymbol{n}} = \nabla d_{\partial\Omega_0}/\|\nabla d_{\partial\Omega_0}\|$ is an unitary extension of the normal field, and $d_{\partial\Omega_0}$ denotes the signed distance function to the boundary $\partial\Omega_0$.

Moreover, the family $\Phi_{\Theta,t}$ enjoys the following properties proved in Dambrine (2002).

Proposition 4.2 *(Variations of geometrical attributes) There exists a constant $C > 0$ such that for all $t \in [0, 1]$:*

1. $\|D\Phi_{\Theta,t} - I_d\|_{L^\infty} + \|D^2\Phi_{\Theta,t}\|_{L^\infty} \le \|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}$;
2. $\|D\Phi_{\Theta,t}^{-1} - I_d\|_{L^\infty} + \|D[D\Phi_{\Theta,t}^{-1}]\|_{L^\infty} \le C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}$;
3. *Let $J(t)$ be the Jacobian* $\det\big(D\Phi_{\Theta,t}\|^t(D\Phi_{\Theta,t})^{-1}n_0\|\big)$. *Then*

$$\begin{aligned} \|J(t) - 1\|_{L^\infty(\partial\Omega_0)} &\le C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}, \\ \|J(t) - 1\|_{\mathcal{C}^1(\partial\Omega_0)} &\le C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}; \end{aligned} \tag{26}$$

4. *If $\boldsymbol{n}_t$ denotes the unitary outer normal vector field on $\Phi_{\Theta,t}(\partial\Omega_0)$,*

$$\|\boldsymbol{n}_t \circ \Phi_{\Theta,t} - \boldsymbol{n}_0\|_{C^1(\partial\Omega_0)} \le C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_{2,\alpha}\ . \tag{27}$$

5. *Moreover, if Θ has the $\mathcal{C}^3$-regularity and satisfies $\|\Theta - Id_{\mathbb{R}^d}\|_3 \le 1/2$, then we have the following estimate*

$$\|\boldsymbol{n}_t \circ \Phi_{\Theta,t} - \boldsymbol{n}_0\|_{\mathcal{C}^2(\partial\Omega_0)} \le C\|\Phi_{\Theta,t} - Id_{\mathbb{R}^d}\|_3\ . \tag{28}$$

Proposition 4.2 recalls the classical results valid for all smooth vector fields used for deformation of the initial shape. On the contrary, the following statement is specific for the chosen field $\boldsymbol{X}_\Theta = m\check{\boldsymbol{n}}$, since it can be verified by simple calculations. Such a result is required, at least for technical reasons. It is not clear at the moment if conditions (29) can be relaxed for the problem under considerations.

PROPOSITION 4.3 *(Properties of $\boldsymbol{X}_\Theta$) There is a constant C dependent only of Ω_0 such that*

$$\forall t \in [0,1], \begin{cases} \left\| m \circ \Phi_{\Theta,t} - m \right\|_{L^2(\partial\Omega_0)} & \leq \quad C \|m\|_{L^2(\partial\Omega_0)} \|\Theta - Id\|_{2,\alpha}, \\ \left\| m \circ \Phi_{\Theta,t} - m \right\|_{H^{1/2}(\partial\Omega_0)} & \leq \quad C \|m\|_{H^{1/2}(\partial\Omega_0)} \|\Theta - Id\|_{2,\alpha}, \\ \left\| m \circ \Phi_{\Theta,t} - m \right\|_{H^1(\partial\Omega_0)} & \leq \quad C \|m\|_{H^1(\partial\Omega_0)} \|\Theta - Id\|_{2,\alpha} \end{cases} \tag{29}$$

where m is given by $m = \langle \boldsymbol{X}_\Theta, \breve{\boldsymbol{n}} \rangle$.

In the previous works on the optimality conditions, the piecewise estimate up to the boundary of the second order derivatives of the state function along the path were required. Such estimates can be provided by the classical Schauder theory for the Dirichlet problem. Now, we mimic the same argument for the Neumann problem. We introduce the necessary notation and definitions. The inverse transport operator is denoted by $\Psi_{\Theta,t} = (\Phi_{\Theta,t})^{-1}$, which allows for working in the fixed domain setting, here the initial domain is Ω_0. This creates an additional difficulty, i.e., the transported solution $u^t = u_t \circ \Phi_{\Theta,t}$ (see below for the definition) does not solve any pure Neumann problem for the Laplacian. Instead, u^t solves the boundary value problem for the perturbed operator $L(t)$. We have the explicit form of the operator $L(t)$,

$$\begin{aligned} L(t)v &= \underbrace{\left[\sum_{i=1}^{n}\sum_{j=1}^{n} \partial_i \Psi^\alpha_{\Theta,t} \partial_j \Psi^\beta_{\Theta,t}\right]}_{} \partial^2_{\alpha,\beta} v + \underbrace{\left[\sum_{i=1}^{n} \partial^2_{i,i} \Psi^\alpha_{\Theta,t}\right]}_{} \partial_\alpha v \ , \\ &= \qquad a_{\alpha,\beta}(t) \qquad \partial^2_{\alpha,\beta} v + \qquad b_\beta(t) \quad \partial_\alpha v \ . \end{aligned} \tag{30}$$

We use the simplified notation for $\mathfrak{D}_t = {}^t D\Phi^{-1}_{\Theta,t}$, and for the transported gradient

$$\mathfrak{D}_t \nabla u^t = \nabla(u_t \circ \Phi_{\Theta,t}). \tag{31}$$

If u_t solves the problem

$$\begin{cases} -\Delta v & = f \quad \text{in } \Omega_t \ , \\ \langle \nabla v, \boldsymbol{n}(t) \rangle & = 0 \quad \text{on } \partial\Omega_t \ , \end{cases} \tag{32}$$

then the transported solution u^t solves the following boundary value problem

$$\begin{cases} -L(t)v & = f \circ \Phi_{\Theta,t} \quad \text{in } \Omega_0 \ , \\ \langle \mathfrak{D}_t \nabla v, \boldsymbol{n}(t) \circ \Phi_{\Theta,t} \rangle & = 0 \quad \text{on } \partial\Omega_0 \ . \end{cases} \tag{33}$$

The normal derivative corresponding to the Neumann boundary condition on $\partial\Omega_t$ is changed into a oblique derivative condition in the direction ${}^t\mathfrak{D}_t \boldsymbol{n}(t) \circ \Phi_{\Theta,t}$ on $\partial\Omega_0$. Since $\boldsymbol{n}(t) \circ \Phi_t = \mathfrak{D}_t \boldsymbol{n} / \|\mathfrak{D}_t \boldsymbol{n}\|$, the vector field can be expressed in terms of the outward normal field $\boldsymbol{n}$ on $\partial\Omega_0$,

$$ {}^t\mathfrak{D}_t \boldsymbol{n}(t) \circ \Phi_{\Theta,t} = \frac{{}^t\mathfrak{D}_t \mathfrak{D}_t \boldsymbol{n}}{\|\mathfrak{D}_t \boldsymbol{n}\|} = \|\mathfrak{D}_t \boldsymbol{n}\| \; \boldsymbol{B}(t) \; \boldsymbol{n} \ , $$

see Section 6 for the definition of $\boldsymbol{B}(t)$.

The first result we need is a uniform estimate in $\mathcal{C}^{2,\alpha}$ along the path, of the transported state function.

Lemma 4.1 *There exists a constant $\eta_0 > 0$, which depends only of Ω_0, and a constant $C = M(f)$, which depends only of η_0, such that for each diffeomorphism Θ with the norm bounded $\|\Theta - Id_{\mathbb{R}^2}\|_{2,\alpha} \leq \eta_0$ and for all $t \in [0,1]$, we have $\|u^t\|_{2,\alpha,\overline{\Omega_0}} \leq M(f)$.*

Proof. We adapt the proof given in Dambrine and Pierre (2000) to the case of the Neumann boundary conditions. First, we introduce the perturbed boundary value problem $(P_{\epsilon,\Theta,t})$ defined by

$$ \begin{cases} -\Delta u + \epsilon u &= f \quad \text{in } \Omega_t, \\ \partial_{\boldsymbol{n}} u &= 0 \quad \text{on } \partial\Omega_t. \end{cases} \tag{34} $$

Let u_ϵ denote the solution of the above equation, by (34) it follows that

$$ \int_{\Omega_t} u_\epsilon = \frac{1}{\epsilon} \int_{\Omega_t} f = 0. $$

It means that for u_ϵ the Poincaré inequality can be used, and there exists a constant C_P, which depends only on the diameter of Ω_t, and can be chosen uniformly with respect to Θ and t, such that

$$ \|u_\epsilon\|_{L^2(\Omega_t)} \leq C_P \|\nabla u_\epsilon\|_{L^2(\Omega_t)}. \tag{35} $$

We also introduce the transported solution $u_\epsilon^t = u_\epsilon \circ \Phi_{\Theta,t}$ which solves

$$ \begin{cases} -L(t)v + \epsilon v &= f \circ \Phi_{\Theta,t} \quad \text{in } \Omega_0, \\ \langle \mathfrak{D}_t \nabla v, \boldsymbol{n}(t) \circ \Phi_{\Theta,t} \rangle &= 0 \quad \text{on } \partial\Omega_0. \end{cases} \tag{36} $$

There exists the lower bound λ, uniform with respect to Θ and t, such that

$$ \lambda |\xi|^2 \leq a_{\alpha,\beta}(t) \xi_\alpha \xi_\beta. $$

From the definitions given in (4.2) and by Proposition 4.2(2), the coefficients $a_{\alpha,\beta}$ and b_β are bounded and there exists a constant M such that

$$\|a_{\alpha,\beta}(t)\|_{0,\alpha,\overline{\Omega_0}}, \|b_\beta(t)\|_{0,\alpha,\overline{\Omega_0}} \leq M.$$

We also have to check that the co-normal direction $\boldsymbol{n}(t)$ is non tangential on the boundary of Ω_0 for $t \in [0,1]$. This can be deduced from the properties of the normal vectors since

$$\langle \boldsymbol{n}(t), \boldsymbol{n}\rangle = 1 - \langle \boldsymbol{n} - \boldsymbol{n}(t), \boldsymbol{n}\rangle \geq \theta_{min} > 0.$$

This a direct application of Proposition 4.2. We use the Schauder estimate (see Gilbarg and Trudinger, 1983) to get the existence of a constant $C = C(d, \alpha, \lambda, \theta_{min}, \Omega_0)$ such that

$$\|u_\epsilon^t\|_{2,\alpha,\overline{\Omega_0}} \leq C\big[\|u_\epsilon^t\|_{L^\infty(\Omega_0)} + \|f\|_{0,\alpha}\big]. \tag{37}$$

Note that by the assumption on the support of f, there is no need to precise the domain where the Hölder norm of f is considered. We use the classical argument to find an upper bound for weak solutions of an elliptic equation (see Gilbarg and Trudinger, 1983) and get

$$\|u_\epsilon^t\|_{L^\infty(\Omega_0)} \leq C\big[\|f\|_{L^q(\Omega_0)} + \|u_\epsilon^t\|_{L^2(\Omega_0)}\big].$$

One has now by the change of variable formula and the upper bound on $D\Phi_{\Theta,t}$ that

$$\|u_\epsilon^t\|^2_{L^2(\Omega_0)} \leq \int_{\Omega_t} (u_\epsilon)^2 \big|D(\Phi_{\Theta,t})^{-1}\big| \leq C\|u_\epsilon\|^2_{L^2(\Omega_t)}$$

After having multiplied (34) by u_ϵ, an integration by parts leads to

$$\|\nabla u_\epsilon\|^2_{L^2(\Omega_t)} + \epsilon\|u_\epsilon\|^2_{L^2(\Omega_t)} = \int_{\Omega_t} f u_\epsilon$$

and we obtain

$$\|\nabla u_\epsilon\|^2_{L^2(\Omega_t)} \ \leq \ \|f\|_{L^2(\Omega_t)} \ \|u_\epsilon\|_{L^2(\Omega_t)}.$$

From the Poincaré inequality, one gets

$$\|u_\epsilon\|^2_{L^2(\Omega_t)} \leq C_P\|f\|_{L^2(\Omega_t)} \ \|u_\epsilon\|_{L^2(\Omega_t)} \ \Rightarrow \ \|u_\epsilon\|_{L^2(\Omega_t)} \leq C_P\|f\|_{L^2(\Omega_t)}$$

hence we can deduce the estimate independent of ϵ

$$\|u_\epsilon^t\|_{2,\alpha,\overline{\Omega_0}} \leq M(f),$$

where $M(f)$ is a constant that depends only on M and η. We pass to the limit with $\epsilon \to 0$ and obtain the required estimate

$$\|u^t\|_{2,\alpha,\overline{\Omega_0}} \leq M(f) \ .$$

■

Then, we can conclude, in the same way as in Dambrine and Pierre (2000), by the compactness of the embedding of the space $\mathcal{C}^{2,\alpha}(\Omega_0)$ into $\mathcal{C}^2(\Omega_0)$ that

$$\omega(\eta) := \sup_{t\in[0,1],\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha}\leq\eta} \|u^t - u_0\|_{\mathcal{C}^2(\overline{\Omega_0})} \to 0 \text{ with } \eta \searrow 0.$$

This means that the function ω is a modulus of continuity. The resulting continuity in $\mathcal{C}^2(\overline{\Omega_0})$ of the transported state function is given in the following proposition.

PROPOSITION 4.4 *There exists a modulus of continuity ω such that for all Θ with $\|\Theta - Id_{\mathbb{R}^d}\|_{2,\alpha} \leq \eta$ and all t in $[0,1]$ one has*

$$\|u^t - u_0\|_{\mathcal{C}^2(\overline{\Omega_0})} \leq \omega(\eta). \tag{38}$$

Using the previous result on the shape differentiability of the functional E, we can deduce easily that e_Θ is twice differentiable, with the second order derivative of the form

$$\begin{aligned} e''_\Theta(t) &= \frac{1}{2}\int_{\partial\Omega_t} \langle\nabla|\nabla u|^2, \boldsymbol{V}\rangle\langle\boldsymbol{V},\boldsymbol{n}\rangle + \int_{\partial\Omega_t}\langle\nabla u', \nabla u\rangle\langle\boldsymbol{V}, \\ &= \frac{1}{2}A(t) + B(t). \end{aligned}$$

We have the simplified expression, since by construction $\operatorname{div}\big(\boldsymbol{X}_\Theta\big) = 0$.

Analysis of $A(t) - A(0)$. First, we recall the definition of A,

$$A(t) = \int_{\partial\Omega_t} m^2\langle\nabla|\nabla u|^2, \breve{\boldsymbol{n}}\rangle\langle\breve{\boldsymbol{n}}, \boldsymbol{n}(t)\rangle.$$

Whence

$$A(t) - A(0) = \int_{\partial\Omega_0} \tilde{m}^2\langle\mathfrak{D}_t\nabla|\mathfrak{D}_t\nabla u^t|^2, \boldsymbol{n}\rangle\langle\boldsymbol{n}, \boldsymbol{n}(t)\circ\Phi_t\rangle J(t) - m^2\langle\nabla|\nabla u_0|^2, \boldsymbol{n}\rangle.$$

Let us denote

$$\begin{cases} c_1(t) &:= \langle\mathfrak{D}_t\nabla|\mathfrak{D}_t\nabla u^t|^2, \boldsymbol{n}\rangle, \\ c_2(t) &:= \langle\boldsymbol{n}, \boldsymbol{n}(t)\circ\Phi_t\rangle, \\ c_3(t) &:= J(t). \end{cases}$$

We can conclude as in Dambrine and Pierre (2000), for $i = 1, 2, 3$ we verify that

$$|c_i(t)| \leq C \text{ and } |c_i(t) - c_i(0)| \leq C\omega(\eta) \; .$$

Then, Proposition 4.3 and simple calculations show,

$$|A(t) - A(0)| = \Big| \int_{\partial\Omega_0} \tilde{m}^2 \Pi_{i=1}^3 c_i(t) - m^2 \Pi_{i=1}^3 c_i(0) \Big| \leq C\omega(\eta) \|m\|^2_{L^2(\partial\Omega_0)}. \tag{39}$$

Analysis of $B(t) - B(0)$. First, we rewrite the term $B(t)$ in the same way as in the proof of Lemma 3.3, hence

$$B(t) = \int_{\Omega_t} |\nabla u'(t)|^2 \; ,$$

where $u'(t)$ solves

$$\begin{cases} -\Delta v &= 0 \text{ in } \Omega_t, \\ \partial_n v &= \mathrm{div}_{\boldsymbol{\tau}} \left(\langle \boldsymbol{X}_\Theta, \boldsymbol{n} \rangle \nabla_{\boldsymbol{\tau}} u \right) := N(0) \text{ on } \partial\Omega_t. \end{cases}$$

After the transport of the domain integral to the critical domain Ω_0, we get the difference (the Jacobian $D\Phi_t \equiv 1$ since $\boldsymbol{X}_\Theta$ is divergence-free)

$$B(t) - B(0) = \int_{\Omega_0} |\mathfrak{D}_t \nabla (u^t)'|^2 - |\nabla u'(0)|^2 \tag{40}$$

where $(u^t)' := u'(t) \circ \Phi_t$ and solves

$$\begin{cases} -L(t)v &= 0 \text{ in } \Omega_0, \\ \langle \nabla v, \|\mathfrak{D}_t \boldsymbol{n}\| \; \boldsymbol{B}(t) \; \boldsymbol{n} \rangle &= N(t) \text{ on } \partial\Omega_t \; , \end{cases}$$

with $N(t)$ defined below, we refer to Section 6 for a justification of the transport formulae (49), (50) and for the definitions of the functions $\alpha(x,t)$ and $\boldsymbol{\beta}(x,t)$),

$$N(t) := \alpha(x,t)\tilde{m} + \langle \boldsymbol{\beta}(x,t), \nabla \tilde{m} \rangle. \tag{41}$$

What is important here is not the exact form of the functions α and $\boldsymbol{\beta}$, the computation of the functions is postponed to the end of the section, but the structure and the regularity of both functions. The classical arguments of elliptic regularity can be used to show that α is $\mathcal{C}^{1,\alpha}$ while $\boldsymbol{\beta}$ is $\mathcal{C}^{2,\alpha}$. Now, we use the classical elliptic estimates :

$$\begin{aligned} \|(u^t)'\|_{H^1(\Omega_0)} &\leq C \|N(t)\|_{H^{-1/2}(\partial\Omega_0)} \\ &\leq C \big[\|\alpha(x,t)\tilde{m}\|_{H^{-1/2}(\partial\Omega_0)} + \|\langle \boldsymbol{\beta}(x,t), \nabla \tilde{m} \rangle\|_{H^{-1/2}(\partial\Omega_0)}. \end{aligned}$$

We now use the following product estimate deduced, by duality, from the $H^{1/2} \times \mathcal{C}^1$ multiplier lemma of Dambrine and Pierre (2000).

LEMMA 4.2 (*Multiplier-product estimate* $H^{-1/2} \times \mathcal{C}^1$)
If $\psi \in H^{-1/2}(\partial\Omega_0) \cap \mathcal{C}^0(\partial\Omega_0)$ *and* $f \in \mathcal{C}^1$*, then there exist a constant* $C(\Omega_0)$ *such that*

$$\|f\psi\|_{H^{-1/2}(\partial\Omega_0)} \leq C(\Omega_0)\|f\|_{\mathcal{C}^1(\partial\Omega_0)}\|\psi\|_{H^{-1/2}(\partial\Omega_0)}. \tag{42}$$

Proof. If ψ is $\mathcal{C}^0$, then by definition

$$\|f\psi\|_{H^{-1/2}(\partial\Omega_0)} = \sup_{\|\varphi\|_{H^{1/2}(\partial\Omega_0)}\leq 1} \int_{\partial\Omega_0} f\psi\varphi.$$

Therefore, for all $\varphi \in H^{1/2}(\partial\Omega_0)$ with $\|\varphi\|_{H^{1/2}(\partial\Omega_0)} = 1$, the duality scalar product is well defined and bounded,

$$\begin{aligned}\langle f\psi, \varphi\rangle = \int_{\partial\Omega_0} f\psi\varphi &= \int_{\partial\Omega_0} \psi f\varphi \ ,\\ &\leq \|\psi\|_{H^{-1/2}(\partial\Omega_0)}\|f\varphi\|_{H^{1/2}(\partial\Omega_0)} \ ,\\ &\leq \|\psi\|_{H^{-1/2}(\partial\Omega_0)}C(\Omega_0)\|f\|_{\mathcal{C}^1(\partial\Omega_0)}\|\varphi\|_{H^{1/2}(\partial\Omega_0)}\end{aligned}$$

which implies (42). ∎

REMARK 4.1 *Note that the corresponding* $\mathcal{C}^0$ *lemma is not true (at least, cannot be shown as easily), because the dual estimation does not have any sense. The product of a continuous function with an element of* $H^{1/2}(\partial\Omega_0)$ *has no reason to belong to the trace space* $H^{1/2}$*. That is why we consider only* $\mathcal{C}^{3,\alpha}$ *shapes.*

Then, we use Lemma 4.2 to show that

$$\begin{aligned}\|(u^t)'\|_{H^1(\Omega_0)} &\leq C\big[\|\alpha(x,t)\|_{\mathcal{C}^1}\|\tilde{m}\|_{H^{-1/2}(\partial\Omega_0)} + \|\boldsymbol{\beta}(x,t)\|_{\mathcal{C}^1}\|\nabla\tilde{m}\|_{H^{-1/2}(\partial\Omega_0)} \ ,\\ &\leq C\big[\|\alpha(x,t)\|_{\mathcal{C}^1}\|\tilde{m}\|_{H^{-1/2}(\partial\Omega_0)} + \|\boldsymbol{\beta}(x,t)\|_{\mathcal{C}^1}\|\tilde{m}\|_{H^{1/2}(\partial\Omega_0)} \ ,\\ &\leq C\|\tilde{m}\|_{H^{1/2}(\partial\Omega_0)} \ .\end{aligned}$$

We are going to verify

$$\big|(u^t)' - u_0'\big| \leq C\omega(\eta)\|\tilde{m}\|_{H^{1/2}(\partial\Omega_0)} \ . \tag{43}$$

The difference $v = (u^t)' - u_0'$ solves the equation

$$-L(t)((u^t)' - u_0') = L(t)u_0' = (L(t) - \Delta)u_0' \ .$$

The boundary condition is a little bit more difficult to get since the direction of derivation is different, hence

$$\begin{aligned}\langle\nabla((u^t)' - u_0'), \|\mathfrak{D}_t\boldsymbol{n}\| \ \boldsymbol{B}(t) \ \boldsymbol{n}\rangle = N(t) - \underbrace{\langle\nabla u_0', \|\mathfrak{D}_t\boldsymbol{n}\| \ \boldsymbol{B}(t) \ \boldsymbol{n}\rangle}_{\langle\nabla u_0', n\rangle + \langle\nabla u_0', (\|\mathfrak{D}_t\boldsymbol{n}\| \ \boldsymbol{B}(t) - I_d)\boldsymbol{n}\rangle},\\ = [N(t) - N(0)] - \langle\nabla u_0', [\|\mathfrak{D}_t\boldsymbol{n}\| \ \boldsymbol{B}(t) - I_d]\boldsymbol{n}\rangle.\end{aligned}$$

From Proposition 4.2 it follows that

$$\|\langle \nabla u_0', [\|\mathfrak{D}_t \boldsymbol{n}\| \ \boldsymbol{B}(t) - I_d]\boldsymbol{n}\rangle\|_{H^{-1/2}(\partial\Omega_0)} \leq C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)} \ .$$

On the other hand

$$\begin{aligned} N(t) - N(0) &= \alpha(x,t)\tilde{m} - \alpha(x,0)m + \langle \boldsymbol{\beta}(x,t), \nabla\tilde{m}\rangle - \langle \boldsymbol{\beta}(x,0), \nabla m\rangle, \\ &= [\alpha(x,t) - \alpha(x,0)]\tilde{m} + \alpha(x,0)[\tilde{m} - m] \\ &\quad + \langle [\boldsymbol{\beta}(x,t) - \boldsymbol{\beta}(x,0)], \nabla\tilde{m}\rangle + \langle \boldsymbol{\beta}(x,0), \nabla[\tilde{m} - m]\rangle. \end{aligned}$$

From the latter expression, we can deduce by (45) and using Proposition 4.3 that

$$\|N(t) - N(0)\|_{H^{-1/2}(\partial\Omega_0)} \leq C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)} \ .$$

As in Dambrine and Pierre (2000) and Dambrine (2000), we get easily that

$$\|[L(t) - \Delta]u_0'\|_{H^{-1}(\Omega_0)} \leq C\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)} \ .$$

We conclude from the classical estimates in Sobolev spaces that

$$\begin{aligned} \|(u^t)' - u_0'\|_{H^1(\Omega_0)} &\leq C\Big[\|[L(t) - \Delta]u_0'\|_{H^{-1}(\Omega_0)} + \\ &\qquad \|\langle \nabla((u^t)' - u_0'), \|\mathfrak{D}_t\boldsymbol{n}\| \ \boldsymbol{B}(t) \ \boldsymbol{n}\rangle\|_{H^{-1/2}(\partial\Omega_0)}\Big], \\ &\leq \ C \ \omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)} \end{aligned}$$

Therefore, we deduce from (40) that

$$\Big|B(t) - B(0)\Big| \leq C\omega(\eta)\|m\|^2_{H^{1/2}(\partial\Omega_0)}. \tag{44}$$

Computation of α and $\boldsymbol{\beta}$: Finally, we determine the expressions for $\alpha(x,t)$ and $\boldsymbol{\beta}(x,t)$. From (49), the transported tangential gradient is given by

$$\big(\nabla_{\boldsymbol{\tau},t} u(t)\big) \circ \Phi_t = \mathfrak{D}_t \nabla u^t - \langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle \mathfrak{D}_t\boldsymbol{n} \ .$$

Hence, we get with $\mathfrak{D}^2_{tt} = D\mathfrak{D}_t$ and $\mu := \big[\langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t)\rangle\big] \circ \Phi_t = \tilde{m}\langle \boldsymbol{n}, \boldsymbol{n}(t) \circ \Phi_{\Theta,t}\rangle$ that $\nabla(\mu\circ\phi) = D\Phi_t(\nabla\mu)\circ\Phi_t$. We make use of the formula $Da\mathbf{v} = \mathbf{aDv} + \mathbf{v}\ {}^{\mathbf{t}}\nabla\mathbf{a}$ to obtain

$$\begin{aligned} &D\Big[\big(\langle \boldsymbol{X}_\Theta, \boldsymbol{n}(t)\rangle \nabla_{\boldsymbol{\tau},t} u(t)\big) \circ \Phi_t\Big] \\ &\quad = \Big[\mathfrak{D}_t\nabla u^t - \langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\mathfrak{D}_t\boldsymbol{n}\Big] \ {}^t\nabla\mu + \mu D\Big[\mathfrak{D}_t\nabla u^t - \langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\mathfrak{D}_t\boldsymbol{n}\Big] \\ &\quad = \Big[\mathfrak{D}_t\nabla u^t - \langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\mathfrak{D}_t\boldsymbol{n}\Big] \ {}^t\nabla\mu + \mu\Big[-\mathfrak{D}_t\boldsymbol{n} \ {}^t\nabla\big(\langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\big) \\ &\qquad\qquad + \mathfrak{D}^2_{tt}\nabla u^t + (\mathfrak{D}_t)^2 D^2 u^t - \langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle(\mathfrak{D}^2_{tt}\boldsymbol{n} + \mathfrak{D}_t D\boldsymbol{n})\Big]. \end{aligned}$$

We then use (50), and the equality $\mathrm{Tr}\big(\mathbf{v_1^t v_2}\big) = \langle \mathbf{v_1}, \mathbf{v_2}\rangle$, to get

$$\begin{aligned}
N(t) = {} & \langle \nabla\mu, (\mathfrak{D}_t)^2\nabla u^t\rangle - \langle \nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle\nabla\mu, (\mathfrak{D}_t)^2\boldsymbol{n}\rangle \\
& + \mu\,\mathrm{Tr}\big(\mathfrak{D}^2_{tt}\nabla u^t + (\mathfrak{D}_t)^2 D^2 u^t\big) \\
& \quad - \mu\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\big[\mathrm{Tr}\big(\mathfrak{D}^2_{tt}\boldsymbol{n} + \mathfrak{D}_t D\boldsymbol{n}\big)\big] - \mu\langle\nabla\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle, \mathfrak{D}_t\boldsymbol{n}\rangle \\
& - \Big[\langle\nabla\mu, \boldsymbol{n}\rangle\big[\langle\mathfrak{D}_t\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle - \langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle\mathfrak{D}_t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle\big] \\
& \quad - \mu\Big(\langle\mathfrak{D}_t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle\nabla\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle, \boldsymbol{n}\rangle + \langle(\mathfrak{D}^2_{tt}\nabla u^t)\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle \\
& \quad + \langle(\mathfrak{D}_t)^2 D^2 u^t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle \\
& - \langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle(\mathfrak{D}^2_{tt}\boldsymbol{n} + \mathfrak{D}_t D\boldsymbol{n})\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle\Big)\Big].
\end{aligned}$$

By the chain rule, $\nabla\mu = \tilde{m}D\Phi_t\nabla\langle\boldsymbol{n}, \boldsymbol{n}(t)\circ\Phi_t\rangle + \langle\boldsymbol{n}, \boldsymbol{n}(t)\rangle D\Phi_t\nabla\tilde{m}$, as a result

$$\left\{\begin{array}{rcl}
\alpha(x,t) & = & \langle D\Phi_t\nabla\langle\boldsymbol{n}, \boldsymbol{n}(t)\circ\Phi_t\rangle, \Big[(\mathfrak{D}_t)^2\nabla u^t + \langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle(\mathfrak{D}_t)^2\boldsymbol{n} \\
& & \quad +\big[\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle\mathfrak{D}_t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle - \langle\mathfrak{D}_t\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\big]\boldsymbol{n}\Big]\rangle \\
& & +\langle\boldsymbol{n}, \boldsymbol{n}(t)\rangle\Big[\mathrm{Tr}\big(\mathfrak{D}^2_{tt}\nabla u^t + (\mathfrak{D}_t)^2 D^2 u^t\big) + \langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle \\
& & \quad \mathrm{Tr}\big(\mathfrak{D}^2_{tt}\boldsymbol{n} + \mathfrak{D}_t D\boldsymbol{n}\big) - \langle\nabla\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle, \mathfrak{D}_t\boldsymbol{n}\rangle \\
& & \quad +\Big(\langle\mathfrak{D}_t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle\nabla\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle, \boldsymbol{n}\rangle + \langle(\mathfrak{D}^2_{tt}\nabla u^t)\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle \\
& & +\langle(\mathfrak{D}_t)^2 D^2 u^t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle \\
& & -\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle(\mathfrak{D}^2_{tt}\boldsymbol{n} + \mathfrak{D}_t D\boldsymbol{n})\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle\Big)\Big] \\
\boldsymbol{\beta}(x,t) & = & \langle\boldsymbol{n}, \boldsymbol{n}(t)\circ\Phi_t\rangle\mathfrak{D}_t\Big[(\mathfrak{D}_t)^2\nabla u^t - \langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle(\mathfrak{D}_t)^2\boldsymbol{n} \\
& & +\big[\langle\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\langle\mathfrak{D}_t\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle - \langle\mathfrak{D}_t\nabla u^t, \boldsymbol{B}(t)\boldsymbol{n}\rangle\big]\boldsymbol{n}\Big].
\end{array}\right.$$

The important observation is that $\alpha(x,t)$ and $\boldsymbol{\beta}(x,t)$ are $\mathcal{C}^1$ functions and there exists $C > 0$ such that for all $t \in [0,1]$

$$\begin{array}{rcl}
\|\alpha(x,t)\|_{\mathcal{C}^\infty}, \|\boldsymbol{\beta}(x,t)\|_{\mathcal{C}^1} & \leq & C\ , \\
\|\alpha(x,t) - \alpha(x,0)\|_{\mathcal{C}^\infty}, \|\boldsymbol{\beta}(x,t) - \boldsymbol{\beta}(x,0)\|_{\mathcal{C}^1} & \leq & C\omega(\eta)\ .
\end{array} \tag{45}$$

To prove this, we observe that the inequalities in (45) are stable under the multiplication and the addition (see Dambrine, 2000). This means that if α_1 and α_2 satisfy (45) then it is also the case for the product $\alpha_1.\alpha_2$ and the sum $\alpha_1 + \alpha_2$. Hence, (45) follows from Propositions 4.2 and 4.4.

5. Analysis of the energy functional E_σ

Shape derivatives of the perimeter. We recall without proofs some classical results on the functional $\mathcal{P}(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$. This functional $\mathcal{P}$ is twice differentiable with the shape derivatives given by

$$\begin{aligned} &D\mathcal{P}(\Omega;\boldsymbol{V}) = \int_{\partial\Omega} H\langle \boldsymbol{V},\boldsymbol{n}\rangle,\\ &D^2\mathcal{P}(\Omega;\boldsymbol{V},\boldsymbol{V}') =\\ &\int_{\partial\Omega} \langle \nabla_{\boldsymbol{\tau}}\langle \boldsymbol{V},\boldsymbol{n}\rangle, \nabla_{\boldsymbol{\tau}}\langle \boldsymbol{V}',\boldsymbol{n}\rangle + \langle \boldsymbol{V},\boldsymbol{n}\rangle\langle \boldsymbol{V}',\boldsymbol{n}\rangle\big[H^2 - \mathrm{Tr}\big({}^t D\boldsymbol{n} D\boldsymbol{n}\big)\big]\\ &+ \int_{\partial\Omega} H\big[\boldsymbol{V}_\tau.D\boldsymbol{n}.\boldsymbol{V}'_\tau + \boldsymbol{n}.D\boldsymbol{V}.\boldsymbol{V}'_\tau + \boldsymbol{n}.D\boldsymbol{V}'.\boldsymbol{V}_\tau\big]. \end{aligned} \tag{46}$$

Here, H denotes the mean curvature of $\partial\Omega$. An important property proved in Dambrine (2000) is the estimate for the second order variation $D^2\mathcal{P}(\Omega;\boldsymbol{V},\boldsymbol{V})$, similar to (5). The estimate takes the form

$$|\mathcal{P}''_\Theta(t) - \mathcal{P}''_\Theta(0)| \le \omega(\|\Theta - I_d\|_{2,\alpha})\|\langle \boldsymbol{V},\boldsymbol{n}\rangle\|_{H^1(\partial\Omega)}. \tag{47}$$

Euler equation for E_σ. From the results of Section 2 and (46), we obtain that any critical shape Ω^*_σ, which minimizes the functional E_σ over the class of domains with fixed volume, v_0 satisfies the following Euler-Lagrange equation

$$\frac{1}{2}|\nabla_{\boldsymbol{\tau}} u_{\Omega^*_\sigma}| + \sigma H + \Lambda^* = 0. \tag{48}$$

We remark that the perimeter term in E_σ suppresses the argument used for the cancellation of the Lagrange multiplier Λ^*. Therefore, there is no obstruction for the existence of the stable critical shapes.

6. Transport of differential operators

This section is devoted to the justification of the formulae we have used in the paper for the transport of tangential differential operators. The starting point is the formula (31) valid for the gradient. Following Sokołowski and Zolésio (1992), one can deduce the formula for the tangential gradient. We introduce the notation. Let ρ denote a $\mathcal{C}^2$ function defined on $\mathbb{R}^d$ and let $\mathbf{A}$ be a $\mathcal{C}^2$ vector field. Assume that $\partial\Omega$ is a $\mathcal{C}^2$ manifold. Suppose that $\boldsymbol{V}$ is a $\mathcal{C}^2$ vector field, and let Φ_t be the flow of $\boldsymbol{V}$, with $\partial\Omega_t = \Phi_t(\partial\Omega)$. We denote by ρ_t (resp. $\mathbf{A_t}$) the restriction of ρ (resp. $\mathbf{A}$) to $\partial\Omega_t$. Let $\nabla_{\boldsymbol{\tau},t}$ (resp. $\mathrm{div}_{\boldsymbol{\tau},\mathrm{t}}(\cdot)$) denote the tangential gradient (resp. divergence) on $\partial\Omega_t$. The tangential gradient is defined as

$$\nabla_{\boldsymbol{\tau},t}\rho_t := \nabla\rho - \langle \nabla\rho, \boldsymbol{n}(t)\rangle \boldsymbol{n}(t).$$

Moreover, we know that

$$\begin{cases} (\nabla\rho)\circ\Phi_t = \mathfrak{D}_t\nabla(\rho\circ\Phi_t),\\ \boldsymbol{n}(t)\circ\Phi_t = \dfrac{\mathfrak{D}_t\boldsymbol{n}}{\|\mathfrak{D}_t\boldsymbol{n}\|}. \end{cases}$$

Therefore, we have

$$(\nabla_{\boldsymbol{\tau},t}\rho_t)\circ\Phi_t = \mathfrak{D}_t\nabla(\rho\circ\Phi_t) - \langle\mathfrak{D}_t\nabla(\rho\circ\Phi_t), \frac{\mathfrak{D}_t\boldsymbol{n}}{\|\mathfrak{D}_t\boldsymbol{n}\|}\rangle\frac{\mathfrak{D}_t\boldsymbol{n}}{\|\mathfrak{D}_t\boldsymbol{n}\|},$$
$$= \mathfrak{D}_t\big[\nabla(\rho\circ\Phi_t) - \langle\nabla(\rho\circ\Phi_t), \frac{{}^t\mathfrak{D}_t\mathfrak{D}_t\boldsymbol{n}}{\|\mathfrak{D}_t\boldsymbol{n}\|^2}\rangle\boldsymbol{n}\big].$$

We follow the notations of Sokołowski and Zolésio (1992) and set

$$\boldsymbol{B}(t) = \frac{{}^t\mathfrak{D}_t\mathfrak{D}_t}{\|\mathfrak{D}_t\boldsymbol{n}\|^2}\,.$$

$\boldsymbol{B}(t)$ is a symmetric matrix function such that $\boldsymbol{B}(0)$ is the identity and $\langle\boldsymbol{B}(t)\boldsymbol{n},\boldsymbol{n}\rangle = 1$ for all t and all Θ. Note also that the leading part of the transported differential operator $L(t)$ is given by the matrix $\mathbf{A}(\mathbf{t}) = \big(\mathbf{a}_{\alpha,\beta}(\mathbf{t})\big)$ defined in (30), and is related to $\boldsymbol{B}(t)$ by $\mathbf{A}(\mathbf{t}) = \|\mathfrak{D}_\mathbf{t}\boldsymbol{n}\|^\mathbf{2}\boldsymbol{B}(\mathbf{t})$. Therefore, we obtain the following formula for the transported tangential gradient:

$$(\nabla_{\boldsymbol{\tau},t}\rho_t)\circ\Phi_t = \mathfrak{D}_t\nabla(\rho\circ\Phi_t) - \langle\nabla(\rho\circ\Phi_t), \boldsymbol{B}(t)\boldsymbol{n}\rangle\mathfrak{D}_t\boldsymbol{n}\,. \tag{49}$$

The tangential divergence is defined by

$$\mathrm{div}_{\boldsymbol{\tau},\mathrm{t}}(\mathbf{A_t}) := \mathrm{div}\big(\mathbf{A}\big) - \langle\mathbf{DA}\boldsymbol{n}(\mathbf{t}), \boldsymbol{n}(\mathbf{t})\rangle.$$

Hence, we get

$$(\mathrm{div}_{\boldsymbol{\tau},\mathrm{t}}(\mathbf{A_t}))\circ\Phi_t = \mathrm{Tr}\big(D\Phi_t^{-1}D(\mathbf{A}\circ\mathbf{\Phi_t})\big) - \langle D(\mathbf{A}\circ\mathbf{\Phi_t})\frac{\mathfrak{D}_t\boldsymbol{n}}{\|\mathfrak{D}_t\boldsymbol{n}\|}, \frac{\mathfrak{D}_t\boldsymbol{n}}{\|\mathfrak{D}_t\boldsymbol{n}\|}\rangle.$$

We have obtained the formula for the transport of the tangential divergence:

$$(\mathrm{div}_{\boldsymbol{\tau},\mathrm{t}}(\mathbf{A_t}))\circ\Phi_t = \mathrm{Tr}\big({}^t\mathfrak{D}_tD(\mathbf{A}\circ\mathbf{\Phi_t})\big) - \langle D(\mathbf{A}\circ\mathbf{\Phi_t})\boldsymbol{n}, \boldsymbol{B}(t)\boldsymbol{n}\rangle. \tag{50}$$

References

Belov, S. and Fuji, N. (1997) Symmetry and sufficient condition of optimality in a domain optimization problem. *Control and Cybernetics*, **26**(1) 45–56.

Courant, R.and Hilbert, D. (1989) *Methods of mathematical physics. Vol. II. Partial differential equations.* Reprint of the 1962 original. A Wiley-Interscience Publication, John Wiley & Sons Inc., New York.

Dambrine, M.(2002) About the variations of the shape Hessian and sufficient conditions of stability for critical shapes. *Revista Real Academia Ciencias, RACSAM*, **96**(1).

Dambrine, M. (2000) Hessiennes de forme et stabilité des formes critiques. *PhD thesis, École Normale Supérieure de Cachan,* Dambrine, M. and Pierre, M. (2000) About stability of equilibrium shapes. *Mathematical Modelling and Numerical Analysis*, **34**(4), 811–834.

Descloux, J. (1990) The bidimensionnal shaping problem without surface tension. Technical report, École Polytechnique Fédérale de Lausanne.

Eppler., K. (2000) Second derivatives and suffient optimality conditions for shape functionals. *Control and Cybernetics*, **29**(2), 485–511.

Gilbarg, D. and Trudinger, N.S. (1983) *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, second edition.

Henrot, A. and Pierre, M. (1989) Un problème inverse en formage des métaux liquides. *RAIRO Modél. Math. Anal. Numér.*, **23**(1), 155–177.

Malanowski, K. (2001) *Stability and sensitivity analysis for optimal control problems with control-state constraints.* Dissertationes Math. (Rozprawy Mat.) **394**.

Murat, F. and Simon J. (1977) Optimal control with respect to the domain. Thèse de l'Université Paris VI.

Sokołowski, J. and Zolésio, J.-P. (1992) *Shape sensitivity analysis. Introduction to shape optimization.* Springer-Verlag, Berlin