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Order-Lipschitzian properties of multifunctions with applications to stability of efficient points

by

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Abstract: We define order-Lipschitzian properties of multifunctions and we investigate local upper order-lipschitzness and order-calmness of efficient points of a set depending upon a parameter.
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1. Introduction

Stability analysis of scalar optimization problems has attained the stage of development permitting the syntheses in the form of monographs or books, see Bonnans and Shapiro (2000), Malanowski (2001). For vector optimization problems formulated in partially ordered spaces, stability analysis is not so advanced. The results obtained thus far depend heavily on properties of cones, which generate order structures of spaces. In investigation of upper types of continuity (Hausdorff, Lipschitzian, Hőlder) of efficient points of a given set $A(u)$ depending upon a parameter u, one of the crucial requirements is that the ordering cone $\mathcal{K}$ have nonempty interior (see e.g. Bednarczuk, 2002a, 2002b).

We define order-Lipschitzian properties of set-valued mappings. Our approach is inspired by that of Papageorgiou (1983) who introduced order-Lipschitz continuity for functions with values in Banach lattices. For other concepts of order continuities of set-valued mappings and functions see e.g. Nikodem (1986), Papageorgiou (1983, 1986), Ke (1996), Sterna-Karwat (1989), Penot and Théra (1982). The definitions we introduce allow us to investigate stability of efficient points without the requirement that the ordering cone $\mathcal{K}$ has a nonempty interior. In Theorems 4.1, 4.2 we prove sufficient conditions for local upper order-Lipschitzness and order-calmness of efficient points of the set $A(u)$ as functions of u. Other order type continuities of efficient points have been investigated, e.g., in Sterna-Krawat (1989), Penot and Sterna-Krawat (1989).

Throughout the paper U and Y are normed vector spaces with open unit balls, B_U, and B_Y, respectively. The space Y is partially ordered by an order $\leq$ generated by a closed convex pointed cone $\mathcal{K} \subset Y$ in the usual way, i.e., $u \leq v$ if and only if $v - u \in \mathcal{K}$. A convex set $\Theta \subset Y$ is a base of $\mathcal{K}$ if $0 \notin cl\Theta$ and $\mathcal{K} = \bigcup\{\lambda\Theta | \lambda \geq 0\}$ closed convex cone $\mathcal{K}$ is *normal* for a given topology of Y if there exists a base $\mathcal{V}$ of $0-$neighbourhoods in Y consisting of saturated (or full) sets V, ie,

$$V = [V] = (V + \mathcal{K}) \cap (V - \mathcal{K}) = \bigcup\{[x, y] \mid x \in V, \quad y \in V\},$$

where $[x, y]$ denotes the order interval with the end-point x, y,

$$[x, y] = (x + \mathcal{K}) \cap (y - \mathcal{K}),$$

(see Peressini, 1967, Schaefer, 1971). By Proposition 1.4 of Peressini (1967), if cone $\mathcal{K}$ is normal, then every order interval is topologically bounded. The converse, however, is not true. In topological vector spaces with normal cones, there exist topologically bounded sets which are not order bounded. In a normed space $(Y, \|\cdot\|)$ the following are equivalent (see Proposition 1.7 of Peressini, 1967):

(i) $\mathcal{K}$ is normal,

(ii) there exists a constant $\gamma > 0$ such that $0 \leq x \leq y$ implies $\gamma\|x\| \leq \|y\|$,

(iii) there exists a constant $\gamma > 0$ such that $\|x + y\| \geq \gamma \max\{\|x\|, \|y\|\}$.

2. Order-Lipschitzian properties of set-valued mappings

Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in $(Y, \|\cdot\|)$, $\mathcal{K} = \{y \in Y \mid y \geq 0\}$. Let $\Gamma : U \rightrightarrows Y$ be a set-valued mapping defined on $(U, \|\cdot\|)$ and taking values in $(Y, \|\cdot\|)$.

Definition 2.1 Γ *is :*

(i) locally upper order-Lipschitz*, or shortly l.u.o-Lipschitz, at u_0 if there exist a constant $r > 0$ and $\ell \in \mathcal{K}$ such that*

$$\Gamma(u) \subset \Gamma(u_0) + [-\ell\|u - u_0\|, \ell\|u - u_0\|] \quad \text{for} \quad \|u - u_0\| \leq r \tag{1}$$

(ii) locally lower order-Lipschitz*, or shortly l.l.o-Lipschitz at u_0 if there exist a constant $r > 0$ and $\ell \in \mathcal{K}$ such that*

$$\Gamma(u_0) \subset \Gamma(u) + [-\ell\|u - u_0\|, \ell\|u - u_0\|] \quad \text{for} \quad \|u - u_0\| \leq r. \tag{2}$$

(iii) order-calm *at (u_0, y_0), $y_0 \in \Gamma(u_0)$, if there exist a neighbourhood V of y_0, a constant $r > 0$ and $\ell \in \mathcal{K}$ such that*

$$\Gamma(u) \cap V \subset \Gamma(u_0) + [-\ell\|u - u_0\|, \ell\|u - u_0\|] \quad \text{for} \quad \|u - u_0\| \leq r. \tag{3}$$

If int$\mathcal{K} \neq \emptyset$, order continuities defined above reduce, respectively, to upper local Lipschitzness, lower local Lipschitzness, and calmness, as defined in e.g. Robinsons (1976), Klatte and Kummer (to appear), Henrion and Outrata (2001). If $\mathcal{K}$ is normal for the topology generated by the norm $\|\cdot\|$, then $[-\ell\|u-u_0\|, \ell\|u-u_0\|] \subset \|\ell\|\|u-u_0\|B_Y$, and consequently, order continuities defined above are stronger than their topological counterparts (see e.g. Bednarczuk, 2002b).

Recall that an ordered vector space Z with order $\leq$ is a vector lattice if $x \vee y := \sup\{x,y\}$ and $x \wedge y := \inf\{x,y\}$ exist. For any $z \in Z$, the modulus of z, $|z|$ is defined as $|z| = \sup\{z, 0\}$. A subset $A \subset Z$ of a vector lattice Z is solid if $x \in A$, $y \in Z$ and $|y| \leq |x|$ implies $y \in A$. A topological vector lattice Z is a vector lattice and a Hausdorff topological vector space (over R) which possesses a base of solid $0-$neighbourhoods. A Banach lattice Z is a normed vector lattice $(Z, \|\cdot\|)$ which is norm complete. For any lattice norm, $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. The classical examples of Banach lattices are the spaces of $p-$th integrable functions, $L^p(\Omega)$, and sequence spaces ℓ^p, with order defined by cones of nonnegative elements. If Y is a topological vector lattice, the positive cone $\mathcal{K} = \{y \in Y \mid y \geq 0\}$, is normal. The converse, however, is not true, and a normal cone does not necessarily generate the lattice structure. For instance, in R^2 equipped with the norm $\|\cdot\|_0$ the cone $\mathcal{K} = \{(x,y) \mid x \geq 0 \quad y = 0\}$ is normal but does not generate the lattice structure.

If Y is a Banach lattice, Definition 2.1 can be rephrased as follows. Γ is l.u.o-Lipschitz, at u_0 if there exist a constant $r > 0$ and $\ell \in \mathcal{K}$ such that for each $y \in \Gamma(u)$, $\|u - u_0\| \leq r$, there exists $y_0 \in \Gamma(u_0)$ such that

$$|y - y_0| \leq \ell \|u - u_0\| . \tag{4}$$

Γ is l.l.o-Lipschitz at u_0 if there exist a constant $r > 0$ and $\ell \in \mathcal{K}$ such that for each $y_0 \in \Gamma(u_0)$, there exists $y \in \Gamma(u)$, $\|u - u_0\| \leq r$, such that

$$|y - y_0| \leq \ell \|u - u_0\| . \tag{5}$$

Γ is order-calm at (u_0, y_0), $y_0 \in \Gamma(u_0)$, if there exist a neighbourhood V of y_0, a constant $r > 0$ and $\ell \in \mathcal{K}$ such that for each $y \in \Gamma(u) \cap V$, $\|u - u_0\| \leq r$, there exists $y_0 \in \Gamma(u_0)$ satisfying

$$|y - y_0| \leq \ell \|u - u_0\| . \tag{6}$$

Some elementary examples illustrating the notions introduced above will now be given.

Example 2.1 *Let $\mathcal{K} \subset R^3$ be given as*

$$\mathcal{K} = \{(x,y,z) \mid z = 0 \ x, y \geq 0\} .$$

The set-valued mapping $\Gamma : R \rightrightarrows R^3$ defined as

$$\Gamma(0) = \{(x,y,z) \mid z = 0 \ 0 \leq x \leq 1 \quad 0 \leq y \leq 1\}, \ \Gamma(u) = \Gamma(0) \cup \{(1, 1, 1 + u)\} ,$$

is locally upper Lipschitz at 0 *in the usual sense but not locally upper order-Lipschitz.*

EXAMPLE 2.2 *Let* $\mathcal{K} \subset R^2$ *be given as*

$$\mathcal{K} = \{(x, y) \mid x = y \; x \geq 0\} .$$

The set-valued mapping $\Gamma : R \rightrightarrows R^2$ *defined as*

$$\Gamma(0) = \{(x, y) \mid 0 \leq x \leq 1 \quad 0 \leq y \leq 1 \quad y \geq -x + 1\} ,$$

$$\Gamma(u) = \Gamma(0) \setminus co\{(0, 0), (1, 1)\} \cup \{(1, 1)\} ,$$

where co stands for the convex hull, is locally lower Lipschitz at 0 *in the usual sense but not locally lower order-Lipschitz.*

PROPOSITION 2.1 *Let* Y *and* U *be normed spaces and let* $\mathcal{K}$ *be a closed convex pointed cone in* Y *.*

1. *If* Γ *is locally upper order-Lipschitz at* u_0 *, then any* $y \in \Gamma(u)$, $\|u - u_0\| \leq r$, *can be represented as* $y = y_0 + \ell\|u - u_0\| - k_y^1$, $y = y_0 - \ell\|u - u_0\| + k_y^2$, *where* $y_0 \in \Gamma(u_0)$, $k_y^1 \in \mathcal{K}$, *and* $k_y^2 \in \mathcal{K}$, *with* $k_y^1 \leq 2\ell\|u - u_0\|$, $k_y^2 \leq 2\ell\|u - u_0\|$.
2. *If* Γ *is locally lower order-Lipschitz at* u_0 *, then any* $y_0 \in \Gamma(u_0)$ *can be represented as* $y_0 = y - \ell\|u - u_0\| + k_y^1$, $y_0 = y - \ell\|u - u_0\| + k_y$, *where* $y \in \Gamma(u)$, $\|u - u_0\| \leq r$, $k_y^1 \in \mathcal{K}$, *and* $k_y^2 \in Y_+$, *with* $k_y^1 \leq 2\ell\|u - u_0\|$, $k_y^2 \leq 2\ell\|u - u_0\|$.

Proof. By definition,

$$-\ell\|u - u_0\| \leq y_0 - y \leq \ell\|u - u_0\| , \quad \|u - u_0\| \leq r, \; .$$

By the left-hand-side inequality, $k_y^1 = y_0 - y + \ell\|u - u_0\| \in \mathcal{K}$, and, by the right-hand-side inequality, $k_y^1 \leq 2\ell\|u - u_0\|$. Other cases can be treated similarly. ■

3. Order-containment property and its rate

Let $\mathcal{K} \subset Y$ be a closed convex and pointed cone in Y. Let $A \subset Y$ be a subset of Y . An element $y \in A$ is *efficient*, $y \in \mathrm{Eff}(A)$, if

$$(A - y) \cap (-\mathcal{K}) = \{0\} .$$

An element $y \in A$ is *locally efficient*, $y \in \mathrm{Eff}_{loc}(A)$, if there exists a $0-$neighbourhood V such that

$$(A \cap (y + V) - y) \cap (-\mathcal{K}) = \{0\} .$$

Let $\ell \in \mathcal{K}$, $\ell \neq 0$. Denote

$$A(\ell) = A \setminus (\mathrm{Eff}(A) + [-\ell, \ell]) .$$

Definition 3.1 *A set A has an $\ell-$*order containment property*, $\ell-(OCP)$, with $\ell\in\mathcal{K}$ if for each $\varepsilon>0$ there exists $\delta>0$ satisfying:*
(C) *for each $y\in A(\varepsilon\ell)$ there exists $\eta_y\in Eff(A)$ such that*

$$y-\eta_y-\delta\ell\in\mathcal{K}\,. \tag{7}$$

Clearly, (7) holds if and only if $y-\eta_y-k\in\mathcal{K}$, for all $k\in\mathcal{K}\,,\ k\leq\delta\ell\,.$

If $\mathcal{K}$ is normal, and (C) holds for $\ell\in\mathcal{K}$, then

$$A\subset \mathrm{cl}(\mathrm{Eff}(A))+\mathcal{K}\,. \tag{8}$$

Indeed, if $y\in A\backslash\mathrm{cl}(\mathrm{Eff}(A))$, there exists $\alpha>0$ such that $(y+\alpha B_Y)\cap\mathrm{cl}(\mathrm{Eff}(A))=\emptyset$. There exists $\varepsilon>0$ such that $\varepsilon\|\ell\|\leq\alpha$, and $y\in A\setminus(\mathrm{Eff}(A)+[-\varepsilon\ell,\varepsilon\ell])$, since $\mathcal{K}$ is normal. By (C), there exists $\eta_y\in\mathrm{Eff}(A)$ satisfying $y-\eta_y\in\mathcal{K}$, which proves (8).

Example 3.1

1. *Let Y be an ordered vector space and let $\mathcal{K}\subset Y$ be a closed convex pointed cone in Y. Let $0\neq\ell\in\mathcal{K}$, and let $\mathcal{K}_1\subset Y$ be a closed convex cone, $\mathcal{K}_1\subset\mathcal{K}$, satisfying the following condition: for each $\varepsilon>0$ there exists $\delta>0$ such that*
$$k-\delta\ell\in\mathcal{K}\quad \textit{for all}\ \ k\in\mathcal{K}_1\setminus[-\varepsilon\ell,\varepsilon\ell]\,. \tag{9}$$
Any order interval $[a,b]_1$, (with respect to $\mathcal{K}_1$),
$$[a,b]_1=(a+\mathcal{K}_1)\cap(b-\mathcal{K}_1),$$
$a,b\in\mathcal{K}$, is $\ell-(OCP)$. To see this, note that $Eff([a,b]_1)=\{a\}$. Let $\varepsilon>0$, and $y\in[a,b]_1\setminus(a+[-\varepsilon\ell,\varepsilon\ell])$. Hence, $y-a\in\mathcal{K}_1$ and $y-a\notin[-\varepsilon\ell,\varepsilon\ell]$. By (9), there exists $\delta>0$ such that
$$y-a-\delta\ell\in\mathcal{K}\,.$$
2. *Let $Y=\ell^2$ and $\mathcal{K}=\ell^2_+$. Consider a closed convex cone $(\ell^2_+)_1\subset\ell^2_+$ of the form*
$$(\ell^2_+)_1=\{k=(k_n)\in\ell^2_+\mid \frac{1}{2n}k_1\leq k_n\leq\frac{3}{2n}k_1\}$$
and the order interval with respect to $(\ell^2_+)_1$,
$$A=[a,b]_1=\{y\in\ell^2\mid a\leq_1 y\leq_1 b\}.$$
Note that $(\ell^2_+)_1$ satisfies condition (9) with e.g. $\ell=(\frac{1}{n})$.
Let $\varepsilon>0$. For any
$$y\in A(\varepsilon\ell)=\{y\in A\mid y_n-a_n>\varepsilon\frac{1}{2n},\ n=1,2,...\}$$
the inequality
$$y-a-\delta\ell\geq0$$
holds for $\delta=\frac{\varepsilon}{2}$.
3. *Let $Y=R^3$ and $\mathcal{K}=\{y\in R^3\mid y_1,y_2\geq0,y_3=0\}$. Let*
$$\mathcal{K}_1=\{y\in\mathcal{K}\mid y_2\leq 3/2y_1\quad y_2\geq1/2y_1\}\,.$$
For the order interval
$$A=[(1,1,0),(2,2,0)]_1$$
the property $\ell-(OCP)$ holds for any $\ell\in\mathcal{K}_\infty$.

Let $\varepsilon > 0$. Denote

$$A(\varepsilon) = A \setminus (\text{Eff}(A) + \varepsilon B_Y).$$

The following properties are related to the one introduced in Definition 3.1.

The containment property (CP) (Bednarczuk, 2000a) holds for a subset $A \subset Y$ if int $\mathcal{K} \neq \phi$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that

(C1) for each $y \in A(\varepsilon)$ there exists $\eta_y \in \text{Eff}(A)$ satisfying

$$y - \eta_y - \delta B_Y \in \mathcal{K}. \tag{10}$$

Let Y^* be the topological dual space of Y with the bilinear duality form $\langle \cdot, \cdot \rangle$. Let Θ be a base of the dual cone $\mathcal{K}^* = \{f \in Y^* \mid \langle f, y \rangle \geq 0 \text{ for all } y \in \mathcal{K}\}$. The dual containment property (DCP) (Bednarczuk, 2002) holds for a subset $A \subset Y$ with respect to Θ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

(C2) for each $y \in A(\varepsilon)$ there exists $\eta_y \in \text{Eff}(A)$ satisfying

$$\theta(y - \eta_y) > \delta \quad \text{for all} \quad \theta \in \Theta. \tag{11}$$

In the proposition below we investigate the relationships between these properties and the $\ell-$order containment property.

PROPOSITION 3.1 *Let $\mathcal{K} \subset Y$ be a normal cone in Y. For any subset $A \subset Y$ the following relations hold:*
(i) If int$\mathcal{K} \neq \emptyset$, then $(C) \to (C1)$,
(ii) If $\mathcal{K}^*$ is based, then $(C) \to (C2)$.

Proof. Let $\varepsilon > 0$ and $y \in A(\varepsilon)$. For any $\ell \in \mathcal{K}$, since $\mathcal{K}$ is normal, there exists $\varepsilon_1 > 0$ such that $y \in A(\varepsilon_1 \ell)$. By (C), there exist $\eta_y \in \text{Eff}(A)$ and $\delta > 0$ such that

$$y - \eta_y - \delta\ell \in \mathcal{K}. \tag{12}$$

(i). Take $\ell \in \text{int}\mathcal{K}$. By (12), there exists $\delta_1 > 0$ such that $y - \eta_y - \delta_1 B_Y \subset \mathcal{K}$.
(ii). Take $\ell \in (\mathcal{K})^+ = \{k \in \mathcal{K} \mid \langle f, k \rangle > 0 \text{ for all } f \in \mathcal{K}^* \setminus \{0\}\}$, and the base Θ of $\mathcal{K}^*$, $\Theta = \{\theta \in \mathcal{K}^* \mid \langle \theta, \ell \rangle = 1\}$. By (12),

$$\theta(y - \eta_y - \delta\ell) \geq 0 \text{ for all } \theta \in \Theta,$$

and consequently,

$$\theta(y - \eta_y) > \tilde{\delta} \text{ for all } \theta \in \Theta, \text{ and some } \tilde{\delta} > 0.$$

which amounts to $(C2)$. ■

Let

$$\mathrm{dom}_\ell(A) = \{\varepsilon > 0 \mid A(\varepsilon\ell) \neq \emptyset\} .$$

Definition 3.2 *The function* $\delta_A^\ell : R \to R$ *is the rate of* $\ell-$*order containment of a set* $A \subset Y$ *if*

$$\delta_A^\ell(\varepsilon) = \inf\{\nu_A^\ell(y) \mid y \in A(\varepsilon\ell)\} ,$$

where, for any $y \in A$,

$$\nu_A^\ell(y) = \sup\{\mu^\ell(y-\eta) \mid \eta \in \mathit{Eff}(A) \cap (y - \mathcal{K})\} ,$$

and, for any $k \in \mathcal{K}$,

$$\mu^\ell(k) = \sup\{\delta \mid k - \delta\ell \in \mathcal{K}\} .$$

We put δ^ℓ, whenever it is clear from the context which set we refer to in δ_A^ℓ. The following properties of the function δ_A^ℓ follow directly from the definition.

1. δ_A^ℓ is a nondecreasing function of ε,
2. if $C_1 \subset C_2$, then $\delta_{C_1}^\ell(\varepsilon) \geq \delta_{C_2}^\ell(\varepsilon)$.

Proposition 3.2 *The following are equivalent:*

(i) $\delta_A^\ell(\varepsilon) > 0$ *for each* $\varepsilon \in \mathit{dom}_\ell(A)$,

(ii) $\ell - (OCP)$ *holds.*

Proof. $(i) \to (ii)$. Let $\varepsilon > 0$. If $\delta^l(\varepsilon) = \tau > 0$, then $\nu^\ell(y) \geq \tau$ for any $y \in A(\varepsilon\ell)$. By definition of ν^ℓ, there exists $\eta_y \in \mathrm{Eff}(A)$ such that $\mu^l(y - \eta_y) > \tau - \varsigma > 0$, for some positive ς. This means that there exists $\delta > \tau - \varsigma$ such that $y - \eta_y - \delta\ell \in \mathcal{K}$. $(ii) \to (i)$. Let $\varepsilon > 0$, and $y \in A(\varepsilon\ell)$. By (C), there exist $\delta > 0$ and $\eta_y \in \mathrm{Eff}(A)$ such that

$$y - \eta_y - \delta\ell \in \mathcal{K} .$$

Hence, $\mu^\ell(y - \eta_y) \geq \delta$, $\nu^\ell(y) \geq \delta$, and

$$\delta_A^\ell(\varepsilon) = \inf\{\nu^\ell(y) \mid y \in A(\varepsilon\ell)\} \geq \delta .$$ ■

Proposition 3.3 *Let* $\mathcal{K} \subset Y$ *be a closed convex pointed cone in* Y. *Let* $A \subset Y$ *be a subset of* Y. *Assume that for each* $y \in \mathit{Eff}(A) + \mathcal{K}$ *the set* $\mathit{Eff}(A) \cap (y - \mathcal{K})$ *is weakly compact. If* $\ell - (OCP)$ *holds for* A, *then for each* $\varepsilon > 0$ *and* $y \in A(\varepsilon\ell)$ *there exists* $\eta_y \in \mathit{Eff}(A)$ *such that*

$$y - \eta_y - \delta_A^\ell(\varepsilon)\ell \in \mathcal{K} .$$

Proof. Let $\varepsilon > 0$, $y \in A(\varepsilon\ell)$. It is enough to show the existence of $\eta_y \in \mathrm{Eff}(A)$ such that

$$\nu^\ell(y) = \mu^\ell(y - \eta_y)\,.$$

Indeed, for each $\eta \in \mathrm{Eff}(A) \cap (y - \mathcal{K})$

$$\mu^\ell(y - \eta) \leq \nu^\ell(y)\,,$$

and for each $\alpha > 0$ there exists $\eta_\alpha \in \mathrm{Eff}(A) \cap (y - \mathcal{K})$ such that

$$\mu^\ell(y - \eta_\alpha) > \nu^\ell(y) - \alpha$$

This means that for $\beta_\alpha = \mu^\ell(y - \eta_\alpha) - \alpha$ we have

$$y - \eta_\alpha = \beta_\alpha \ell + k_\alpha\,, \quad \text{where } k_\alpha \in \mathcal{K}, \quad \beta_\alpha \to \nu^\ell(y)\,, \quad \text{as} \quad \alpha \to 0\,.$$

By assumption, $\{\eta_\alpha\}$ contains a weakly convergent subnet with the limit point $\eta_0 \in \mathrm{Eff}(A)$, then

$$y - \eta_0 - \nu^\ell(y)\ell = k_0 \in \mathcal{K}\,,$$

since $\mathcal{K}$ is weakly closed. This ends the proof. ■

4. Order-Lipschitz continuity of efficient points

Let $\mathcal{M} : U \rightrightarrows Y$ be a set-valued mapping defined as

$$\mathcal{M}(u) = \mathrm{Eff}(\Gamma(u))\,.$$

The set-valued mapping $\mathcal{M}$ is called the minimal point mapping. Order-type continuities of $\mathcal{M}$ have been investigated in Penot and Sterna-Krawat (1989), Sterna-Krawat (1986, 1989).

THEOREM 4.1 *Let $(Y, \|\cdot\|)$ and $(U, \|\cdot\|)$ be normed spaces. Let $\mathcal{K} \subset Y$ be a closed convex pointed cone in Y . Assume that*
(i) for any $y \in Eff(\Gamma(u_0)) + \mathcal{K}$ the set $Eff(\Gamma(u_0)) \cap (y - \mathcal{K})$ is weakly compact,
(ii) Γ is l.u.o-Lipschitz at u_0 , and l.l.o-Lipschitz at u_0 , with constant $\ell \in \mathcal{K}$,
(iii) $\delta^\ell_{\Gamma(u_0)}(\varepsilon) \geq c\varepsilon$, $c > 0$.
The minimal point multifunction $\mathcal{M}$ is l.u.o-Lipschitz at u_0 .

Proof. By (ii) , for all $\|u - u_0\| \leq r$ we have

$$\Gamma(u) \subset \{\mathrm{Eff}(\Gamma(u_0)) + [-(\tfrac{4}{c} + 1)\ell\|u - u_0\|, (\tfrac{4}{c} + 1)\ell\|u - u_0\|]\} \cup$$

$$\{\Gamma(u_0) \setminus (\mathrm{Eff}(\Gamma(u_0)) + [-\tfrac{4}{c}\ell\|u - u_0\|, \tfrac{4}{c}\ell\|u - u_0\|])\} + [-\ell\|u - u_0\|, \ell\|u - u_0\|].$$

We show that if

$$y \in \Gamma(u) \cap \{\{\Gamma(u_0) \setminus (\mathrm{Eff}(\Gamma(u_0)) + [-\tfrac{4}{c}\ell\|u-u_0\|, \tfrac{4}{c}\ell\|u-u_0\|])\} \\ +[-\ell\|u-u_0\|, \ell\|u-u_0\|]\},$$

for all $\|u-u_0\| \leq r$, then $y \notin \mathrm{Eff}(\Gamma(u))$. Indeed, take any $y \in \Gamma(u)$, $\|u-u_0\| \leq r$, such that

$$y \in \{\Gamma(u_0) \setminus (\mathrm{Eff}(\Gamma(u_0)) + [-\tfrac{4}{c}\ell\|u-u_0\|, \tfrac{4}{c}\ell\|u-u_0\|])\} + [-\ell\|u-u_0\|, \ell\|u-u_0\|].$$

Then $y = \gamma + \xi$, where

$$\begin{aligned} \gamma &\in \Gamma(u_0) \setminus (\mathrm{Eff}(\Gamma(u_0)) + [-\tfrac{4}{c}\ell\|u-u_0\|, \tfrac{4}{c}\ell\|u-u_0\|]), \\ \xi &\in [-\ell\|u-u_0\|, \ell\|u-u_0\|], \\ \xi &= \ell\|u-u_0\| - k_\xi, \quad k_\xi \in \mathcal{K} \quad k_\xi \leq 2\ell\|u-u_0|. \end{aligned}$$

By Proposition 3.3, there exists $\eta_\gamma \in \mathrm{Eff}(\Gamma(u_0))$ such that

$$\gamma - \eta_\gamma - k \in \mathcal{K} \quad \text{for all} \quad k \in \mathcal{K}, \quad k \leq \delta^{\ell}_{\Gamma(u_0)}(\tfrac{4}{c}\|u-u_0\|)\ell \tag{13}$$

By the lower order-Lipschitz continuity of Γ at u_0 there are $z \in \Gamma(u)$, $\|u-u_0\| \leq r$, $k_z \in \mathcal{K}$ such that

$$\eta_\gamma = z + \ell\|u-u_0\| - k_z, \|u-u_0\| \leq r, \quad k_z \leq 2\ell\|u-u_0\| \ .$$

In consequence,

$$\begin{aligned} y - z &= \gamma - z + \ell\|u-u_0\| - k_\xi \\ &= [\gamma - \eta_\gamma] + \ell\|u-u_0\| - k_z + \ell\|u-u_0\| - k_\xi, \end{aligned}$$

and by *(iii), since* $k_z + k_\xi \leq 4\ell\|u-u_0\| \leq \delta^\ell(\tfrac{4}{c}\|u-u_0\|)\ell$,

$$y - z \in \mathcal{K} \setminus \{0\}.$$

This proves that for all $\|u-u_0\| \leq r$ *the following inclusion holds*

$$\mathcal{M}(u) \subset \mathit{Eff}(\Gamma(u_0)) + [-(\tfrac{4}{c}+1)\ell\|u-u_0\|, (\tfrac{4}{c}+1)\ell\|u-u_0\|]. \qquad \blacksquare$$

Remark 1 *By examining the proof one can see that we exploited l.l.o-Lipschitz property of* Γ *only partially. Namely, only right-hand-side inequality of (2) from Definition 2.1 was used. In the following example we show that, in Theorem 4.1, the order-Lipschitz continuity of* Γ *cannot be dropped.*

Example 4.1 *Let* Y, $\mathcal{K}$, *and* Γ *be as in Example 2.2. Then*

$$\mathcal{M}(0) = \{(x,y) \mid x \geq 0 \ \ y \geq 0 \ \ y = -x+1\},$$

and

$$\mathcal{M}(u) = (\mathcal{M}(0) \setminus (\tfrac{1}{2}, \tfrac{1}{2})) \cup \{(1,1)\} \quad u \neq 0.$$

In the theorem below we investigate the order-calmness of $\mathcal{M}$.

THEOREM 4.2 *Let $(Y,\|\cdot\|)$ and $(U,\|\cdot\|)$ be normed spaces. Let $\mathcal{K}\subset Y$ be a closed convex pointed normal cone in Y and let $\Gamma(u_0)$ be convex. Assume that $y_0\in Eff_{loc}(\Gamma(u_0))$, i.e., $y_0\in Eff(\Gamma(u_0)\cap V_1)$, where V_1 is a neighbourhood of y_0 and*

(i) Γ is order-calm at (u_0,y_0), with constant $\ell\in\mathcal{K}$, and neighbourhood V_2 of y_0, and l.l.o-Lipschitz at u_0, with constant $\ell\in\mathcal{K}$,

(ii) for any $y\in Eff(\Gamma(u_0)\cap V)+\mathcal{K}$ the set $Eff(\Gamma(u_0)\cap V)\cap(y-\mathcal{K})$ is weakly compact, where $V=V_1\cap V_2$,

(iii) $\delta^{\ell}_{\Gamma(u_0)\cap V}(\varepsilon)\geq c\varepsilon$, $c>0$.

The minimal point multifunction $\mathcal{M}$ is order-calm at (u_0,y_0).

Proof. Let $\tilde V$ be a $0-$neighbourhood such that $(y_0+\tilde V)+\tilde V\subset V$. Without losing generality we can assume that $r\|\ell\|B_Y\subset\tilde V$. By (ii),

$$\begin{aligned}\Gamma(u)\cap\tilde V\subset\{&Eff(\Gamma(u_0)\cap V)+[-(\tfrac{4}{c}+1)\ell\|u-u_0\|,(\tfrac{4}{c}+1)\ell\|u-u_0\|]\}\cup\\ &\{\Gamma(u_0)\cap V\setminus(Eff(\Gamma(u_0)\cap V)+[-\tfrac{4}{c}\ell\|u-u_0\|,\tfrac{4}{c}\ell\|u-u_0\|])\}\\ &+[-\ell\|u-u_0\|,\ell\|u-u_0\|]\},\end{aligned}$$

for $\|u-u_0\|\leq r$. We show that if $y\in\Gamma(u)\cap\tilde V\cap\{\{\Gamma(u_0)\cap V\setminus(Eff(\Gamma(u_0)\cap V)+[-\frac{4}{c}\ell\|u-u_0\|,\frac{4}{c}\ell\|u-u_0\|])\}+[-\ell\|u-u_0\|,\ell\|u-u_0\|]\}$, $\|u-u_0\|\leq r$, then $y\notin Eff(\Gamma(u))\cap\tilde V$.

Indeed, take any $y\in\Gamma(u)\cap\tilde V$, $\|u-u_0\|\leq r$, such that

$$\begin{aligned}y\in\{&\Gamma(u_0)\cap V\setminus(Eff(\Gamma(u_0)\cap V)+[-\tfrac{4}{c}\ell\|u-u_0\|,\tfrac{4}{c}\ell\|u-u_0\|])\}\\ &+[-\ell\|u-u_0\|,\ell\|u-u_0\|].\end{aligned}\tag{14}$$

Then, $y=\gamma+\xi$, where

$$\begin{aligned}\gamma&\in\Gamma(u_0)\cap V\setminus(Eff(\Gamma(u_0)\cap V)+[-\tfrac{4}{c}\ell\|u-u_0\|,\tfrac{4}{c}\ell\|u-u_0\|]),\\ \xi&\in[-\ell\|u-u_0\|,\ell\|u-u_0\|],\\ \xi&=\ell\|u-u_0\|-k_\xi,\quad k_\xi\in\mathcal{K},\quad k_\xi\leq2\ell\|u-u_0\|.\end{aligned}$$

By Proposition 3.3, there exists $\eta_\gamma\in Eff(\Gamma(u_0)\cap V)$ such that

$$\gamma-\eta_\gamma-k\in\mathcal{K}\quad\text{for all}\quad k\in\mathcal{K},\quad k\leq\delta^{\ell}_{\Gamma(u_0)\cap V}(\tfrac{4}{c}\|u-u_0\|)\ell.\tag{15}$$

By the lower order-Lipschitz continuity of Γ at u_0 there are $z\in\Gamma(u)$, $k_z\in\mathcal{K}$ such that

$$\eta_\gamma=z+\ell\|u-u_0\|-k_z,\quad k_z\leq2\ell\|u-u_0\|.$$

In consequence,

$$\begin{aligned}y-z&=\gamma-z+\ell\|u-u_0\|-k_\xi\\ &=[\gamma-\eta_\gamma]+\ell\|u-u_0\|-k_z+\ell\|u-u_0\|-k_\xi,\end{aligned}$$

and by (iii), since $k_z + k_\xi \leq 4\ell\|u-u_0\| \leq \delta^\ell(\frac{4}{c}\|u-u_0\|)\ell$,

$$y - z \in \mathcal{K} \setminus \{0\}.$$

Since $\Gamma(u_0)$ *is convex, for all* u *such that* $\|u-u_0\| \leq r$,

$$\begin{array}{rcl} \mathcal{M}(u) \cap \tilde{V} & \subset & Eff_{loc}(\Gamma(u_0)) + [-(\frac{4}{c}+1)\ell\|u-u_0\|, (\frac{4}{c}+1)\ell\|u-u_0\|] \\ & \subset & \mathcal{M}(u_0) + [-(\frac{4}{c}+1)\ell\|u-u_0\|, (\frac{4}{c}+1)\ell\|u-u_0\|]. \end{array}$$ ■

5. Conclusions

The order-Lipschitz continuity of set-valued mappings introduced here is stronger than the usual Lipschitz continuity. In finite-dimensional case, roughly speaking, it allows Γ *to vary only in directions parallel to aff*$\mathcal{K}$*. On the other hand, to derive sufficient conditions for efficient points to have order-Lipschitz continuity of efficient points we need only standard assumptions on* $\mathcal{K}$.

In assumption (ii) of Theorem 4.1 we require that the order containment rate is at least linear for small arguments. If the order containment rate is of higher order, then one can prove order-Hőlder behaviour of the minimal point multifunction.

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