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Approximation of optimal control problems with bound constraints by control parameterization

by

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Abstract: We consider nonlinear optimal control problems with bound constraints for the controls. Under the assumption that the optimal control is continuous and has finitely many smooth boundary arcs, we show that the system of optimality conditions can be reduced to a system of operator equations. Based on this system we investigate convergence of approximations by control parameterization.

Keywords: nonlinear optimal control, bound constraints, Newton's method, control parameterization.

1. Introduction

Solutions of optimal control problems with control constraints are often continuous and have finitely many boundary arcs. This structure is essential if shooting techniques are applied for the numerical solution of the control problem (see Bulirsch, 1971). Recently, such structural assumptions have been used to investigate parametric nonlinear control problems (see Maurer, Pesch, 1998, Malanowski, Maurer,1996, 1998, Malanowski, 1998).

First order approximations of nonlinear control problems can be obtained by somewhat weaker assumptions. Theoretical and numerical results for Euler approximations can be found in Dontchev, Hager (1993, 2000), Malanowski et al. (1997), Alt (1997, 2001), and Dontchev, Hager, Malanowski (2000). Results on first order Ritz type discretizations can be found in Felgenhauer (1999a, b).

Under the assumption that the derivative of the optimal control has bounded variation, a second-order Runge-Kutta approximation for control problems with convex control constraints is investigated in Dontchev, Hager, Veliov (2000). Higher order Ritz type approximations for nonlinear control problems are studied in Felgenhauer (1998), where, however, a method of order p requires that the optimal control be of class C^p.

In the present paper we use a structural assumption which requires that the optimal control be continuous and piecewise of class C^p. As in multiple shooting we do not discretize the control problem directly. Based on the structural assumption we first reduce the system of necessary optimality conditions to a system of equations. Then we show how for this system higher order approximations can be obtained. The aim of the present paper is to focus on the basic ideas. In order to keep the analysis as simple as possible we consider only control problems with a simple scalar bound constraint.

The paper is organized as follows. In Section 2 we introduce the optimal control problem and the structural assumption for the optimal control, which is used to reduce the system of necessary optimalty conditions to a system of equations. In Section 3 this system is formulated as an operator equation, and we show that this equation is regular, if a strong second-order sufficient optimality is satisfied. Section 4 shortly discusses the application of Newton's method to this operator equation. In Section 5 a simple dicretization of the operator equation based on control parameterization is discussed. Section 6 gives a convergence analysis of solutions of the discretized equations. The results show that under the structural assumption one can obtain higher order approximations.

2. Optimality conditions

We consider the following optimal control problem with a simple bound constraint:

$$\text{(OC)} \qquad \mathrm{Min}_{(x,u)} J(x,u) = \int_{t_a}^{t_f} \varphi(x(t),u(t))\,dt + \phi(x(t_f))$$

subject to

$$\begin{aligned}
&\dot{x}(t) = \psi(x(t),u(t)) && \text{for a.a. } t \in [t_a,t_f]\,,\\
&x(t_a) = a, &&\\
&u(t) \le b && \text{for a.a. } t \in [t_a,t_f]\,,\\
&x \in W^{1,\infty}(t_a,t_f;\mathbb{R}^n)\,,\ u \in L^\infty(t_a,t_f;\mathbb{R}^m)\,, &&
\end{aligned}$$

where $\varphi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\phi: \mathbb{R}^n \to \mathbb{R}$, and $\psi: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. We shall not treat the most general case and assume that the control variable is scalar, i.e., $m = k = 1$. We further assume that:

(C1) There exists a (local) solution $(\tilde{x},\tilde{u})$ of (OC).

(C2) For some $p \ge 1$, the mappings φ, ϕ, ψ, and g are $p+1$ times Fréchet differentiable in all arguments, and the respective derivatives are locally Lipschitz continuous in x, u.

In order to formulate the necessary optimality conditions we denote by H the Hamiltonian defined by

$$H(x,u,\lambda) = \varphi(x,u) + \langle \lambda, \psi(x,u) \rangle\,,$$

and we denote by $\tilde{H}$ the augmented Hamiltonian defined by

$$\tilde{H}(x,u,\lambda,\mu) = H(x,u,\lambda) + \mu(u-b)\,.$$

We assume that for the local solution $(\tilde{x},\tilde{u})$ of (OC) there exist Lagrange multipliers $\tilde{\lambda} \in W^{1,\infty}(t_a,t_f;\mathbb{R}^n)$, $\tilde{\mu} \in L^\infty(t_a,t_f;\mathbb{R})$ such that $(\tilde{x},\tilde{u},\tilde{\lambda},\tilde{\mu})$ satisfy the following system of first order necessary optimality conditions:

$$\begin{aligned}
&\dot{\lambda}(t)^T = -H_x(x(t),u(t),\lambda(t)) && \text{for a.a. } t \in [t_a,t_f]\,,\\
&\lambda(t_f)^T = \phi_x(x(t_f))\,, &&\\
&\tilde{H}_u(x(\cdot),u(\cdot),\lambda(\cdot),\mu(\cdot)) = 0 && \text{for a.a. } t \in [t_a,t_f]\,,\\
&\mu(t) \ge 0\,, \quad \mu(t)(u(t)-b) = 0 && \text{for a.a. } t \in [t_a,t_f]\,.
\end{aligned} \tag{1}$$

Moreover, we assume that the active set or boundary part of the inequality constraint $u(t) \le b$ consists of finitely many boundary arcs. For simplicity we assume that there is only one boundary arc. More precisely, we assume:

(CS) There exists one junction point $\tilde{s}$ such that

$$\tilde{u}(t) < b \;\forall t \in [t_a,\tilde{s}[\,, \quad \tilde{u}(t)) = b \;\forall t \in [\tilde{s},t_f]\,,$$

and $\tilde{u} \in C(t_a,t_f;\mathbb{R}^1)$, $\tilde{u}\big|_{[t_a,\tilde{s}]} \in C^p(t_a,\tilde{s};\mathbb{R}^1)$, where $p \ge 1$.

Example 2.1 *We consider the following simple control problem:*

$$\text{(OCex)} \quad \min_{(u,x)} \; \frac{1}{2}\int_0^1 \left(x(t)^3 + u(t)^2\right)\,dt$$

subject to

$$\begin{aligned}
&\dot{x}(t) = u(t) && \textit{for a.a. } t \in [0,1]\,,\\
&x(0) = 4\,, &&\\
&u(t) \le -2 && \textit{for a.a. } t \in [0,1]\,.
\end{aligned}$$

The optimal control shown in Fig.1 has the structure defined by (CS).

In the multiple shooting approach inequalities defined by system (1) are replaced by suitable equations and the junction point is treated as additional variable. We use a similar approach associating with (1) a new system of equations, and with the junction point as additional variable (compare Maurer, Pesch, 1995, Malanowski, Maurer, 1996, 1998, Malanowski, 1998, for a related approach in sensitivity analysis).

Based on the structural assumption (CS) we compute two functions $\tilde{u}^{(1)} \in C^p(t_a,\tilde{s};\mathbb{R}^1)$, $u^{(2)} \in C^p(\tilde{s},t_f;\mathbb{R}^1)$ defining the optimal control by

$$\tilde{u}(t) = \begin{cases} \tilde{u}^{(1)}(t)\,, & t \in [t_a,\tilde{s}]\,,\\ \tilde{u}^{(2)}(t)\,, & t \in [\tilde{s},t_f]\,. \end{cases}$$

Let the functions $\tilde{x}^{(1)}$, $\tilde{u}^{(1)}$, $\tilde{x}^{(2)}$, $\tilde{u}^{(2)}$ be defined by

$$\begin{aligned}
\tilde{x}^{(1)}(t) = \tilde{x}(t)\,,\; \tilde{u}^{(1)}(t) = \tilde{u}(t)\,, \quad \forall t \in [t_a,\tilde{s}]\,,\\
\tilde{x}^{(2)}(t) = \tilde{x}(t)\,,\; \tilde{u}^{(2)}(t) = \tilde{u}(t)\,, \quad \forall t \in [\tilde{s},t_f]\,.
\end{aligned} \tag{2}$$

Figure 1. Optimal control for Problem (OCex)

and let the functions $\tilde{\lambda}^{(1)}$, $\tilde{\lambda}^{(2)}$, $\tilde{\mu}^{(2)}$ be defined by

$$\begin{aligned} &\tilde{\lambda}^{(1)}(t) = \tilde{\lambda}(t) && \forall t \in [t_a, \tilde{s}]\,, \\ &\tilde{\lambda}^{(2)}(t) = \tilde{\lambda}(t)\,, \quad \tilde{\mu}^{(2)}(t) = \tilde{\mu}(t)\,, && \forall t \in [\tilde{s}, t_f]\,. \end{aligned} \tag{3}$$

By Assumptions (C1)–(C2), (CS), $(\tilde{x}^{(1)}, \tilde{u}^{(1)}, \tilde{x}^{(2)}, \tilde{u}^{(2)}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \tilde{\mu}^{(2)}, \tilde{s})$ is a solution of the system defined by

$$\begin{aligned} &\dot{\lambda}^{(1)}(t)^T = -H_x(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) \quad \forall t \in [t_a, s]\,, \\ &\lambda^{(1)}(s) = \lambda^{(2)}(s)\,, \\ &H_u(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) = 0 \quad \forall t \in [t_a, s]\,, \\ &\dot{x}^{(1)}(t) = \psi(x^{(1)}(t), u^{(1)}(t)) \quad \forall t \in [t_a, s]\,, \\ &x^{(1)}(t_a) = a\,, \\ &\dot{\lambda}^{(2)}(t)^T = -\tilde{H}_x(x^{(2)}(t), u^{(2)}(t), \lambda^{(2)}(t), \mu^{(2)}(t)) \quad \forall t \in [s, t_f]\,, \\ &\lambda^{(2)}(t_f)^T = \phi_x(x^{(2)}(t_f))\,, \\ &\dot{x}^{(2)}(t) = \psi(x^{(2)}(t), u^{(2)}(t)) \quad \forall t \in [s, t_f]\,, \\ &\tilde{H}_u(x^{(2)}(t), u^{(2)}(t), \lambda^{(2)}(t), \mu^{(2)}(t)) = 0 \quad \forall t \in [s, t_f]\,, \\ &x^{(2)}(s) = x^{(1)}(s)\,, \\ &u^{(2)}(t) = b \quad \forall t \in [s, t_f]\,. \end{aligned} \tag{4}$$

and

$$u^{(1)}(s) = u^{(2)}(s))\,. \tag{5}$$

A drawback of this system in view of discretizations and application of Newton's method is that the intervals $[t_a, \tilde{s}]$ and $[\tilde{s}, t_f]$ depend on the variable junction point $\tilde{s}$. We therefore slightly extend these intervals in such a way that $(\tilde{z}, \tilde{s})$ defines a unique solution of the resulting system. These extensions require the strict Legendre-Clebsch condition:

(C3) There exists $\alpha > 0$ such that
$$H_{uu}(\tilde{x}^{(1)}(t), \tilde{u}^{(1)}(t), \tilde{\lambda}^{(1)}(t)) \geq \alpha$$
for all $t \in [t_a, \tilde{s}]$.

LEMMA 2.1 *Let Assumptions* (C1)–(C3) *and* (CS) *be satisfied for some* $p \geq 1$. *For* $\bar{t}_1, \bar{t}_2 > 0$ *sufficiently small there exist unique extensions of* $\tilde{u}^{(1)}$, $\tilde{x}^{(1)}$, $\tilde{\lambda}^{(1)}$ *to* $[\tilde{s}, \tilde{s} + \bar{t}_1]$, *and unique extensions of* $\tilde{u}^{(2)}$, $\tilde{x}^{(2)}$, $\tilde{\lambda}^{(2)}$, $\tilde{\mu}^{(2)}$ *to* $[\tilde{s} - \bar{t}_2, \tilde{s}]$ *such that the extended functions solve system* (4) *on* $[t_a, \tilde{s} + \bar{t}_1]$, *resp.* $[\tilde{s} - \bar{t}_2, t_f]$.

Proof. By virtue of Assumption (C3), it follows from the implicit function theorem that for every $t \in [t_a, \tilde{s}]$ there exists a radius $\varepsilon(t) > 0$ such that for all $(x, \lambda) \in B_{\varepsilon(t)}(\tilde{x}^{(1)}(t), \tilde{\lambda}^{(1)}(t))$ there exists a locally unique solution $u(x, \lambda)$ of $H_u(x, u, \lambda) = 0$, which is a p times Fréchet differentiable function of the parameter (x, λ). By compactness of $[t_a, \tilde{s}]$, we can choose $\varepsilon(t)$ independently of t. In particular, we have

$$\tilde{u}^{(1)}(t) = u(\tilde{x}^{(1)}(t), \tilde{\lambda}^{(1)}(t)) \quad \forall t \in [t_a, \tilde{s}]\,.$$

This implies that $\tilde{x}^{(1)}$, $\tilde{\lambda}^{(1)}$ are solutions of the differential equations

$$\begin{aligned}
&\dot{\lambda}^{(1)}(t)^T = -H_x(x^{(1)}(t), u(x^{(1)}(t), \lambda^{(1)}(t)), \lambda^{(1)}(t)) \quad \forall t \in [t_a, \tilde{s}]\,,\\
&\lambda^{(1)}(\tilde{s}) = \tilde{\lambda}^{(2)}(\tilde{s})\,,\\
&\dot{x}^{(1)}(t) = \psi(x^{(1)}(t), u(x^{(1)}(t), \lambda^{(1)}(t))) \quad \forall t \in [t_a, \tilde{s}]\,,\\
&x^{(1)}(\tilde{s}) = \tilde{x}^{(2)}(\tilde{s})\,.
\end{aligned}$$

By the standard results of the theory of differential equations (see e.g. Knobloch, Kappel, 1974, Chap. III, Th. 2.1) there exists $\bar{t}_1 > 0$, such that the solutions exist and are unique on $[\tilde{s}, \tilde{s}+\bar{t}_1]$. This further implies that $\tilde{u}^{(1)}(t) = u(\tilde{x}^{(1)}(t), \tilde{\lambda}^{(1)}(t))$ has a unique extension on $[\tilde{s}, \tilde{s} + \bar{t}_1]$.

For $t \in [\tilde{s}, t_f]$, $u^{(2)}(t)$ is uniquely determined by $u^{(2)}(t) = b$, and $\mu^{(2)}$ is uniquely determined by

$$\mu^{(2)}(t) = \mu^{(2)}(t, \lambda) = -H_u(x(t), b, \lambda(t))\,.$$

This implies that $\tilde{x}^{(2)}$, $\tilde{\lambda}^{(2)}$ are solutions of the differential equations

$$\begin{aligned}
&\dot{\lambda}^{(2)}(t)^T = -H_x(x^{(2)}(t), b, \lambda^{(2)}(t), \mu^{(2)}(t)) \quad \forall t \in [\tilde{s}, t_f]\,,\\
&\lambda^{(2)}(\tilde{s}) = \tilde{\lambda}^{(1)}(\tilde{s})\,,\\
&\dot{x}^{(2)}(t) = \psi(x^{(2)}(t), b) \quad \forall t \in [\tilde{s}, t_f]\,,\\
&x^{(2)}(\tilde{s}) = \tilde{x}^{(1)}(\tilde{s})\,.
\end{aligned}$$

Again by standard results of the theory of differential equations there exists $\bar{t}_2 > 0$ such that the solutions exist and are unique on $[\tilde{s} - \bar{t}_2, \tilde{s}]$. This further implies that $\tilde{\mu}^{(2)}$ has a unique extension on $[\tilde{s} - \bar{t}_2, \tilde{s}]$. ■

We denote $\sigma_1 = \tilde{s} + \bar{t}_1$, $\sigma_2 = \bar{s} - \bar{t}_2$. In the following we study the system defined by the extended equations of system (4),

$$\begin{aligned} &\dot{\lambda}^{(1)}(t)^T + H_x(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) = 0 \\ &\lambda^{(1)}(s) - \lambda^{(2)}(s) = 0\,, \\ &H_u(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) = 0\,, \\ &\dot{x}^{(1)}(t) - \psi(x^{(1)}(t), u^{(1)}(t)) = 0\,, \\ &x^{(1)}(t_a) - a = 0\,, \end{aligned} \tag{6}$$

for all $t \in [t_a, \sigma_1]$,

$$\begin{aligned} &\dot{\lambda}^{(2)}(t)^T + \tilde{H}_x(x^{(2)}(t), u^{(2)}(t), \lambda^{(2)}(t), \mu^{(2)}(t)) = 0\,, \\ &\lambda^{(2)}(t_f)^T - \phi_x(x^{(2)}(t_f))^T = 0\,, \\ &\tilde{H}_u(x^{(2)}(t), u^{(2)}(t), \lambda^{(2)}(t), \mu^{(2)}(t)) = 0\,, \\ &\dot{x}^{(2)}(t) - \psi(x^{(2)}(t), u^{(2)}(t)) = 0\,, \\ &x^{(2)}(s) - x^{(1)}(s) = 0\,, \\ &u^{(2)}(t) - b = 0\,, \end{aligned} \tag{7}$$

for all $t \in [\sigma_2, t_f]$, and

$$u^{(1)}(s) - u^{(2)}(s) = 0\,. \tag{8}$$

We show that this system defines a regular operator equation.

3. Regularity

We write system (6)–(8) as an operator equation $F(z, s) = 0$, and show that $F'(\tilde{z}, \tilde{s})$ is invertible, where

$$\tilde{z} = (\tilde{x}^{(1)}, \tilde{u}^{(1)}, \tilde{x}^{(2)}, \tilde{u}^{(2)}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \tilde{\mu}^{(2)})$$

is defined by the solution of (6)–(7). To this end we define spaces $Z = Z_1 \times Z_2 \times Z_3 \times Z_4$ with

$$\begin{aligned} Z_1 &= C^1(t_a, \sigma_1; \mathbb{R}^n) \times C(t_a, \sigma_1; \mathbb{R}^1)\,, \\ Z_2 &= C^1(\sigma_2, t_f; \mathbb{R}^n) \times C(\sigma_2, t_f; \mathbb{R}^1)\,, \\ Z_3 &= C^1(t_a, \sigma_1; \mathbb{R}^n)\,, \\ Z_4 &= C^1(\sigma_2, t_f; \mathbb{R}^n) \times C(\sigma_2, t_f; \mathbb{R}^1)\,, \end{aligned}$$

and $W = W_1 \times W_2 \times W_3 \times W_4$ with

$$
\begin{aligned}
W_1 &= C(t_a, \sigma_1; \mathbb{R}^n) \times \mathbb{R}^n \times C(t_a, \sigma_1; \mathbb{R}), \\
W_2 &= C(t_a, \sigma_1; \mathbb{R}^n) \times \mathbb{R}^n, \\
W_3 &= C(\sigma_2, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times C(\sigma_2, t_f; \mathbb{R}), \\
W_4 &= C(\sigma_2, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times C(\sigma_2, t_f; \mathbb{R}),
\end{aligned}
$$

and the operator $F\colon Z \times \mathbb{R} \to W \times \mathbb{R}$ by

$$
F(z, s) = \begin{pmatrix} F_a(z, s) \\ F_b(z, s) \end{pmatrix}, \tag{9}
$$

where F_a is defined by (6), (7), and F_b is defined by (8).

In view of application of Newton's method to this system and stable discretizations of this system we have to guarantee that the system is regular, i.e., that $F'(\tilde{z}, \tilde{s})$ exists and is a continuous linear operator.

Since by Assumption (CS) $\tilde{u}$ is continuous, we have $\dot{x}^{(1)}(\tilde{s}) = \dot{x}^{(2)}(\tilde{s})$, and $\dot{\lambda}^{(1)}(\tilde{s}) = \dot{\lambda}^{(2)}(\tilde{s})$. This implies

$$
\frac{\partial}{\partial s} F_a(\tilde{z}, \tilde{s}) = 0 .
$$

Therefore, F' has the structure

$$
F'(\tilde{z}, \tilde{s}) = \begin{pmatrix} \frac{\partial}{\partial z} F_a(\tilde{z}, \tilde{s}) & 0 \\ \frac{\partial}{\partial z} F_b(\tilde{z}, \tilde{s}) & \frac{\partial}{\partial s} F_b(\tilde{z}, \tilde{s}) \end{pmatrix},
$$

and $F'(\tilde{z}, \tilde{s})$ is regular, if

$$
\frac{\partial}{\partial z} F_a(\tilde{z}, \tilde{s})^{-1} \text{ exists,} \tag{10}
$$

and

$$
\frac{\partial}{\partial s} F_b(\tilde{z}, \tilde{s}) \neq 0 . \tag{11}
$$

Condition (11) is equivalent to
(C4) $\quad \dot{u}^{(1)}(\tilde{s}) - \dot{u}^{(2)}(\tilde{s}) \neq 0.$
This condition requires that the two arcs $u^{(1)}$, $u^{(2)}$ have a nontangential junction at $\tilde{s}$ (compare Assumption (A3) in Maurer, Pesch, 1995 and Fig. 1).

Condition (10) is equivalent to the fact that there exists a constant c such that for each $w = (w_1, \ldots, w_{11}) \in W$ the system

$$
\frac{\partial}{\partial z} F_a(\tilde{z}, \tilde{s}) z = w \tag{12}
$$

has a unique solution $z(w)$ with $\|z(w)\| \leq c\,\|w\|$. In the following we show that this is true if a second-order optimality condition for problem (OC) is satisfied.

To simplify notations, the argument of functions evaluated at the point $(\tilde{x}^{(i)}(t), \tilde{u}^{(i)}(t), \ldots))$, $i = 1, 2$, will be denoted by ${}^{(i)}[t]$. Then, the system (12) is eqivalent to

$$\begin{aligned}
&\dot{\lambda}^{(1)}(t) + H_{xx}^{(1)}[t]x^{(1)}(t) + H_{xu}^{(1)}[t]u^{(1)}(t) + \psi_x^{(1)}[t]^T \lambda^{(1)}(t) = w_1(t)\,,\\
&\lambda^{(1)}(\tilde{s}) - \lambda^{(2)}(\tilde{s}) = w_2\,,\\
&H_{ux}^{(1)}[t]x^{(1)}(t) + H_{uu}^{(1)}[t]u^{(1)}(t) + \psi_u^{(1)}[t]^T \lambda^{(1)}(t) = w_3(t)\,,\\
&\dot{x}^{(1)}(t) - \psi_x^{(1)}[t]x^{(1)}(t) - \psi_u^{(1)}[t]u^{(1)}(t) = w_4(t)\,,\\
&x^{(1)}(t_a) = w_5\,,
\end{aligned} \tag{13}$$

for all $t \in [t_a, \sigma_1]$, and

$$\begin{aligned}
&\dot{\lambda}^{(2)}(t) + \tilde{H}_{xx}^{(2)}[t]x^{(2)}(t) + \tilde{H}_{xu}^{(2)}[t]u^{(2)}(t) + \psi_x^{(2)}[t]^T \lambda^{(2)}(t) = w_6(t)\,,\\
&\lambda^{(2)}(t_f) - \phi_{xx}(\tilde{x}^{(2)}(t_f))^T x^{(2)}(t_f) = w_7\,,\\
&\tilde{H}_{ux}^{(2)}[t]x^{(2)}(t) + \tilde{H}_{uu}^{(2)}[t]u^{(2)}(t) + \psi_u^{(2)}[t]^T \lambda^{(2)}(t) + \mu^{(2)}(t) = w_8(t)\,,\\
&\dot{x}^{(2)}(t) - \psi_x^{(2)}[t]x^{(2)}(t) - \psi_u^{(2)}[t]u^{(2)}(t) = w_9(t)\,,\\
&x^{(2)}(\tilde{s}) - x^{(1)}(\tilde{s}) = w_{10}\,,\\
&u^{(2)}(t) = w_{11}(t)\,,
\end{aligned} \tag{14}$$

for all $t \in [\sigma_2, t_f]$. We define

$$\begin{aligned}
\tilde{Z}_1 &= W_2^1(t_a, \sigma_1; \mathbb{R}^n) \times L^2(t_a, \sigma_1; \mathbb{R}^1)\,,\\
\tilde{Z}_2 &= W_2^1(\sigma_2, t_f; \mathbb{R}^n) \times L^2(\sigma_2, t_f; \mathbb{R}^1)\,,\\
\tilde{Z}_3 &= W_2^1(t_a, \sigma_1; \mathbb{R}^n)\,,\\
\tilde{Z}_4 &= W_2^1(\sigma_2, t_f; \mathbb{R}^n) \times L^2(\sigma_2, t_f; \mathbb{R}^1)\,,\\
\tilde{W}_1 &= L^2(t_a, \sigma_1; \mathbb{R}^n) \times \mathbb{R}^n \times L^2(t_a, \sigma_2; \mathbb{R})\,,\\
\tilde{W}_2 &= L^2(t_a, \sigma_2; \mathbb{R}^n) \times \mathbb{R}^n\,,\\
\tilde{W}_3 &= L^2(\sigma_2, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times L^2(\sigma_2, t_f; \mathbb{R}^1)\,,\\
\tilde{W}_4 &= L^2(\sigma_2, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times L^2(\sigma_2, t_f; \mathbb{R})\,.
\end{aligned}$$

If we restrict system (13)–(14) to the interval $[t_a, \tilde{s}]$, resp. $[\tilde{s}, t_f]$, then the resulting system defines the necessary optimality conditions for the following quadratic control problem:

$$\begin{aligned}
(\mathrm{OQ})_w \quad & \mathrm{Min}_{(x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)}) \in \tilde{Z}_1 \times \tilde{Z}_2} J_Q(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}, w)\\
& \text{subject to}\\
& \dot{x}^{(1)}(t) = \psi_x^{(1)}[t]x^{(1)}(t) + \psi_u^{(1)}[t]u^{(1)}(t) + w_4(t) \quad \forall t \in [t_a, \tilde{s}]\,,\\
& x^{(1)}(t_a) = w_5,\\
& \dot{x}^{(2)}(t) = \psi_x^{(2)}[t]x^{(2)}(t) + \psi_u^{(2)}[t]u^{(2)}(t) + w_9(t) \quad \forall t \in [\tilde{s}, t_f]\,,\\
& x^{(2)}(\tilde{s}) - x^{(1)}(\tilde{s}) = w_{10},\\
& u^{(2)}(t) = w_{11}(t) \quad \forall t \in [\tilde{s}, t_f]\,,
\end{aligned}$$

where

$$
\begin{aligned}
&J_Q(x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)}, w) \\
&\quad = \frac{1}{2}\int_{t_a}^{\tilde{s}} \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix}^T Q^{(1)}(t) \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix} dt \\
&\quad - \int_{t_a}^{\tilde{s}} \left[w_1(t)^T x^{(1)}(t) + w_3(t)^T u^{(1)}(t) \right] dt \\
&\quad + \frac{1}{2}\int_{\tilde{s}}^{t_f} \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix}^T Q^{(2)}(t) \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix} dt \\
&\quad - \int_{\tilde{s}}^{t_f} \left[w_6(t)^T x^{(2)}(t) + w_8(t)^T u^{(2)}(t) \right] dt \\
&\quad + x^{(2)}(t_f)^T \phi_{xx}(\tilde{x}^{(2)}(t_f)) x^{(2)}(t_f) - w_2^T x^{(1)}(\tilde{s}) - w_7^T x^{(2)}(t_f)\,,
\end{aligned}
$$

and

$$
Q^{(i)}(t) = \begin{pmatrix} H_{xx}^{(i)}[t] & H_{xu}^{(i)}[t] \\ H_{ux}^{(i)}[t] & H_{uu}^{(i)}[t] \end{pmatrix}, \quad i = 1, 2\,.
$$

We first prove an auxiliary result. Let the operator $G\colon Z_1 \times Z_2 \to W_2 \times W_4$ be defined by the constraints of Problem $(\mathrm{OQ})_w$, i.e.,

$$
G(x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)}) = \begin{pmatrix} \dot{x}^{(1)}(\cdot) - \psi_x^{(1)}[\cdot]x^{(1)}(\cdot) - \psi_u^{(1)}[\cdot]u^{(1)}(\cdot) \\ x^{(1)}(t_a) \\ \dot{x}^{(2)}(\cdot) - \psi_x^{(2)}[\cdot]x^{(2)}(\cdot) - \psi_u^{(2)}[\cdot]u^{(2)}(\cdot) \\ x^{(2)}(\tilde{s}) - x^{(1)}(\tilde{s}) \\ u^{(2)}(\cdot) \end{pmatrix},
$$

and let the operator $\tilde{G}\colon \tilde{Z}_1 \times \tilde{Z}_2 \to \tilde{W}_2 \times \tilde{W}_4$ be defined in the same way.

LEMMA 3.1 *If Assumption* (C3) *holds, then the operators* G*, and* $\tilde{G}$ *are surjective.*

Proof. Let $v := (w_4, w_5, w_9, w_{10}, w_{11}) \in \tilde{W}_2 \times \tilde{W}_4$ be arbitrary. To prove the assertion for $\tilde{G}$, we have to find $(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in \tilde{Z}_1 \times \tilde{Z}_2$ such that $G(x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)}) = v$. The last equation of this system is

$$
u^{(2)}(t) = w_{11}(t)\,. \tag{15}
$$

Setting $u^{(1)} = 0$, we obtain an initial value problem having unique solutions $x^{(1)}$ and $x^{(2)}$. For $v := (w_4, w_5, w_9, w_{10}, w_{11}) \in W_2 \times W_4$ it follows from (15) that $u^{(2)} \in C(\sigma_2, t_f; \mathbb{R}^1)$. Hence $(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in Z_1 \times Z_2$, which proves the assertion for G. ■

Using Lemma 3.1 it can be shown that Problem $(\mathrm{OQ})_w$ has a unique solution $z(w)$ for each $w \in W$, if the following strong second-order condition is satisfied (compare e.g. Malanowski, 1998), where $u_{\tilde{s}}^{(1)} = u^{(1)}|_{[t_a,\tilde{s}]}$, $u_{\tilde{s}}^{(2)} = u^{(2)}|_{[\tilde{s},t_f]}$:

(C5) There exists $\gamma > 0$ such that

$$\frac{1}{2}\int_{t_a}^{\tilde{s}} \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix}^T Q^{(1)}(t) \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix} dt$$

$$+\frac{1}{2}\int_{\tilde{s}}^{t_f} \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix}^T Q^{(2)}(t) \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix} dt$$

$$+\frac{1}{2}x^{(2)}(t_f)^T \phi_{xx}(\tilde{x}^{(2)}(t_f))x^{(2)}(t_f) \geq \gamma \left(\|u_{\tilde{s}}^{(1)}\|_2^2 + \|u_{\tilde{s}}^{(2)}\|_2^2\right)$$

for all $(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in \tilde{Z}_1 \times \tilde{Z}_2$ satisfying

$$\begin{aligned}
&\dot{x}^{(1)}(t) = \psi_x^{(1)}[t]x^{(1)}(t) + \psi_u^{(1)}[t]u^{(1)}(t) \quad \forall t \in [t_a, \tilde{s}]\,,\\
&x^{(1)}(t_a) = 0\,,\\
&\dot{x}^{(2)}(t) = \psi_x^{(2)}[t]x^{(2)}(t) + \psi_u^{(2)}[t]u^{(2)}(t) \quad \forall t \in [\tilde{s}, t_f]\,,\\
&x^{(2)}(\tilde{s}) = x^{(1)}(\tilde{s})\,,\\
&u^{(2)}(t) = 0 \quad \forall t \in [\tilde{s}, t_f]\,.
\end{aligned}$$

We can now show regularity of $F'(\tilde{z}, \tilde{s})$.

THEOREM 3.1 *Let Assumptions* (C1)–(C5) *and* (CS) *be satisfied. Then* $F'(\tilde{z}, \tilde{s})$ *is regular.*

Proof. Assumption (C4) implies (11), so that it remains to prove (10), i.e., we have to show that system (13)–(14) has a unique solution. Let $w \in W$ be arbitrary. By Malanowski (1998), Lemma 4.1, system (14) restricted to the interval $[\tilde{s}, t_f]$, has a unique solution $(x^{(2)}, u^{(2)}, \lambda^{(2)}, \mu^{(2)})$. We show that this solution can be uniquely extended to $[\sigma_2, \tilde{s}]$. System (14) defines $u^{(2)}$ on $[\sigma_2, t_f]$ by $u^{(2)}(t) = w_{11}(t)$, and $\mu^{(2)}$ by

$$\mu^{(2)}(t) = w_8(t) - \tilde{H}_{ux}^{(2)}[t]x^{(2)}(t) + \tilde{H}_{uu}^{(2)}[t]u^{(2)}(t) - \psi_u^{(2)}[t]^T \lambda^{(2)}(t)\,.$$

Inserting these expressions in the state and adjoint equations in (14) it follows that $x^{(2)}, \lambda^{(2)}$ exist and are uniquely defined on $[\sigma_2, t_f]$. The assertion for system (13) can be shown in the same way. ■

The result of Theorem 3.1 can be directly obtained (in the same way as Lemma 4.1 in Malanowski, 1998), if (C5) is replaced by the following stronger Assumption:

($\widetilde{\text{C5}}$) There exists $\gamma > 0$ such that

$$B(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \geq \gamma \left(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2\right)$$

for all $(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in \tilde{Z}_1 \times \tilde{Z}_2$ satisfying

$$\begin{aligned}
&\dot{x}^{(1)}(t) = \psi_x^{(1)}[t]x^{(1)}(t) + \psi_u^{(1)}[t]u^{(1)}(t) \quad \forall t \in [t_a, \sigma_1]\,,\\
&x^{(1)}(t_a) = 0\,,\\
&\dot{x}^{(2)}(t) = \psi_x^{(2)}[t]x^{(2)}(t) + \psi_u^{(2)}[t]u^{(2)}(t) \quad \forall t \in [\sigma_2, t_f]\,,\\
&x^{(2)}(\tilde{s}) = x^{(1)}(\tilde{s})\,,\\
&u^{(2)}(t) = 0 \quad \forall t \in [\sigma_2, t_f]\,,
\end{aligned}$$

where the quadratic form B is defined by

$$\begin{aligned} B(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \quad = \quad & \frac{1}{2}\int_{t_a}^{\sigma_1} \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix}^T Q^{(1)}(t) \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix} dt \\ & + \frac{1}{2}\int_{\sigma_2}^{t_f} \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix}^T Q^{(2)}(t) \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix} dt \\ & + \frac{1}{2} x^{(2)}(t_f)^T \phi_{xx}(\tilde{x}^{(2)}(t_f)) x^{(2)}(t_f) \,. \end{aligned}$$

EXAMPLE 3.1 *The optimal trajectory $\tilde{x}$ of the control Problem (OCex) in Example 2.1 is positive. Therefore, Assumption ($\widetilde{\text{C5}}$) is satisfied.*

The stronger Assumption ($\widetilde{\text{C5}}$) will be used in the following to derive error estimates for discretizations of system (6)–(7).

4. Application of Newton's method

The result of Theorem 3.1 implies that Newton's method for operator equations can be applied to system (9). If we denote

$$z^{(k)} = (x^{(1,k)}, u^{(1,k)}, x^{(2,k)}, u^{(2,k)}, \lambda^{(1,k)}, \lambda^{(2,k)}, \mu^{(2,k)}) \,,$$

then in each iteration step of Newton's method we have to solve the system

$$\begin{aligned} F_a(z^{(k)}, s^{(k)}) + F_a'(z^{(k)}, s^{(k)})(z - z^{(k)}, s - s^{(k)}) = 0 \,, \\ F_b(z^{(k)}, s^{(k)}) + F_b'(z^{(k)}, s^{(k)})(z - z^{(k)}, s - s^{(k)}) = 0 \,. \end{aligned} \tag{16}$$

The first equation is independent of s and defines the next iterate $z^{(k+1)} = z^{(k)} + \Delta z^{(k)}$. It can be solved by solution of the following quadratic control problem:

$$\begin{aligned} (\text{OQ})_k \quad & \text{Min}_{\Delta x^{(1)}, \Delta u^{(1)}, \Delta x^{(2)}, \Delta u^{(2)} \in \tilde{Z}_1 \times \tilde{Z}_2} J_k(\Delta x^{(1)}, \Delta u^{(1)}, \Delta x^{(2)}, \Delta u^{(2)}) \\ & \text{subject to} \\ & \Delta\dot{x}^{(1)}(t) = \psi_x^{(1,k)}[t]\Delta x^{(1)}(t) + \psi_u^{(1,k)}[t]\Delta u^{(1)}(t) + \psi^{(1,k)}[t] \ \forall t \in [t_a, \sigma_1] \,, \\ & \Delta x^{(1)}(t_a) = a, \\ & \Delta\dot{x}^{(2)}(t) = \psi_x^{(2,k)}[t]\Delta x^{(2)}(t) + \psi_u^{(2,k)}[t]\Delta u^{(2)}(t) + \psi^{(2,k)}[t] \ \forall t \in [\sigma_2, t_f] \,, \\ & \Delta x^{(2)}(\tilde{s}) = \Delta x^{(1)}(\tilde{s}), \\ & u^{(2,k)}(t) - b + \Delta u^{(2)}(t) = 0 \qquad \forall t \in [\sigma_2, t_f] \,, \end{aligned}$$

where the argument of functions evaluated at the point $(\tilde{x}^{(i,k)}(t), \tilde{u}^{(i,k)}(t), \ldots))$, $i = 1, 2$, is denoted by ${}^{(i,k)}[t]$, and

$$J_k(x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)}) = \int_{t_a}^{\sigma_1} \left[\varphi_x^{(1,k)}[t]\Delta x^{(1)}(t) + \varphi_u^{(1,k)}[t]\Delta u^{(1)}(t)\right] dt$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{t_a}^{\sigma_1}\begin{pmatrix}\Delta x^{(1)}(t)\\ \Delta u^{(1)}(t)\end{pmatrix}^T Q_k^{(1)}(t)\begin{pmatrix}\Delta x^{(1)}(t)\\ \Delta u^{(1)}(t)\end{pmatrix} dt\\
&+\int_{\sigma_2}^{t_f}\left[\varphi_x^{(2,k)}[t]\Delta x^{(2)}(t)+\varphi_u^{(2,k)}[t]\Delta u^{(2)}(t)\right] dt\\
&+\frac{1}{2}\int_{\sigma_2}^{t_f}\begin{pmatrix}\Delta x^{(2)}(t)\\ \Delta u^{(2)}(t)\end{pmatrix}^T Q_k^{(2)}(t)\begin{pmatrix}x^{(2)}(t)\\ u^{(2)}(t)\end{pmatrix} dt\\
&+\Delta x^{(2)}(t_f)^T\phi_{xx}(\tilde{x}^{(2)}(t_f))\Delta x^{(2)}(t_f)\,,
\end{aligned}
$$

with

$$
Q_k^{(i)}(t)=\begin{pmatrix}H_{xx}^{(i,k)}[t] & H_{xu}^{(i,k)}[t]\\ H_{ux}^{(i,k)}[t] & H_{uu}^{(i,k)}[t]\end{pmatrix},\quad i=1,2\,.
$$

The second equation in (16) is

$$
u^{(1,k+1)}(s^{(k)})-u^{(2,k+1)}(s^{(k)})+(\dot{u}^{(1,k)}(s^{(k)})-\dot{u}^{(2,k)}(s^{(k)}))(s-s^{(k)})=0\,.
$$

By Assumption (C4) this equation has the unique solution

$$
s^{(k+1)}=s^{(k)}-\frac{u^{(1,k+1)}(s^{(k)})-u^{(2,k+1)}(s^{(k)})}{\dot{u}^{(1,k)}(s^{(k)})-\dot{u}^{(2,k)}(s^{(k)})}\,. \tag{17}
$$

In this way we obtain a sequential quadratic programming method for the solution of system (9), where in each iteration step we first solve problem $(\mathrm{OQ})_k$ to obtain $z^{(k+1)}$ and then we compute $s^{(k+1)}$ from (17).

5. Discretization by control parameterization

The reduction of the solution of Problem (OC) to the solution of the operator equation (9) allows to obtain similar results as in Felgenhauer, 1998, but for the more general case, where the optimal control satisfies the structural assumption (CS), i.e., the optimal control is only piecewise of class C^p. We present here only the main ideas for a simple approximation of system (9). As in Sirisena, Chou (1979) we use a discretization defined by control parameterization, i.e., only the control functions are discretized while it is assumed that the system and adjoint equations are solved exactly. The discretization of system (9) discussed in the following is motivated by a control parameterization method for the quadratic control problems $(\mathrm{OQ})_k$ of the preceding section.

Let $N\in\mathbb{N}$, $N\geq 2$, and let $h_1=(\sigma_1-t_a)/N$, $h_2=(t_f-\sigma_2)/N$ be the mesh spacings and

$$
t_i^{(1)}=t_a+ih_1\,,\ i=0,\dots,N\,,\quad t_j^{(2)}=\sigma_2+jh_2\,,\ j=0,\dots,N\,,
$$

the nodes of the discretization. We approximate the controls $u^{(1)}$, $u^{(2)}$ piecewise by polynomials of degree k. With

$$I^{(1)} = \left\{ [t_j^{(1)}, t_{j+1}^{(1)}] \,\middle|\, j = 0, \dots, N-1 \right\} ,$$
$$I^{(2)} = \left\{ [t_j^{(2)}, t_{j+1}^{(2)}] \,\middle|\, j = 0, \dots, N-1 \right\}$$

we introduce finite dimensional spaces

$$U_N^{(1)} = \left\{ u\colon [t_a, \sigma_1] \to \mathbb{R}^1 \mid u|_I \text{ is a polynomial of degree } k \ \forall I \in I^{(1)} \right\}$$

and

$$U_N^{(2)} = \left\{ u\colon [\sigma_2, t_f] \to \mathbb{R}^1 \mid u|_I \text{ is a polynomial of degree } k \ \forall I \in I^{(2)} \right\} .$$

We use additional interpolation nodes

$$\tau_{ij}^{(1)} = t_i^{(1)} + j h_1 / k \,, \quad i = 0, \dots N-1 \,,\ j = 0, \dots, k \,,$$

and

$$\tau_{ij}^{(2)} = t_i^{(2)} + j h_2 / k \,, \quad i = 0, \dots N-1 \,,\ j = 0, \dots, k \,.$$

We use Lagrange polynomials as basis functions for $U_N^{(1)}$ and $U_N^{(2)}$. For $i = 0, \dots N-1$, we denote by $L_{i,j}^{(1)}$, $j = 0, \dots, k$, the Lagrange polynomials defined on $[t_i^{(1)}, t_{i+1}^{(1)}] = [\tau_{i,0}^{(1)}, \tau_{i,k}^{(1)}]$ having the property

$$L_{ij}^{(1)} = \begin{cases} 1 \,, & j = i \,, \\ 0 \,, & j \neq i \,, \end{cases}$$

and by $L_{i,j}^{(2)}$, $j = 0, \dots, k$, the Lagrange polynomials defined on $[t_i^{(2)}, t_{i+1}^{(2)}] = [\tau_{i,0}^{(2)}, \tau_{i,k}^{(2)}]$ having the property

$$L_{ij}^{(2)} = \begin{cases} 1 \,, & j = i \,, \\ 0 \,, & j \neq i \,. \end{cases}$$

Further we define discretization operators $\Delta_N^{(1)}\colon L^2([t_a, \sigma_1]) \to U_N^{(1)}$, where $v := \Delta_N^{(1)}(u)$ is uniquely defined by

$$v(t_i^{(1)}) = u(t_i^{(1)}) \,, \quad i = 0, \dots N \,,$$
$$v(\tau_{ij}^{(1)}) = u(\tau_{ij}^{(1)}) \,,\ i = 0, \dots N-1 \,,\ j = 0, \dots, k \,,$$

and $\Delta_N^{(2)}\colon L^2([\sigma_2, t_f]) \to U_N$, where $v := \Delta_N^{(2)}(u)$ is uniquely defined by

$$v(t_i^{(2)}) = u(t_i^{(2)}) \,, \quad i = 0, \dots N \,,$$
$$v(\tau_{ij}^{(2)}) = u(\tau_{ij}^{(2)}) \,,\ i = 0, \dots N-1 \,,\ j = 0, \dots, k \,.$$

Then, for $u^{(1)} \in C^{k+1}([t_a, \sigma_1])$, $u^{(2)} \in C^{k+1}([\sigma_2, t_f])$ the error estimates

$$\|u^{(1)} - \Delta_N^{(1)} u^{(1)}\|_\infty \le c\, h^{k+1}, \quad \|\dot{u}^{(1)} - \Delta_N^{(1)} \dot{u}^{(1)}\|_\infty \le c\, h^k, \tag{18}$$

and

$$\|u^{(2)} - \Delta_N^{(2)} u^{(2)}\|_\infty \le c\, h^{k+1}, \quad \|\dot{u}^{(2)} - \Delta_N^{(2)} \dot{u}^{(2)}\|_\infty \le c\, h^k, \tag{19}$$

hold, where $h = \max\{h_1, h_2\}$. Finally, we define a discretization of (6) by

$$\begin{aligned}
&\dot{\lambda}^{(1)}(t)^T + H_x(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) = 0\\
&\lambda^{(1)}(s) - \lambda^{(2)}(s) = 0,\\
&\int_{t_a}^{\sigma_1} H_u(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) L_{ij}^{(1)}(t)\, dt = 0,\\
&\qquad i = 0, \dots N-1,\ j = 0, \dots, k,\\
&\dot{x}^{(1)}(t) - \psi(x^{(1)}(t), u^{(1)}(t)) = 0,\\
&x^{(1)}(t_a) - a = 0,
\end{aligned} \tag{20}$$

for all $t \in [t_a, \sigma_1]$, and a discretization of (7) by

$$\begin{aligned}
&\dot{\lambda}^{(2)}(t)^T + \tilde{H}_x(x^{(2)}(t), u^{(2)}(t), \lambda^{(2)}(t), \mu^{(2)}(t)) = 0,\\
&\lambda^{(2)}(t_f)^T - \phi_x(x^{(2)}(t_f))^T = 0,\\
&\int_{\sigma_2}^{t_f} \tilde{H}_u(x^{(2)}(t), u^{(2)}(t), \lambda^{(2)}(t), \mu^{(2)}(t)) L_{ij}^{(2)}(t)\, dt = 0,\\
&\qquad i = 0, \dots N-1,\ j = 0, \dots, k,\\
&\dot{x}^{(2)}(t) - \psi(x^{(2)}(t), u^{(2)}(t)) = 0,\\
&x^{(2)}(s) - x^{(1)}(s) = 0,\\
&u^{(2)}(t) - b = 0,
\end{aligned} \tag{21}$$

for all $t \in [\sigma_2, t_f]$. In order to obtain a discretization F_N of the operator F we define spaces $Z_N = Z_{N,1} \times Z_{N,2} \times Z_{N,3} \times Z_{N,4}$ by

$$\begin{aligned}
Z_{N,1} &= C^1(t_a, \sigma_1; \mathbb{R}^n) \times U_N^{(1)},\\
Z_{N,2} &= C^1(\sigma_2, t_f; \mathbb{R}^n) \times U_N^{(2)},\\
Z_{N,3} &= C^1(t_a, \sigma_1; \mathbb{R}^n),\\
Z_{N,4} &= C^1(\sigma_2, t_f; \mathbb{R}^n) \times U_N^{(2)},
\end{aligned}$$

and $W_N = W_{N,1} \times W_{N,2} \times W_{N,3} \times W_{N,4}$ by

$$\begin{aligned}
W_{N,1} &= C(t_a, \sigma_1; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^{(k+1)N},\\
W_{N,2} &= C(t_a, \sigma_1; \mathbb{R}^n) \times \mathbb{R}^n,\\
W_{N,3} &= C(\sigma_2, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^{(k+1)N},\\
W_{N,4} &= C(\sigma_2, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times U_N^{(2)},
\end{aligned}$$

where $U_N^{(1)}$ is provided with the norm of $C(t_a, \sigma_1; \mathbb{R}^1)$ and $U_N^{(2)}$ is provided with the norm of $C(\sigma_2, t_f; \mathbb{R}^1)$. Further, we define the operator $F_N: Z_N \times \mathbb{R} \to W_N \times \mathbb{R}$ by

$$F_N(z,s) = \begin{pmatrix} F_{N,a}(z,s) \\ F_{N,b}(z,s) \end{pmatrix}, \tag{22}$$

where F_a is defined by (20), (21), and F_b is defined by

$$u^{(1)}(s) - u^{(2)}(s) = 0\,. \tag{23}$$

In the following section we show that for sufficiently large N the discretized equation $F_N(z,s)$ has a solution and we derive error estimates.

6. Convergence analysis

Convergence results for discretizations of operator equations can be found e.g. in Allgower et al. (1986), Deuflhard, Potra (1992), results for discretizations of generalized equations can be found e.g. in Alt (1997), Dontchev, Hager, Malanowski (2000), Malanowski et al. (1997). For the convergence analysis of the equations considered here Theorem 2.2 of Malanowski et al. (1997) is most suitable. In the following we use a special case of this result based on a modification due to Felgenhauer (1998, 1999a). If we apply Theorem 7 of (1999a) to operator equations, we obtain the following result.

THEOREM 6.1 *Let V and $V' \subset V^*$ be Banach spaces, let $F: V \to V'$ be Fréchet differentiable and let $\tilde{v}$ be a solution of $F(v) = 0$. Suppose that for $N \geq \tilde{N}$ subspaces $V_N \subset V$, $V'_N \subset V'$ and operators $F_N: V_N \to V'_N$ are given with:*

(a) For each N one can find $v_N \in V_N$ such that $\|F_N(v_N)\| \to 0$ for $N \to \infty$.

(b) There exists $r > 0$, $L > 0$ such that the operators F'_N are Fréchet differentiable with

$$\|F'_N(v_1) - F'_N(v_2)\| \leq L\,\|v_1 - v_2\| \quad \forall v_1, v_2 \in B_r(v_N)\,.$$

Then for sufficiently large N equation $F_N(v) = 0$ has a solution v_N^* with

$$\|v_N^* - \tilde{v}\| \leq c(\|F_N(v_N)\| + \|v_N - \tilde{v}\|)$$

with a constant c independent of N.

Proof. Apply Theorem 7 of Felgenhauer (1999a) with $s_h = s_N = F_N(v_N)$ and $K = \{0\}$. ■

REMARK 6.1 Theorem 2.2 of Malanowski et al. (1997) and Theorem 7 of Felgenhauer (1999a) assume a global Lipschitz condition for the operators F'_N, but it can be easily seen that it is enough to require the local assumption (b) (compare also Alt, 1997, Theorem 3.2, Dontchev, Hager, Malanowski, 2000, Theorem 3.1).

In the following we apply Theorem 6.1 with $\tilde{v} = (\tilde{z}, \tilde{s})$, where

$$\tilde{z} = (\tilde{x}^{(1)}, \tilde{u}^{(1)}, \tilde{x}^{(2)}, \tilde{u}^{(2)}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \tilde{\mu}^{(2)})$$

is the solution of (6)–(7), and $v_N = (\tilde{z}_N, \tilde{s})$, where

$$\tilde{z}_N = (\tilde{x}^{(1)}, \Delta_N^{(1)} \tilde{u}^{(1)}, \tilde{x}^{(2)}, \Delta_N^{(2)} \tilde{u}^{(2)}, \tilde{\lambda}^{(1)}, \tilde{\lambda}^{(2)}, \Delta_N^{(2)} \tilde{\mu}^{(2)}) \,.$$

Assumption (b) of Theorem 6.1 can be easily checked. The following Lemma shows that Assumption (a) is satisfied.

LEMMA 6.1 *Let Assumptions* (C1)–(C2) *and* (CS) *be satisfied for some* $p \geq k+1$*, where* k *is the order of polynomials used to approximate the controls. Then*

$$\|F_N(\tilde{z}_N, \tilde{s})\| \leq c h^{k+1}$$

with a constant c *independent of* N*.*

Proof. Since $\tilde{z}$ is a solution of (6)–(7), the first component of $F_N(\tilde{z}_N, \tilde{s})$ is

$$F_{N,1} = H_x(x^{(1)}(t), \Delta_N^{(1)} \tilde{u}^{(1)} u^{(1)}(t), \lambda^{(1)}(t)) - H_x(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) \,.$$

By Assumption (C2) and (18) we therefore obtain

$$\|F_{N,1}\| \leq c_1 h^{k+1}$$

with a constant c_1 independent of N. The second component of $F_N(\tilde{z}_N, \tilde{s})$ vanishes. Again, since $\tilde{z}$ is a solution of (6)–(7), the third component of $F_N(\tilde{z}_N, \tilde{s})$ is

$$\begin{aligned} F_{N,3} = \int_{t_a}^{\sigma_1} & \big[H_u(x^{(1)}(t), u^{(1)}(t), \lambda^{(1)}(t)) \\ & - H_x(x^{(1)}(t), \Delta_N^{(1)} \tilde{u}^{(1)} u^{(1)}(t), \lambda^{(1)}(t))\big] L_{ij}^{(1)}(t)\, dt = 0 \,, \\ & i = 0, \ldots N-1 \,,\ j = 0, \ldots, k \,, \end{aligned}$$

which by Assumption (C2) and (18) implies

$$\|F_{N,3}\| \leq c_3 h^{k+1}$$

with a constant c_3 independent of N. For the remaining components of $F_{N,a}$ we obtain the desired error estimates by the same argumentation. Since $F_b(\tilde{z}, \tilde{s}) = 0$ we obtain

$$F_{N,b}(\tilde{z}_N, \tilde{s}) = \Delta_N^{(1)} u^{(1)}(s) - u^{(1)}(s) - (\Delta_N^{(2)} u^{(2)}(s) - u^{(2)}(s)) \,.$$

By (18), (19) this implies

$$\|F_{N,b}(\tilde{z}_N, \tilde{s})\| \leq c_b h^{k+1}$$

with a constant c_b independent of N. ■

It remains to prove Assumption (c) of Theorem 6.1, i.e., the operators $F'_N(\tilde{z}_N,\tilde{s})^{-1}$ exist and are uniformly bounded. To this end we proceed as in Section 3. $F'_N(\tilde{z}_N,\tilde{s})$ has the structure

$$F'_N(\tilde{z}_N,\tilde{s}) = \begin{pmatrix} \frac{\partial}{\partial z}F_{N,a}(\tilde{z}_N,\tilde{s}) & 0 \\ \frac{\partial}{\partial z}F_{N,b}(\tilde{z}_N,\tilde{s}) & \frac{\partial}{\partial s}F_{N,b}(\tilde{z}_N,\tilde{s}) \end{pmatrix}.$$

Therefore, the operators $F'_N(\tilde{z}_N,\tilde{s})$ are uniformly regular, if

$$\left\| \frac{\partial}{\partial z}F_{N,a}(\tilde{z},\tilde{s})^{-1} \right\| \le c_a \tag{24}$$

and

$$\left| \frac{\partial}{\partial s}F_{N,b}(\tilde{z},\tilde{s}) \right| \ge c_b \tag{25}$$

with constants c_a, c_b independent of N.

Condition (25) is equivalent to

$$\left| \frac{d}{dt}\Delta_N^1\tilde{u}^{(1)}(\tilde{s}) - \frac{d}{dt}\Delta_N^2 u^{(2)}(\tilde{s}) \right| \ge c_b\,. \tag{26}$$

By Assumption (C4) we have

$$\tilde{c} := \left| \frac{d}{dt}\tilde{u}^{(1)}(\tilde{s}) - \frac{d}{dt}\tilde{u}^{(2)}(\tilde{s}) \right| > 0\,.$$

It therefore follows from (18), (19) that

$$\left| \frac{d}{dt}\Delta_N^1\tilde{u}^{(1)}(\tilde{s}) - \frac{d}{dt}\Delta_N^2 u^{(2)}(\tilde{s}) \right| \ge \tilde{c} - ch^{k-1}\,.$$

This shows that (26) is satified for sufficiently large N.

Condition (24) is equivalent to the fact that there exists a constant c_a independent of N such that for each $w = (w_1,\ldots,w_{11}) \in W_N$ the system

$$\frac{\partial}{\partial z}F_{N,a}(\tilde{z}_N,\tilde{s})z = w \tag{27}$$

has a unique solution $z(w)$ with

$$\|z(w)\| \le c_a\|w\|\,. \tag{28}$$

To show this we proceed in the same way as in Section 3. We first introduce spaces $\tilde{Z}_N = \tilde{Z}_{N,1}\times\tilde{Z}_{N,2}\times\tilde{Z}_{N,3}\times\tilde{Z}_{N,4}$ by

$$\begin{aligned}
\tilde{Z}_{N,1} &= C^1(t_a,\sigma_1;\mathbb{R}^n)\times U_N^{(1)}\,,\\
\tilde{Z}_{N,2} &= C^1(\sigma_2,t_f;\mathbb{R}^n)\times U_N^{(2)}\,,\\
\tilde{Z}_{N,3} &= C^1(t_a,\sigma_1;\mathbb{R}^n)\,,\\
\tilde{Z}_{N,4} &= C^1(\sigma_2,t_f;\mathbb{R}^n)\times U_N^{(2)}\,,
\end{aligned}$$

and $\tilde{W}_N = \tilde{W}_{N,1} \times \tilde{W}_{N,2} \times \tilde{W}_{N,3} \times \tilde{W}_{N,4}$ by

$$\begin{aligned}
\tilde{W}_{N,1} &= C(t_a,\sigma_1;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^{(k+1)N}\,,\\
\tilde{W}_{N,2} &= C(t_a,\sigma_1;\mathbb{R}^n) \times \mathbb{R}^n\,,\\
\tilde{W}_{N,3} &= C(\sigma_2,t_f;\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^{(k+1)N}\,,\\
\tilde{W}_{N,4} &= C(\sigma_2,t_f;\mathbb{R}^n) \times \mathbb{R}^n \times U_N^{(2)}\,,
\end{aligned}$$

where $U_N^{(1)}$ is provided with the norm of $L^2(t_a,\sigma_1;\mathbb{R}^1)$ and $U_N^{(2)}$ is provided with the norm of $L^2(\sigma_2,t_f;\mathbb{R}^1)$.

To simplify notations, the argument of functions evaluated at the point $(\tilde{x}^{(i)}(t), \Delta_N^{(1)}\tilde{u}^{(i)}(t),\ldots))$, $i=1,2$, will be denoted by ${}^{(i,N)}[t]$. System (27) defines the necessary optimality conditions for the following quadratic control problem:

$$\begin{aligned}
(\mathrm{OQ})_{N,w}\ &\mathrm{Min}_{(x^{(1)},x^{(2)},u^{(1)},u^{(2)})\in\tilde{Z}_{1,N}\times\tilde{Z}_{2,N}} J_N(x^{(1)},u^{(1)},x^{(2)},u^{(2)},w)\\
&\text{subject to}\\
&\dot{x}^{(1)}(t) = \psi_x^{(1,N)}[t]x^{(1)}(t) + \psi_u^{(1,N)}[t]u^{(1)}(t) + w_4(t) \qquad \forall t\in[t_a,\sigma_1]\,,\\
&x^{(1)}(t_a) = w_5,\\
&\dot{x}^{(2)}(t) = \psi_x^{(2,N)}[t]x^{(2)}(t) + \psi_u^{(2,N)}[t]u^{(2)}(t) + w_9(t) \qquad \forall t\in[\sigma_2,t_f]\,,\\
&x^{(2)}(\tilde{s}) - x^{(1)}(\tilde{s}) = w_{10},\\
&u^{(2)}(t) = w_{11}(t) \qquad \forall t\in[\sigma_2,t_f]\,,
\end{aligned}$$

where

$$\begin{aligned}
&J_N(x^{(1)},x^{(2)},u^{(1)},u^{(2)},w)\\
&= \frac{1}{2}\int_{t_a}^{\sigma_1} \begin{pmatrix} x^{(1)}(t)\\ u^{(1)}(t)\end{pmatrix}^T Q^{(1,N)}(t) \begin{pmatrix} x^{(1)}(t)\\ u^{(1)}(t)\end{pmatrix} dt\\
&\quad - \int_{t_a}^{\sigma_1} [w_1(t)^T x^{(1)}(t) + w_3(t)^T u^{(1)}(t)]dt\\
&\quad + \frac{1}{2}\int_{\sigma_2}^{t_f} \begin{pmatrix} x^{(2)}(t)\\ u^{(2)}(t)\end{pmatrix}^T Q^{(2,N)}(t) \begin{pmatrix} x^{(2)}(t)\\ u^{(2)}(t)\end{pmatrix} dt\\
&\quad - \int_{\sigma_2}^{t_f} [w_6(t)^T x^{(2)}(t) - w_8(t)^T u^{(2)}(t)]dt\\
&\quad + x^{(2)}(t_f)^T \phi_{xx}(\tilde{x}^{(2)}(t_f))x^{(2)}(t_f) - w_2^T x^{(1)}(\tilde{s}) - w_7^T x^{(2)}(t_f)\,,
\end{aligned}$$

and

$$Q^{(i,N)}(t) = \begin{pmatrix} H_{xx}^{(i,N)}[t] & H_{xu}^{(i,N)}[t]\\ H_{ux}^{(i,N)}[t] & H_{uu}^{(i,N)}[t]\end{pmatrix}, \quad i=1,2\,.$$

As in Section 3 it can be shown, that Problem $(\mathrm{OQ})_{N,w}$, and hence system (27), has a unique solution $z(w)$ for each $w\in\tilde{W}_N$ with (28), if the operators G_N

defined by the constraints of Problem $(\mathrm{OQ})_{w,N}$ are uniformly surjective and if a strong second-order condition is satisfied. We first prove uniform surjectivity. The operators $G_N\colon Z_{1,N} \times Z_{2,N} \to W_{2,N} \times W_{4,N}$ are defined by

$$G_N(x^{(1)}, x^{(2)}, u^{(1)}, u^{(2)}) = \begin{pmatrix} \dot{x}^{(1)}(\cdot) - \psi_x^{(1)}[\cdot]x^{(1)}(\cdot) - \psi_u^{(1)}[\cdot]u^{(1)}(\cdot) \\ x^{(1)}(t_a) \\ \dot{x}^{(2)}(\cdot) - \psi_x^{(2)}[\cdot]x^{(2)}(\cdot) - \psi_u^{(2)}[\cdot]u^{(2)}(\cdot) \\ x^{(2)}(\tilde{s}) - x^{(1)}(\tilde{s}) \\ u^{(2)}(\cdot) \end{pmatrix},$$

and the operators $\tilde{G}_N\colon \tilde{Z}_{1,N} \times \tilde{Z}_{2,N} \to \tilde{W}_{2,N} \times \tilde{W}_{4,N}$ are defined in the same way. The discrete counterpart of Lemma 3.1 is

LEMMA 6.2 *If Assumption* (C3) *holds, then the operators G_N are uniformly surjective, i.e., for each $w \in W_{2,N}\times W_{4,N}$ there exists $z = (x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in Z_{1,N} \times Z_{2,N}$ such that $G_N(z) = w$ and*

$$\|z\| \le c\,\|w\|\,,$$

where the constant c is independent of N and z. Moreover, the operators $\tilde{G}_N$ are uniformly surjective, i.e., for each $w \in \tilde{W}_{2,N} \times \tilde{W}_{4,N}$ there exists $z = (x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in \tilde{Z}_{1,N} \times \tilde{Z}_{2,N}$ such that $\tilde{G}_N(z) = w$ and

$$\|z\|_2 \le c\,\|w\|_2\,,$$

where the constant c is independent of N and z.

Proof. Let $w := (w_4, w_5, w_9, w_{10}, w_{11}) \in W_2 \times W_4$ be arbitrary. We first have to find $z = (x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in \tilde{Z}_1 \times \tilde{Z}_2$ such that $G(z) = w$. The last equation of this system is $u^{(2)} = w_{11}$. We set $u^{(1)} = 0$. Then $\|u^{(1)}\| \le \|w\|$, $\|u^{(2)}\| \le \|w\|$, and we obtain an initial value problem having unique solutions $x^{(1)}$ and $x^{(2)}$ with

$$\|x^{(1)}\| \le c_1\,\|(w_4, w_5)\|\,, \quad \|x^{(2)}\| \le c_2\,\|(w_9, w_{10}, w_{11})\|\,,$$

where the constants c_1, c_2 are independent of N and z. The assertion for $\tilde{G}_N$ follows in the same way. ■

The strong second-order condition required for $(\mathrm{OQ})_{N,w}$ is

$(\widetilde{\mathrm{C5}})_N$ There exists $\bar{\gamma} > 0$ independent of N such that

$$B_N(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \ge \bar{\gamma}\left(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2\right)$$

for all $(x^{(1)}, u^{(1)}, x^{(2)}, u^{(2)}) \in \tilde{Z}_{N,1} \times \tilde{Z}_{N,2}$ satisfying

$$\begin{aligned} &\dot{x}^{(1)}(t) = \psi_x^{(1)}[t]x^{(1)}(t) + \psi_u^{(1)}[t]u^{(1)}(t) \quad \forall t \in [t_a, \sigma_1]\,, \\ &x^{(1)}(t_a) = 0\,, \\ &\dot{x}^{(2)}(t) = \psi_x^{(2)}[t]x^{(2)}(t) + \psi_u^{(2)}[t]u^{(2)}(t) \quad \forall t \in [\sigma_2, t_f]\,, \\ &x^{(2)}(\tilde{s}) = x^{(1)}(\tilde{s})\,, \\ &u^{(2)}(t) = 0 \quad \forall t \in [\sigma_2, t_f]\,, \end{aligned}$$

where the quadratic form B_N is defined by

$$\begin{aligned} B_N(x^{(1)},u^{(1)},x^{(2)},u^{(2)}) = & \frac{1}{2}\int_{t_a}^{\sigma_1} \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix}^T Q^{(1,N)}(t) \begin{pmatrix} x^{(1)}(t) \\ u^{(1)}(t) \end{pmatrix} dt \\ & + \frac{1}{2}\int_{\sigma_2}^{t_f} \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix}^T Q^{(2,N)}(t) \begin{pmatrix} x^{(2)}(t) \\ u^{(2)}(t) \end{pmatrix} dt \\ & + \frac{1}{2} x^{(2)}(t_f)^T \phi_{xx}(\tilde{x}^{(2)}(t_f)) x^{(2)}(t_f)\,. \end{aligned}$$

We show that this condition follows from $(\widetilde{\mathrm{C5}})$. Let $K := \mathrm{kern}\,(\tilde{G})$ with the operator $\tilde{G}$ of Lemma 3.1. Then $(\widetilde{\mathrm{C5}})$ can be stated in the form

$$\begin{aligned} & B(x^{(1)},u^{(1)},x^{(2)},u^{(2)}) \geq \gamma \left(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2 \right) \\ & \qquad \forall (x^{(1)},u^{(1)},x^{(2)},u^{(2)}) \in K\,. \end{aligned}$$

It follows from Assumption (C2) and (18) that for $i = 1, 2$

$$\|Q^{(i,N)}(t) - Q^{(i)}(t)\| \leq \tilde{c}_1 h^{k+1}\,,$$

where the constant $\tilde{c}_1$ is independent of N. This implies that for sufficiently large N

$$\begin{aligned} & B_N(x^{(1)},u^{(1)},x^{(2)},u^{(2)}) \geq \tfrac{\gamma}{2} \left(\|u^{(1)}\|_2^2 + \|u^{(2)}\|_2^2 \right) \\ & \qquad \forall (x^{(1)},u^{(1)},x^{(2)},u^{(2)}) \in K\,. \end{aligned}$$

Since $\mathrm{kern}\,(\tilde{G}) \subset K$, it follows that for sufficiently large N, $(\widetilde{\mathrm{C5}})_N$ is satisfied with $\bar{\gamma} = \gamma/2$.

We have shown that all Assumptions of Theorem 6.1 are satisfied for the discretization described in the last section. We therefore obtain the following convergence result:

THEOREM 6.2 *Let Assumptions* (C1)–(C4), $(\widetilde{\mathrm{C5}})_N$ *and* (CS) *be satisfied for some* $p \geq k+1$, *where* k *is the order of polynomials used to approximate the controls. Then for sufficiently large* N *the discretized system* $F_N(z,s) = 0$ *has a solution* (z_N^*, s_N^*) *with*

$$\|(z_N^*, s_N^*) - (\tilde{z}, \tilde{s})\| \leq c h^{k+1}\,,$$

where the constant c *is independent of* N.

For the solution of the discretized equations we can use the Newton method described in Section 4, where the controls are discretized according to Section 5.

Similar results can be obtained for more general control problems and more general discretizations. We refer to Felgenhauer (1998), where most of the results needed for the convergence analysis can be found.

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