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Shape optimization of thermoviscoelastic contact problems

by

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Abstract: This paper is concerned with a shape optimization problem of a viscoelastic body in unilateral dynamic contact with a rigid foundation. The contact with Coulomb friction is assumed to occur at a portion of the boundary of the body. The nonpenetration condition is described in terms of velocities. The thermal deformation is taken into account. Using the material derivative method as well as the results concerning the regularity of solutions to dynamic variational thermoviscoelastic problem the directional derivative of the cost functional is calculated. A necessary optimality condition is formulated.

Keywords: dynamic thermoviscoelastic contact problem, shape optimization, sensitivity analysis, necessary optimality condition.

1. Introduction

This paper deals with the formulation of a necessary optimality condition for a shape optimization problem of a viscoelastic body in unilateral dynamic contact with a rigid foundation. The contact with a given friction, described by Coulomb law (Duvaut, Lions, 1972), is assumed to occur at a portion of the boundary of the body. The contact condition is described in terms of velocities. This first order approximation seems to be physically realistic for the case of a small distance between the body and the obstacle, and for small time intervals. The friction coefficient is assumed to be bounded. Since the friction represents significant heat source, heat generation and flow are taken into account. The equlibrium state of this thermoviscoelastic contact problem is described by a coupled hyperbolic - parabolic system. This system consists of an hyperbolic hemivariational inequality of the second order, governing a displacement field, and a parabolic equation governing a heat flow. The existence of solutions to dynamic hemivariational inequalities is shown in Goeleven, Miettinen, Panagiotopoulos (1999), Han, Sofonea (2002), Jarušek, Eck (1999), Jarušek (1996).

The existence of solutions to the coupled thermoviscoelastic contact problems is shown in Jarušek, Eck (1999) using the Schouder fixed - point theorem.

The shape optimization problem for a viscoelastic body in contact consists in finding, in a contact region, such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. It is assumed, that the volume of the body is constant.

Shape optimization of static contact problems was considered, among others, in Haslinger, Neittaanmäki (1988), Klabring, Haslinger (1993), Myśliński (1991, 1992, 1994), Sokołowski, Zolésio (1992). In Haslinger, Neittaanmäki (1988), Klabring, Haslinger (1993) the existence of optimal solutions and convergence of the finite-dimensional approximation was shown. Moreover, it was shown that optimal shape of the bodies in contact implies almost constant normal contact stress. In Myśliński (1991, 1992, 1994), Sokołowski, Zolésio (1992) the necessary optimality conditions were formulated using the material derivative approach (see Zolésio, 1992). Numerical results are reported in Haslinger, Neittaanmäki (1988), Myśliński (1994). The necessary optimality conditions for shape optimization problems of dynamic contact problems were formulated in Jarušek, Krbec, Rao, Sokołowski (2002), Myśliński (2000). In Jarušek, Krbec, Rao, Sokołowski (2002) the conical differentiabilty of solutions to parabolic variational inequality was shown. The material derivative approach was used in Myśliński (2000) to formulate a necessary optimality condition for the shape optimization problem of viscoelastic bodies in unilateral contact with a given friction. The shape optimization for thermoelastic problems was considered in Myśliński, Tröltzsch (1999). In this paper a necessary optimality condition was formulated for thermoelastic problem described by a coupled elliptic - parabolic system with a nonlinear boundary condition. The existence of solutions to optimal design problems for systems described by parabolic or hyperbolic variational inequalities was shown in Denkowski, Migórski (1998), Gasiński (2000).

In the present paper we study this shape optimization problem for a viscoelastic body in unilateral dynamical contact with friction and heat flow. Using material derivative method, Sokołowski, Zolésio (1992), as well as the results of regularity of solutions to the dynamic variational inequality, Jarušek, Eck (1999), Jarušek (1996), we calculate the directional derivative of the cost functional and we formulate a necessary optimality condition for this problem.

We shall use the following notation : $\Omega \subset R^2$ will denote a bounded domain with Lipschitz continuous boundary Γ. The time variable will be denoted by t and the time interval $I = (0, \mathcal{T}), \mathcal{T} > 0$. By $H^k(\Omega)$, $k \in (0, \infty)$, we will denote the Sobolev space of functions having derivatives in all directions of the order k belonging to $L^2(\Omega)$, Adams (1975). For an interval I and a Banach space B, $L^p(I; B)$, $p \in (1, \infty)$, denotes the usual Bochner space, Duvaut, Lions (1972). $\dot{u} = \frac{du}{dt}$ and $\ddot{u} = \frac{d^2u}{dt^2}$ will denote first and second order derivatives of function u with respect to t, respectively. $\dot{u}_N$ and $\dot{u}_T$ will denote normal and tangential components of function $\dot{u}$, respectively. $Q = I \times \Omega$, $\gamma_i = I \times \Gamma_i$, $i = 1, 2, 3,$

where Γ_i are disjoint pieces of the boundary Γ.

2. Contact problem formulation

Consider deformations of an elastic body occupying domain $\Omega \subset R^2$. The boundary Γ of domain Ω is Lipschitz continuous. The body is subject to body forces $f = (f_1, f_2)$. Moreover, surface tractions $p = (p_1, p_2)$ are applied to a portion Γ_1 of the boundary Γ. We assume that the body is clamped along the portion Γ_0 of the boundary Γ. The contact conditions are prescribed on the portion Γ_2 of the boundary Γ. Moreover $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j, i, j = 0, 1, 2$, $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$.

We denote by $u = (u_1, u_2)$, $u = u(t, x)$, $t \in [0, \mathcal{T}]$, $0 < \mathcal{T} < \infty$, $x \in \Omega$, a displacement of the body and by $\theta = \theta(t, x)$ an absolute temperature of the strip. $\sigma = \{\sigma_{ij}(u(t, x))\}$, $i, j = 1, 2$, denotes the stress field in the body. We shall consider viscoelastic bodies obeying the Kelvin-Voigt law, Duvaut, Lions (1972), Haslinger, Neittaanmäki (1988), Hlavacek et al. (1986), Telega (1987):

$$\begin{aligned} &\sigma_{ij}(u(t,x)) = c^0_{ijkl}(x) e_{kl}(u) + c^1_{ijkl}(x) e_{kl}(\dot{u}) \ \ x \in \Omega, \\ &e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}), \end{aligned} \tag{1}$$

$i, j, k, l = 1, 2$, $u_{k,l} = \frac{\partial u_k}{\partial x_l}$. We use here the summation convention over repeated indices, Duvaut, Lions (1972). $c^0_{ijkl}(x)$ and $c^1_{ijkl}(x)$, $i, j, k, l = 1, 2$, are components of Hook's tensor. It is assumed that the elasticity coefficients c^0_{ijkl} and c^1_{ijkl} are Lipschitz continuous with respect to the space variable. Moreover, they satisfy usual symmetry, boundedness and ellipticity conditions, Duvaut, Lions (1972), Jarušek (1999), Haslinger, Neittaanmäki (1988), Halvacek et al. (1986), Jarušek, Eck (1999) i.e.,

$$c^\iota_{ijkl}(x) = c^\iota_{jikl}(x) = c^\iota_{klij}(x), \tag{2}$$

$$m^\iota_0 \xi_{ij}\xi_{kl} \leq c^\iota_{ijkl}(x)\xi_{ij}\xi_{kl} \leq M^\iota_0 \xi_{ij}\xi_{kl}, \tag{3}$$

for all symetric second order tensors ξ_{ij} and all $x \in \Omega$ with constants $0 < m^\iota_0 \leq M^\iota_0$, $\iota = 0, 1$.

In an equlibrium state a stress field σ and a temperature field θ satisfy the system, Duvaut, Lions (1972), Haslinger, Neittaanmäki (1988), Jarušek, Eck (1999), Jarušek (1996), of equations:

$$\ddot{u}_i - \sigma_{ij}(u(t,x))_{,j} = f_i(x) + (b_{ij}\theta)_{,j} \quad \text{in} \quad (0, \mathcal{T}) \times \Omega, \ \ i, j = 1, 2, \tag{4}$$

where $\sigma_{ij}(u(t,x))_{,j} = \frac{\partial \sigma_{ij}(u(t,x))}{\partial x_j}$, $i, j = 1, 2$. b_{ij}, $i, j = 1, 2$, denotes a thermal expansion tensor, symmetric, bounded, elliptic and Lipschitz continuous with respect to the space variable. The temperature flow is governed by:

$$\dot{\theta} - (c_{ij}\theta_{,j})_{,i} + b^1_{ij}\dot{u}_{i,j} = 0 \quad \text{in } (0, \mathcal{T}) \times \Omega, \tag{5}$$

where c_{ij}, $i,j=1,2$, is a tensor of thermal conductivity, Lipschitz continuous with respect to the space variable and satisfying usual symmetry, boundedness and ellipticity conditions. b^1_{ij}, $i,j=1,2$, denotes a thermal deformation tensor, symmetric, bounded, elliptic and Lipschitz continuous with respect to the space variable. The dependance of this tensor on current temperature is neglected. The following initial conditions are given

$$u_i(0,x)=u_0, \quad \text{and} \quad \dot{u}_i(0,x)=u_1, \quad i=1,2, \quad \text{in} \quad \Omega, \tag{6}$$

$$\theta(0,x)=\theta_0 \quad \text{in } \Omega. \tag{7}$$

u_0, u_1, θ_0 are given functions. The following boundary conditions are given

$$\begin{aligned} &u_i(x)=0 \quad \text{on } (0,\mathcal{T})\times\Gamma_0, \; i=1,2, \\ &\sigma_{ij}(u)n_j=p_i \text{ on } (0,\mathcal{T})\times\Gamma_1, \; i,j=2, \end{aligned} \tag{8}$$

$$\dot{u}_N\le 0, \; \sigma_N\le 0, \; \dot{u}_N\sigma_N=0, \; \text{on } (0,\mathcal{T})\times\Gamma_2, \tag{9}$$

$$\begin{aligned} &\dot{u}_T=0 \quad \Rightarrow \; |\, \sigma_T \,|\le \mathcal{F} \,|\, \sigma_N \,|, \\ &\dot{u}_T\ne 0 \quad \Rightarrow \quad \sigma_T=-\mathcal{F}\,|\,\sigma_N\,|\,\frac{\dot{u}_T}{|\,\dot{u}_T\,|} \text{ on } (0,\mathcal{T})\times\Gamma_2. \end{aligned} \tag{10}$$

Here we use the notation: $u_N=u_in_i$, $\sigma_N=\sigma_{ij}n_in_j$, $(u_T)_i=u_i-u_Nn_i$, $(\sigma_T)_i=\sigma_{ij}n_j-\sigma_Nn_i$ $i,j=1,2$, where $n=(n_1,n_2)$ is the unit outward versor to the boundary Γ. $\mathcal{F}$ denotes the friction coefficient. Moreover, the heat flux on the boundary of the strip Ω is equal to:

$$\frac{\partial\theta}{\partial n}(t,x)=\kappa(\theta_g-\theta), \quad \text{on } (0,\mathcal{T})\times(\Gamma_0\cup\Gamma_1), \tag{11}$$

$$\frac{\partial\theta}{\partial n}(t,x)=\kappa(\theta_g-\theta)+\mathcal{F}\,|\,\sigma_N\,||\,\dot{u}_T\,|, \quad \text{on } (0,\mathcal{T})\times\Gamma_2. \tag{12}$$

κ is a given nonnegative, bounded, measurable function of the space variable and θ_g is a given external temperature. Since for $u\to\infty$ the term describing the frictional heat generation may increase faster than all the other terms, it creates the main difficulty in the investigation of the coupled problem. To assure the existence of solutions to the problem (1)–(12), we replace the frictional heat generation term $\mathcal{F}\,|\,\sigma_N\,||\,\dot{u}_T\,|$ by a function $j(x,\mathcal{F}\,|\,\sigma_N\,|,|\,\dot{u}_T\,|)$. This function is assumed monotone in the second and third argument and satisfying the linear growth condition

$$j(x,\mathcal{F}\,|\,\sigma_N\,|,|\,\dot{u}_T\,|)\le const_1(1+\mathcal{F}\,|\,\sigma_N\,|+|\,\dot{u}_T\,|), \tag{13}$$

with some positive constant $const_1\in R$. Moreover, for $g_n\rightharpoonup g$ in $L^2(\Gamma_2)$ and $h_n\to h$ in $L^2(\Gamma_2)$ the function j satisfies for a.e. $x\in\Omega$

$$j(.,g_n,h_n)\rightharpoonup j(.,g,h) \text{ in } L^\alpha(\Gamma_2), \text{ for } \alpha>\frac{8}{5}. \tag{14}$$

We shall consider problem (4)–(12) in the variational form. Let us assume,

$$
\begin{aligned}
& f \in H^{1/4}(I;(H^1(\Omega;R^2)^*) \cap L^2(I;(H^{1/2}(\Omega;R^2))^*),\\
& p \in L^2(I;(H^{-1/2}(\Gamma_1;R^2)),\ \ u_0 \in H^{3/2}(\Omega;R^2),\ \ u_1 \in H^{3/2}(\Omega;R^2),\\
& \theta_0 \in L^2(\Omega;R),\ \ \theta_g \in L^2(I;H^{-1/2}(\Gamma;R)),\\
& \mathcal{F} \geq 0,\ \mathcal{F} \in L^\infty(\Gamma_2;R^2),\ \mathcal{F}(.,x) \text{ is continuous for a.e.} x \in \Gamma_2,
\end{aligned}
\tag{15}
$$

are given. The compatibility conditions $u_{0|\Gamma_0 \cup \Gamma_2} = 0,\ u_{1|\Gamma_0} = 0$ are assumed to be satisfied. The space $L^2(Q;R^2)$ and the Sobolev spaces $H^{1/4}(I;(H^1(\Omega;R^2)^*)$ as well as $(H^{1/2}(\Gamma_1);R^2)$ are defined in Adams (1975), Duvaut, Lions (1972). Let us introduce:

$$H = H^{1/2}(I;H^1(\Omega;R^2)),\ F = \{z \in H\ :\ z_i = 0 \text{ on } (0,\mathcal{T}) \times \Gamma_0, i = 1,2\}, \tag{16}$$

$$K = \{\in F\ :\ \dot{z}_N \leq 0 \text{ on } (0,\mathcal{T}) \times \Gamma_2\ \}. \tag{17}$$

Let us also introduce the bilinear forms: $a^i(.,.)\ :\ F \times F \to R$, $i = 0,1$, given by

$$
\begin{aligned}
a^0(u,v) &= \int_Q c^0_{ijkl} e_{ij}(u) e_{kl}(v) dxdt,\\
a^1(u,v) &= \int_Q c^1_{ijkl} e_{ij}(u) e_{kl}(v) dxdt.
\end{aligned}
\tag{18}
$$

The problem (1)–(12) is equivalent to the following variational problem, Jarušek, Eck (1999), Jarušek (1996): find a pair (u,θ), such that $u \in L^2(I;H^1(\Omega;R^2)) \cap H^{1/2}(I;H^1(\Omega;R^2)) \cap K$, $\dot{u} \in L^2(I;H^1(\Omega;R^2)) \cap H^{1/2}(I;L^2(\Omega;R^2)) \cap K$, and $\ddot{u} \in L^2(I;H^{-1}(\Omega;R^2)) \cap (H^{1/2}(I;L^2(\Omega;R^2)))^*$, $\theta \in H^{1/2,1}(Q)$, $\dot{\theta} \in (H^{1/2,1}(Q))^*$ satisfying the initial conditions (6)–(7) as well as the following system, Jarušek, Eck (1999), Jarušek (1996),

$$
\begin{aligned}
& \int_Q \ddot{u}(v - \dot{u}) dxdt + a^0(u, v - \dot{u}) + a^1(\dot{u}, v - \dot{u}) - \int_Q b_{ij} \theta e_{ij}(v - \dot{u}) dxdt + \\
& \quad \int_0^{\mathcal{T}} \int_{\Gamma_2} j(\mu \mid \sigma_N \mid, (\mid v_T \mid - \mid \dot{u}_T \mid))\, dsdt \geq \\
& \quad \int_Q f v dxdt + \int_{\gamma_2} p v dsdt \quad \forall v \in K,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
& \int_Q \dot{\theta} \varphi dxdt + \int_Q (c_{ij} \theta_{,j} \varphi_{,i} + b^1_{ij} \dot{u}_{i,j} \varphi) dxdt + \int_0^{\mathcal{T}} \int_\Gamma \kappa(\theta_g - \theta) \varphi dsdt = \\
& \int_0^{\mathcal{T}} \int_{\Gamma_2} j(\mu \mid \sigma_N \mid, \mid \dot{u}_T \mid) \varphi dsdt \quad \forall \varphi \in H.
\end{aligned}
\tag{20}
$$

The existence of solutions to system (19)–(20) was shown in Jarušek, Eck (1999).

THEOREM 2.1 *Assume : (i) the data are smooth enough, i.e. (13), (14), (15) are satisfied (ii)* Γ_2 *is of class* $C^{1,1}$ *(iii) the friction coefficient is small enough. Than there exists a solution to the problem (19)–(20).*

Proof. The proof is based on the fixed point theorem of Schouder and is given in Jarušek, Eck (1999). First, for a given temperature field $\bar{\theta}$, a unique solution u to the contact problem (19) is shown to exist. The proof consists in penalization of the inequality (19), friction regularization and employment of localization and shifting technique due to Lions and Magenes (see Jarušek, 1996). Next, for the found displacement $u(\bar{\theta})$, the existence of a unique temperature field $\theta(u(\bar{\theta}))$ satisfying the heat equation (20) is proved. This procedure defines an operator $\Phi : \bar{\theta} \to \theta$. Since the operator Φ maps an appriopiate convex bounded set into itself and is completely continuous it implies the existence of a fixed point of this operator as well as the existence of a solution to the thermoviscoelastic contact problem (19)–(20). ■

We confine ourselves to consideration of the contact problem with a prescribed friction, i.e.,

$$\mathcal{F} \mid \sigma_N \mid = \sigma_T \leq 1. \tag{21}$$

The condition (10) is replaced by the following one,

$$\dot{u}_T \sigma_T + \mid \dot{u}_T \mid = 0, \ \mid \sigma_T \mid \leq 1 \quad \text{on } I \times \Gamma_2. \tag{22}$$

For the sake of simplicity we set $c_{ij} = 1$, $i, j = 1, 2$, in (5). We shall consider the temperature flow governing equation (5) with $b_{ij}^1 = 0$, $i, j = 1, 2$. Moreover, we assume, that the boundary condition (12) does not contain the term depending on a friction coefficient, i.e., it has the form

$$\frac{\partial \theta}{\partial n}(t, x) = \kappa(\theta_g - \theta) + g(\theta, t, x), \quad \text{on } (0, \mathcal{T}) \times \Gamma_2. \tag{23}$$

Let $\hat{\Omega} \subset R^2$ be a domain such that $\Omega \subset \hat{\Omega}$. We assume that $g(., ., .) : R \times [0, \mathcal{T}] \times \hat{\Omega} \to R$ satisfies the following assumptions, Myśliński, Tröltzsch (1999):

$$\begin{array}{ll}
(A1) & \text{g(.,.,.) is continuously differentiable w.r. to } (\theta, t, x) \\
 & \mid g(0, t, x) \mid \leq \psi_0 \forall (t, x) \in [0, \mathcal{T}] \times \hat{\Omega} \\
 & \mid g_\theta(\theta, t, x) \mid + \mid g_t(\theta, t, x) \mid \leq \psi_M \forall (\theta, t, x \\
 & \in [-M, M] \times [0, \mathcal{T}] \times \hat{\Omega} \\
 & \text{here, } \psi_0 \text{ and } \psi_M \text{ are certain real constants.} \\
(A2) & g_\theta(\theta, t, x) \leq 0 \ \forall (\theta, x, t) \in R \times [0, \mathcal{T}] \times \hat{\Omega} \\
(A3) & \text{Compatibility conditions: } \theta_0(x) = \theta_0 \\
 & \text{ is constant on } \hat{\Omega} \text{ and } g(\theta_0(x), 0, x)) = 0.
\end{array} \tag{24}$$

$g_\theta(\theta, t, x)$ denotes the derivative of function g with respect to θ. By introducing a set

$$\Lambda = \{\lambda \in L^2(I; L^\infty(\Gamma_2; R^2)) \ : | \ \lambda \ |\leq 1 \ \text{ on } I \times \Gamma_2\}, \tag{25}$$

and taking into account (22), (23) we transform the system (20)–(21) into : find $u \in K$, $\lambda \in \Lambda$ and $\theta \in H$ such that,

$$\int_Q \ddot{u}(v-\dot{u})dxdt + a^0(u, v-\dot{u}) + a^1(\dot{u}, v-\dot{u}) - \int_Q b_{ij}\theta e_{ij}(v-\dot{u})dxdt - \\ \int_{\gamma_2} \lambda_T(v_T - \dot{u}_T)dxd\tau \geq \int_Q fvdxdt + \int_{\gamma_2} pvdxdt \quad \forall v \in K, \tag{26}$$

$$\int_{\gamma_2} \sigma_T \dot{u}_T dsd\tau \leq \int_{\gamma_2} \lambda_T \dot{u}_T dsd\tau \quad \forall \lambda_T \in \Lambda, \tag{27}$$

$$\int_Q \dot{\theta}\varphi dxdt + \int_Q \theta_{,j}\varphi_{,i} dxdt + \int_0^{\mathcal{T}} \int_{\Gamma_0 \cup \Gamma_1} \kappa(\theta_g - \theta)\varphi dsdt + \\ \int_0^{\mathcal{T}} \int_{\Gamma_2} (\kappa(\theta_g - \theta) + g(\theta, t, x))\varphi dsdt = 0 \quad \forall \varphi \in H. \tag{28}$$

3. Formulation of the shape optimization problem

We are going to consider a family $\{\Omega_s\}$ of the domains Ω_s depending on parameter s. For each Ω_s we formulate a variational problem corresponding to (26)–(28). In this way we obtain a family of the variational problems depending on s and for this family we shall study a shape optimization problem, i.e., we minimize with respect to s a cost functional associated with the solutions to (26)–(28).

We shall consider the domain Ω_s as an image of a reference domain Ω under a smooth mapping $\mathbf{T}_s$. To describe the transformation $\mathbf{T}_s$ we shall use the speed method Sokołowski, Zolésio (1992). Let us denote by $V(s,x)$ the sufficiently regular vector field depending on parameter $s \in [0, \vartheta), \vartheta > 0$:

$$V(.,.) \ : \ [0, \vartheta) \times R^2 \to R^2,$$

$$V(s,.) \in C^2(R^2, R^2) \ \ \forall s \in [0, \vartheta), \ \ V(., x) \in C([0, \vartheta), R^2) \ \ \forall x \in R^2. \tag{29}$$

Let $\mathbf{T}_s(V)$ denote the family of mappings: $\mathbf{T}_s(V) : R^2 \ni X \to x(t, X) \in R^2$ where the vector function x(.,X) = x(.) satisfies the system of ordinary differential equations:

$$\frac{d}{d\tau}x(\tau, X) = V(\tau, x(\tau, X)), \tau \in [0, \vartheta), \quad x(0, X) = X \in R. \tag{30}$$

We denote by $D\mathbf{T}_s$ the Jacobian of the mapping $\mathbf{T}_s(V)$ at a point $X \in R^2$. We denote by $D\mathbf{T}_s^{-1}$ and $^\star D\mathbf{T}_s^{-1}$ the inverse and the transpose inverse of the Jacobian $D\mathbf{T}_s$, respectively. $J_s = \det D\mathbf{T}_s$ will denote the determinant of the Jacobian $D\mathbf{T}_s$. The family of domains $\{\Omega_s\}$ depending on parameter $s \in [0, \vartheta), \vartheta > 0$, is defined as follows: $\Omega_0 = \Omega$,

$$\begin{aligned}\Omega_s = \mathbf{T}_s(\Omega)(V) = \{ & x \in R^2 \ : \ \exists X \in R^2 \\ & \text{such that, } x = x(s, X), \text{ where the function } x(., X) \text{ satisfies} \\ & \text{equation (30) for } 0 \leq \tau \leq s \ \}.\end{aligned} \tag{31}$$

Let us consider problem (26)–(28) in the domain Ω_s. Let F_s, K_s, Λ_s be defined, by (16), (17), (23), respectively, with Ω_s instead of Ω. We shall write $u_s = u(\Omega_s)$, $\sigma_s = \sigma(\Omega_s)$ and $\theta_s = \theta(\Omega_s)$. The problem (26)–(28) in the domain Ω_s takes the form: find $u_s \in K_s$, $\lambda_s \in \Lambda_s$ and $\theta_s \in H_s$ such that

$$\begin{aligned}\int_{Q_s} & \ddot{u}_s(v - \dot{u}_s)dxdt + a^0(u_s, v - \dot{u}_s) + a^1(\dot{u}_s, v - \dot{u}_s) - \\ & \int_{Q_s} b_{ij}\theta_s e_{ij}(v - \dot{u}_s)dxdt - \int_{\gamma_{s2}} \lambda_{Ts}(v_T - \dot{u}_{Ts})dxd\tau \ \geq \\ & \int_{Q_s} fvdxdt + \int_{\gamma_{s2}} pvdxdt \quad \forall v \in H^{1/2}(I; H^1(\Omega_s; R^2)) \cap K \ ,\end{aligned} \tag{32}$$

$$\int_{\gamma_{s2}} \sigma_{sT}\dot{u}_{sT}dsd\tau \leq \int_{\gamma_{s2}} \lambda_{sT}\dot{u}_{sT}dsd\tau \quad \forall \lambda_{sT} \in \Lambda_s, \tag{33}$$

$$\begin{aligned}\dot{\theta}_s\varphi dxdt + \int_{Qs} \theta_{s,j}\varphi_{,i}dxdt + \int_0^{\mathcal{T}} \int_{\Gamma_{s0}\cup\Gamma_{s1}} \kappa(\theta_g - \theta)dsdt + \\ \int_0^T \int_{\Gamma_{s2}} (\kappa(\theta_g - \theta_s) + g(\theta_s, t, x))\varphi dsdt = 0 \quad \forall \varphi \in H_s.\end{aligned} \tag{34}$$

Let us formulate the optimization problem. By $\hat{\Omega} \subset R^2$ we denote a domain such that $\Omega_s \subset \hat{\Omega}$ for all $s \in [0, \vartheta), \vartheta > 0$. Let $\phi \in M$ be a given function. The set M is determined by

$$M = \{\phi \in L^\infty(I; H_0^2(\hat{\Omega}; R^2)) : \phi \leq 0 \ \text{ on } I \times \hat{\Omega}, \| \ \phi \ \|_{L^\infty(I; H_0^2(\hat{\Omega}; R^2))} \leq 1\}. \tag{35}$$

Let us introduce, for given $\phi \in M$, the following cost functional:

$$J_\phi(u_s) = \int_{\gamma_{s2}} \sigma_{sN}\phi_{sN}dzd\tau, \tag{36}$$

where ϕ_{sN} and σ_{sN} are normal components of ϕ_s and σ_s, respectively, depending on parameter s. Note that the cost functional (36) approximates the normal contact stress, Haslinger, Neittaanmäki (1988), Klabring, Haslinger (1993), Myśliński (1994). We shall consider a such family of domains $\{\Omega_s\}$ that every Ω_s, $s \in [0, \vartheta)$, $\vartheta > 0$, has constant volume $c > 0$, i.e., every Ω_s belongs to the constraint set U given by

$$U = \{\Omega_s \; : \; \int_{\Omega_s} dx = c\}. \tag{37}$$

We shall consider the following shape optimization problem:

For given $\phi \in M$, find the boundary Γ_{2s} of the domain Ω_s occupied by the body, minimizing the cost functional (36) subject to $\Omega_s \in U$. (38)

The set U given by (37) is assumed to be nonempty. $(u_s, \lambda_s, \theta_s) \in K_s \times \Lambda_s \times H_s$ satisfy (32)–(34). Note that the goal of the shape optimization problem (38) is to find such boundary Γ_2 of the domain Ω occupied by the body that the normal contact stress is minimized. Remark, that the cost functional (36) can be written in the following form, Haslinger, Neittaanmäki (1988), Telega (1987):

$$\begin{aligned}\int_{\gamma_{s2}} \sigma_{sN}\phi_{sN} ds d\tau = \int_{Q_s} \ddot{u}_s \phi_s dx d\tau + \int_{Q_s} \sigma_{sij}(u_s) e_{kl}(\phi_s) dx d\tau \; - \\ \int_{Q_s} f\phi_s dx d\tau - \int_{Q_s} b_{ij}\theta_s e_{ij}(\phi_s) dx d\tau \; - \\ \int_{\gamma_{s1}} p_s \phi_s dz d\tau - \int_{\gamma_{s2}} \sigma_{sT}\phi_{sT} dz d\tau.\end{aligned} \tag{39}$$

We shall assume that there exists at least one solution to the optimization problem (38). It implies the compactness assumption for the set (37) in a suitable topology. For detailed discussion concerning the conditions assuring the existence of optimal solutions see Haslinger, Neittaanmäki (1988), Sokołowski, Zolésio (1992).

4. Shape derivatives of contact problem solution

In order to calculate the Euler derivative (see Sokołowski, Zolésio, 1992) of the cost functional (36) we have to determine the shape derivatives $(u', \lambda', \theta') \in F \times \Lambda \times H$ of a solution $(u_s, \lambda_s, \theta_s) \in K_s \times \Lambda_s \times H_s$ of the system (32)–(34). Let us recall from Sokołowski, Zolésio (1992):

DEFINITION 4.1 *The shape derivative $u' \in F$ of the function $u_s \in F_s$ is determined by:*

$$(\tilde{u_s})_{|\Omega} = u + su' + o(s), \tag{40}$$

where $\| o(s) \|_F /s \to 0$ *for* $s \to 0$, $u = u_0 \in F$, $\tilde{u}_s \in F(R^2)$ *is an extension of the function* $u_s \in F_s$ *into the space* $F(R^2)$. $F(R^2)$ *is defined by (10) with* R^2 *instead of* Ω.

In order to calculate the shape derivatives $(u', \lambda', \theta') \in F \times \Lambda \times H$ of a solution $(u_s, \lambda_s, \theta_s) \in K_s \times \Lambda_s \times H_s$ of the system (32)–(34), first we calculate the material derivatives $(u^\bullet, \lambda^\bullet, \theta^\bullet) \in F \times \Lambda \times H$ of the solution $(u_s, \lambda_s, \theta_s) \in K_s \times \Lambda_s \times H_s$ to the system (32)–(34). Let us recall the notion of material derivative, Sokołowski, Zolésio (1992):

DEFINITION 4.2 *The material derivative* $u^\bullet \in F$ *of the function* $u_s \in K_s$ *at a point* $X \in \Omega$ *is determined by:*

$$\lim_{s \to 0} \| [(u_s \circ \mathbf{T}_s) - u]/s - u^\bullet \|_F = 0, \tag{41}$$

where $u \in K$, $u_s \circ \mathbf{T}_s \in K$ *is an image of function* $u_s \in K_s$ *in the space* F *under the mapping* $\mathbf{T}_s$.

Taking into account Definition 4.2 we can calculate material derivatives of a solution to the system (32)–(34):

LEMMA 4.1 *The material derivatives* $(u^\bullet, \lambda^\bullet, \theta^\bullet) \in K_1 \times \Lambda \times H$ *of a solution* $(u_s, \lambda_s, \theta_s) \in K_s \times \Lambda_s \times H_s$ *to the system (32)–(34) are determined as a unique solution to the following system:*

$$\begin{aligned}
&\int_Q \{\ddot{u}^\bullet \eta + \ddot{u}\eta^\bullet + \ddot{u}\eta \operatorname{div} V(0) + \sigma_{ij}(u^\bullet) e_{kl}(\eta) + \sigma_{ij}(u) e_{kl}(\eta^\bullet) - \\
&\quad \sigma_{ij}(u)[{}^T DV(0) D\eta + DV(0)^T D\eta] - \sigma_{ij}(\eta)[{}^T DV(0) Du + DV(0)^T Du] - \\
&\quad f^\bullet \eta - f\eta^\bullet + (\sigma_{ij}(u) e_{kl}(\eta) - f\eta) \operatorname{div} V(0)\} dx d\tau - \\
&\int_Q [b_{ij}\theta^\bullet e_{ij}(\eta) + b_{ij}\theta e_{ij}(\eta^\bullet) - b_{ij}\theta[{}^T DV(0) D\eta + DV(0)^T D\eta] + \\
&\quad b_{ij}\theta e_{ij}(\eta) div V(0)] dx dt - \\
&\int_{\gamma_1} (p^\bullet \eta + p\eta^\bullet + p\eta D) dx d\tau - \int_{\gamma_2} \{\lambda^\bullet \eta_T + \lambda \eta_T^\bullet + \lambda \dot{\eta}_T D\} dx d\tau \geq 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \eta \in K_1,
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\int_{\gamma_2} [(\lambda^\bullet - \mu)\dot{u}_T + (\lambda - \mu^\bullet)\dot{u}_T + (\lambda - \mu)\dot{u}_T^\bullet + (\lambda - \mu)\dot{u}_T D] dx d\tau \geq 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \mu \in L_1,
\end{aligned} \tag{43}$$

$$\int_0^{\mathcal{T}} \int_\Omega \{[-\theta^\bullet \frac{\partial\varphi}{\partial t} - \theta \frac{\partial\varphi^\bullet}{\partial t} divV(0)] + [\nabla\theta^\bullet\nabla\varphi + \nabla\theta\nabla\varphi^\bullet] + [\mathrm{div}V(0)I - ({}^T DV(0) + DV(0))\nabla\theta\nabla\varphi]\}dxdt = \int_0^{\mathcal{T}} \int_\Gamma \{\frac{\partial g}{\partial\theta}\theta^\bullet\varphi + g\nabla\varphi V(0) + \nabla g\varphi V(0) + g\varphi(\mathrm{div}V(0) - (DVn,n))\}dzdt + \int_\Omega \theta_0\varphi(x,0)\mathrm{div}V(0)dx \quad \forall\varphi\in H, \tag{44}$$

where V(0) = V(0,X), DV(0) denotes the Jacobian matrix of the matrix V(0) and div *denotes a divergence operator. Moreover*

$$K_1 = \{\xi\in F \ : \ \xi = u - DVu \ \ on\ \gamma_0,\ \xi n \geq nDV(0)u \ \ on\ A_1,\ \xi n = nDV(0)u \ \ on\ A_2\ \}, \tag{45}$$

$$A_0 = \{x\in\gamma_2 \ : \ \dot{u}_N = 0\},\ \ A_1 = \{x\in B \ : \ \sigma_N = 0\},\quad A_2 = \{x\in B \ : \ \sigma_N < 0\}, \tag{46}$$

$$B_0 = \{x\in\gamma_2 \ : \ \lambda_T = 1,\ \dot{u}_T \neq 0\},\quad B_1 = \{x\in\gamma_2 \ : \ \lambda_T = -1,\ \dot{u}_T = 0\},\quad B_2 = \{x\in\gamma_2 \ : \ \lambda_T = 1,\ :\ \dot{u}_T = 0\}, \tag{47}$$

$$L_1 = \{\xi\in\Lambda \ : \ \xi\geq 0 \ on\ B_2,\ \xi\leq 0\ on\ B_1,\ \xi = 0\ on\ B_0\ \}, \tag{48}$$

and D is given by

$$D = div\, V(0) - (DV(0)n,n). \tag{49}$$

Proof. The proof is based on approach proposed in Sokołowski, Zolésio (1992). First we transport the system (32)–(34) to the fixed domain Ω. Let $u^s = u_s \circ \mathbf{T}_s \in F$, $u = u_0 \in F$, $\lambda^s = \lambda_s \circ \mathbf{T}_s \in \Lambda$, $\lambda = \lambda_0 \in \Lambda$. Since in general $u^s \notin K(\Omega)$, we introduce a new variable $z^s = D\mathbf{T}_s^{-1}u^s \in K$. Moreover, $\dot{z} = \dot{u} - DV(0)u$, Jarušek (1996), Sokołowski, Zolésio (1988). Using this new variable z^s as well as the formulae for transformation of the function and its gradient into reference domain Ω, Sokołowski, Zolésio (1988, 1992), we write the system (32)–(34) in the reference domain Ω. Using the estimates on time derivative of function u, Jarušek (1996), the Lipschitz continuity of u, λ and θ satisfying (32)–(34) with respect to s can be proved. Applying to this system the result concerning the differentiability of solutions to variational inequality, Sokołowski, Zolésio (1988, 1992), we can expect that the material derivative $(u^\bullet, \lambda^\bullet, \theta^\bullet) \in K_1 \times \Lambda \times H$ satisfies the system (42) - (44). Moreover, from the ellipticity condition of the elasticity coefficients by a standard argument, Sokołowski, Zolésio (1988), it follows that $(u^\bullet, \lambda^\bullet, \theta^\bullet) \in K_1 \times \Lambda \times H$ is a unique solution to the system (42)–(44). ■

Recall, Sokołowski, Zolésio (1992), that if the shape derivative $u' \in F$ of the function $u_s \in F_s$ exists, then the following condition holds:

$$u' = u^{\bullet} - \nabla u V(0). \tag{50}$$

From the regularity result in Jarušek (1996), Myśliński, Tröltzsch (1999), it follows that:

$$\bigtriangledown u V(0) \in F,\ \bigtriangledown \lambda_T V(0) \in \Lambda,\ \bigtriangledown \theta V(0) \in H, \tag{51}$$

where the spaces F, H and L are determined by (16) and (23), respectively. Integration by parts of the system (42) - (44), and consideration of (50), (51), lead to a similar system to (42) - (44), determining the shape derivative $(u', \lambda'_T, \theta') \in F \times L \times H$ of the solution $(u_s, \lambda_{sT}, \theta_s) \in K_s \times L_s \times H_s$ of the system (32)–(34):

$$\begin{aligned}
&\int_Q [\ddot{u}'\eta + \ddot{u}\eta' + (DV(0) +^T DV(0))\ddot{u}\eta]\, dxd\tau + \int_\gamma \ddot{u}\eta V(0) n ds d\tau + \\
&\quad \int_Q \sigma_{ij}(u') e_{kl} \eta\, dxdt - \int_Q [b_{ij}\theta' e_{ij}(\eta) + b_{ij}\nabla\theta V(0) e_{ij}(\eta')] dxdt + \\
&\quad \int_{\gamma_2} \{\lambda' \dot{\eta}_T + \lambda \dot{\eta}'_T\} dxd\tau + I_1(\dot{u}, \eta) + I_2(\lambda, u, \eta) \geq 0 \quad \forall \eta \in N_1,
\end{aligned} \tag{52}$$

$$\int_{\gamma_2} [\dot{u}'_T(\mu - \lambda) - \dot{u}_T \lambda'] dxd\tau + I_3(u, \mu - \lambda) \geq 0 \quad \forall \mu \in L_1, \tag{53}$$

$$\begin{aligned}
&\int_0^{\mathcal{T}} \int_\Omega \{-\theta' \frac{\partial \varphi}{\partial t} + \nabla\theta' \nabla\varphi\} dxdt + \\
&\qquad \int_0^{\mathcal{T}} \int_\Gamma \{\theta \frac{\partial \varphi}{\partial t} V(0) n + \nabla\theta\nabla\varphi V(0) n\} dsdt = 0,
\end{aligned} \tag{54}$$

$$N_1 = \{\eta \in F \ :\ \eta = \lambda - DuV(0), \quad \lambda \in K_1\}, \tag{55}$$

$$\begin{aligned}
I_1(\varphi, \phi) = &\int_\gamma \{\sigma_{ij}(\varphi) e_k l \phi - f\phi - \\
&((\nabla p n)\phi + (p\nabla\phi) n + p\phi H) V(0) n\} dxd\tau,
\end{aligned} \tag{56}$$

$$\begin{aligned}
I_2(\mu, \varphi, \phi) = &\int_{\gamma_2} \{(\nabla\mu) n \nabla\phi + \mu(\nabla(\nabla\varphi n))\varphi + \\
&\mu\nabla\dot{\varphi}_T H + \mu\nabla\varphi n\} V(0) n dxd\tau,
\end{aligned} \tag{57}$$

$$I_3(\varphi, \mu - \lambda) = \int_{\gamma_2} (\varphi n)(\mu - \lambda) + \varphi(\nabla \mu n) - \varphi(\nabla \lambda n) + \varphi(\mu - \lambda)\tilde{H}]V(0) n dx d\tau, \tag{58}$$

where $\tilde{H}$ denotes the mean curvature of the boundary Γ, Sokołowski, Zolésio (1992).

5. Necessary optimality condition

Our goal is to calculate the directional derivative of the cost functional (36) with respect to the parameter s. We will use this derivative to formulate necessary optimality condition for the optimization problem (38). First, let us recall from Sokołowski, Zolésio (1992) the notion of Euler derivative of the cost functional depending on domain Ω:

DEFINITION 5.1 *Euler derivative $dJ(\Omega; V)$ of the cost functional J at a point Ω in the direction of the vector field V is given by:*

$$dJ(\Omega; V) = \limsup_{s \to 0} [J(\Omega_s) - J(\Omega)]/s. \tag{59}$$

The form of the directional derivative $dJ_\phi(u; V)$ of the cost functional (36) is given in:

LEMMA 5.1 *The directional derivative $dJ_\phi(u; V)$ of the cost functional (36), for $\phi \in M$ given, at a point $u \in K$ in the direction of vector field V is determined by :*

$$dJ_\phi(u; V) = \int_Q [\ddot{u}'\eta + \ddot{u}\eta' + (DV(0) +^T DV(0))\ddot{u}\eta] dx d\tau + \int_\gamma \ddot{u}\eta V(0) n \, ds dt + \int_Q \sigma'_{ij} e_{kl}(\phi) dx dt + \int_\Gamma [(\sigma_{ij} e_{kl}(\phi) - f\phi) V(0) n] ds - \int_{\Gamma_1} (\triangledown p \phi V(0) + p \triangledown \phi V(0) + p\phi D) ds - \int_{\Gamma_2} \sigma'_T \phi_T ds + I_1(u, \phi) - I_2(\lambda, u, \phi), \tag{60}$$

where σ' is a shape derivative of the function σ_s with respect to s. This derivative is defined by (40). $\triangledown p$ is a gradient of function p with respect to x. Moreover $V(0) = V(0,X)$, ϕ_T and σ_T are tangent components of functions ϕ and σ, respectively, as well as D is given by (49).

Proof. Taking into account (36), (39), as well as the formula for transformation of the gradient of the function defined on domain Ω_s into the reference domain Ω, Sokołowski, Zolésio (1992), and using the mapping (29)–(30) we can express the cost functional (36) defined on domain Ω_s in the form of the functional $J_\phi(u^s)$ defined on domain Ω, determined by:

$$
\begin{aligned}
J_\phi(u^s) = & \int_Q \ddot{u}_s \phi \; detD\mathbf{T}_s dxd\tau + \\
& \int_Q [\sigma_{ij}(D\mathbf{T}_s u^s) \; e_{kl}(D\mathbf{T}_s \phi_t^s) - f^s \; \phi) \det D\mathbf{T}_s dxd\tau - \\
& \int_Q b_{ij}\theta^s e_{ij}(D\mathbf{T}_s\phi) dxd\tau - \\
& \int_{\gamma_1} p^s \phi \parallel \det D\mathbf{T}_s{}^\star D\mathbf{T}_s^{-1} n \parallel dsd\tau - \\
& \int_{\gamma_2} \lambda_{sT}\phi_T \parallel \det D\mathbf{T}_s{}^\star D\mathbf{T}_s^{-1} n \parallel dsd\tau,
\end{aligned}
\tag{61}
$$

where $u^s = u_s \circ \mathbf{T}_s \in F$, $u = u_0 \in F$ and $\lambda = \lambda_0 \in \Lambda$. By (59) we have:

$$
dJ_\phi(u; V) = \limsup_{t \to 0} [J_\phi(u^s) - J_\phi(u)]/s. \tag{62}
$$

Note that it follows by standard arguments, Haslinger, Neittaanmäki (1988), Myśliński, Tröltzsch (1999), Sokołowski, Zolésio (1992), that the triple $(u_s, \lambda_{sT}, \theta_s) \in K_s \times L_s \times H_s$, $s \in [0, \vartheta)$, $\vartheta > 0$, satisfying the system (32)–(34) is Lipschitz continuous with respect to the parameter s. Passing to the limit with $s \to 0$ in (62) and taking into account the formulae for derivatives of $D\mathbf{T}_s^{-1}$ and $\det D\mathbf{T}_s$ with respect to the parameter s, Sokołowski, Zolésio (1992), and (40) we obtain (60). ∎

In order to eliminate the shape derivatives u', λ', θ' from (60) we introduce an adjoint state $(r, q, p) \in K_2 \times L_2 \times L_3$ defined as follows:

$$
\int_Q \ddot{r}\zeta dxd\tau + \int_Q \sigma_{ij}(\zeta) e_{kl}(\phi + r) dxd\tau + \int_{\gamma_2} \dot{\zeta}_T (q - \lambda)\zeta dxd\tau = 0 \quad \forall \zeta \in K_2, \tag{63}
$$

with

$$
r(\mathcal{T}, x) = 0, \quad \dot{r}_(\mathcal{T}, x) = 0,
$$

$$
\int_{\gamma_2} (\dot{r}_T + \dot{\phi}_T - \dot{u}_T)\delta dxd\tau = 0, \;\; \forall \delta \in L_2, \tag{64}
$$

$$
\int_0^{\mathcal{T}} \int_\Omega -\frac{\partial p}{\partial t}\varphi dxdt + \int_0^{\mathcal{T}} \int_\Omega \nabla p \nabla \varphi dxdt - \int_0^{\mathcal{T}} \int_{\Gamma_2} \frac{\partial g}{\partial \theta} p\varphi dsdt - \tag{65}
$$

$$
\int_\Omega \nabla \varphi q dx + \int_\Gamma q\varphi n ds = 0 \quad \forall \varphi \in H, \tag{66}
$$

and $p(\mathcal{T}, x) = 0$. Moreover

$$K_2 = \{\zeta \in K_1 \ : \ \zeta n = 0 \text{ on } A_0\}, \tag{67}$$

$$L_2 = \{\delta \in \Lambda \ : \ \delta = 0 \ \text{ on } A_0 \cap B_0 \ \}. \tag{68}$$

Since $\phi \in M$ is a given element, then by the same arguments as used to show the existence of solution $(u, \lambda, \theta) \in K \times L \times H$ to the system (32)–(34) we can show the existence of the solution $(r, q, p) \in K_2 \times L_2 \times H$ to the system (63) - (65). From (60), (52), (53), (63), (64) and (65) we obtain

$$dJ_\phi(u; V) = I_1(u, \phi + r) + I_2(\lambda, u, \phi + r) + I_3(u, q - \lambda). \tag{69}$$

The necessary optimality condition has a standard form (see Haslinger, Neittaanmäki, 1988; Hang, Choi, Komkov, 1986; Hlavacek et al., 1986; Sokołowski, Zolésio, 1992):

THEOREM 5.1 *There exists a Lagrange multiplier $\mu \in R$ such that for all vector fields V determined by (29), (30) the following condition holds:*

$$dJ_\phi(u; V) + \mu \int_\Gamma V(0) n ds \geq 0, \tag{70}$$

where $dJ_\phi(u; V)$ is given by (69).

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