of length n. From this we obtain, in particular, that a Baer lattice is upper semimodular.

(2) Every interval and every principal dual ideal of a Baer lattice are likewise Baer lattices.

(3) In a Baer lattice the following exchange property holds: if u, v are join-irreducible elements, b an arbitrary element and $v \leqslant b \vee u$ but $v \nleqslant b \vee u'$ (u' denotes the uniquely determined lower neighbor of u), then $u \leqslant b \vee v$. In the special case of AC-lattices we get from this the Steinitz–MacLane exchange property in its lattice-theoretic form.

(4) In a Baer lattice the Theorem of Kurosh–Ore holds: if an element b can be represented as a join of finitely many join-irreducible elements, then two minimal representations of b as a join of join-irreducible elements have the same number of components.

(5) Calling the number of components in a minimal representation of an element b as a join of join-irreducible elements the rank of b we can show: For a Baer lattice L the subset $F(L)$ of all elements of finite rank is an ideal. Using this notion of rank, a simple necessary and sufficient condition can be given for $F(L)$ to be a standard ideal in the sense of [2]. For the special case of finite-modular AC-lattices we obtain as a corollary that $F(L)$ is always standard (see [3]).

The above-mentioned results suggest that it might be possible to prove many results of [5] on AC-lattices also for Baer lattices.

References

[1] R. Baer, *A unified theory of projective spaces and finite abelian groups*, Trans. Amer. Math. Soc. 52 (1942), 283–343.

[2] G. Grätzer and E. T. Schmidt, *Standard ideals in lattices*, Acta Math. Acad. Sci. Hung. 12 (1961), 17–86.

[3] M. F. Janowitz, *On the modular relation in atomistic lattices*, Fund. Math. 66 (1970), 337–346.

[4] B. Jónsson and G. S. Monk, *Representations of primary arguesian lattices*, Pacific J. Math. 30 (1969), 95–139.

[5] F. Maeda and S. Maeda, *Theory of symmetric lattices*, Springer-Verlag, Berlin–Heidelberg–New York 1970.

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DIRECT LIMITS AND FILTERED COLIMITS ARE STRONGLY EQUIVALENT IN ALL CATEGORIES

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Herrlich–Strecker ([6], p. 151, Def. 22.1), defines "direct limits" as special cases of "filtered colimits" in the sense of MacLane ([8], p. 207). Here we show that the two concepts are strongly equivalent: i.e. every filtered diagram (of any category) can be transformed into a directed diagram, in a rather natural and constructive way, so that the same objects and the same arrows are used and not only the colimit objects but also the colimiting cocones of the two diagrams coincide (if any of them exists). This implies that the images (1) of the two diagrams coincide. In other words, the two diagrams will be "cofinal" (or, more categorically, final).

We use the word "diagram" as a synonim for "functor". We shall refer to the monographs by Herrlich–Strecker [6] and by MacLane [8] as "Herrlich–Strecker" and "MacLane".

DEFINITION 1 (Herrlich–Strecker, Def. 22.1). A *directed partial order* is a pair $(R, \leqslant)$ where R is a class such that any finite subset of R has an upper bound in $(R, \leqslant)$. (Note that this implies that R is nonempty!) Partial orders are considered to be categories.

A *directed diagram* is a functor $(R, \leqslant) \xrightarrow{D} \mathscr{C}$ from a directed partial order into a category $\mathscr{C}$.

A *direct limit* is a colimit of a directed diagram.

DEFINITION 2 (MacLane, p. 207)., A category I is *filtered* if any finite diagram $\mathscr{V} \xrightarrow{V} I$ (i.e. any functor $V\colon \mathscr{V} \to I$ such that $\mathscr{V}$ is finite) has an upper bound in I. (By an *upper bound* of V we understand a cocone $(f_i)_{i \in \mathrm{Ob}\,\mathscr{V}}$ compatible with V, i.e. "commuting over V"). (This implies that I is nonempty since $\emptyset \xrightarrow{\emptyset} I$ (cf. MacLane, p. 229) is a finite diagram.)

(1) *Image*: cf. Mac Lane, p. 243, Ex. 4. (The image of a functor need not be a category but only a *partial category*.)

A *filtered diagram* is a functor $I \xrightarrow{D} \mathscr{C}$ from a filtered category I.

A *filtered colimit* is a colimit of a filtered diagram.

Notation. The symbol $(f_i)_{i \in A}$ denotes a mapping with domain A correlating with each $i \in A$ the value f_i (Herrlich–Strecker, p. 10, "family").

The colimit of a diagram $I \xrightarrow{D} \mathscr{C}$ is denoted by $\mathrm{Colim}\,D$. That is, $\mathrm{Colim}\,D$ is a cocone $(f_i)_{i \in \mathrm{Ob}\,I}$ of arrows of $\mathscr{C}$. Therefore $\mathrm{Colim}\,D\colon \mathrm{Ob}\,I \to \mathrm{Mor}\,\mathscr{C}$ is a mapping; and, given another functor $K \xrightarrow{T} I$, the composite mapping $T \circ \mathrm{Colim}\,D\colon \mathrm{Ob}\,K \to \mathrm{Mor}\,\mathscr{C}$ is a new cocone (actually commuting over the composite diagram $T \circ D$, cf. MacLane, p. 43).

Composition is written in the order:

$$\begin{tikzcd}
& \cdot \arrow[dr, "g"] & \\
\cdot \arrow[ur, "f"] \arrow[rr, "fg"] & & \cdot
\end{tikzcd}$$

Remark. It is true that to every category I there is a fairly obvious final (2) functor $R \xrightarrow{T} I$ whose domain R is a partial order, but this construction does not preserve important properties of I; e.g. filteredness is almost always lost: there is a filtered category I such that the corresponding R is not filtered (not directed). (Moreover, to every regular cardinal α there is a filtered category I such that R is not even α-filtered (α-directed).)

In contrast, the construction of the present paper is intended to be "natural"; at least it preserves filteredness.

THEOREM 1. *To every filtered category I there is a directed partial order $(R, \leqslant)$ together with a final* (3) *functor $(R, \leqslant) \xrightarrow{T} I$.*

By finality, for any diagram (functor) $F\colon I \to \mathscr{C}$ the colimits of F and $T \circ F$ coincide. More precisely, T is such that the properties (i)–(v) *below hold for any $F\colon I \to \mathscr{C}$:*

(i) *F has a colimit iff $T \circ F$ has one.*

(ii) *The colimit objects coincide.*

(iii) $\mathrm{Colim}(T \circ F) = T \circ \mathrm{Colim}\,F$, *i.e., if the colimiting cocone of F is* $\mathrm{Colim}\,F = (f_i)_{i \in \mathrm{Ob}\,I}$, *then*

$$\mathrm{Colim}(T \circ F) = (f_{T(r)})_{r \in R}$$

(which means that the colimiting cocones coincide via T).

(2) "Final": cf. MacLane, p. 213; "Final" = "Kofinal": cf. Gabriel–Ulmer [5], Def. 2.12.

(3) Cf. MacLane, p. 213.

(iv) *For any choice function $T^{-1}\colon \mathrm{Ob}\,I \to R$ such that $T^{-1} \circ T = \mathrm{Id}_{\mathrm{Ob}\,I}$ we have:*

$$\mathrm{Colim}\,F = T^{-1} \circ \mathrm{Colim}(T \circ F).$$

(v) *$(R, \leqslant)$, T and a T^{-1} can be defined constructively without using the axiom of choice.*

Proof. The idea of the proof is the following: Let I be an arbitrary filtered category. Why not take all finite subcategories of I together with the "subcategory of" relation? Will this be directed? Now, the union of two finite subcategories might generate an infinite subcategory. To avoid this, we use finite diagrams of I instead of finite subcategories.

Let R consist of all finite diagrams over I. Let the ordering $\leqslant$ be the "subdiagram of" relation. Such an $(R, \leqslant)$ is a "typical example" of directed preorders. Since I is filtered, all these diagrams have upper bounds (in I). Therefore, let the functor $R \xrightarrow{T} I$ correlate with each diagram V an upper bound of V.

Now, how are we to prove that T is a functor? There is an easy way out: Let R consist only of those finite diagrams which have a colimit. Let T correlate with a diagram its colimit. (This modification does no harm since each finite diagram having an upper bound can be extended to a greater one having a colimit. Therefore the new R is "cofinal" in the old one and remains directed.) T can be defined on the morphisms in this spirit: If V is a subdiagram of V_1 then there is a unique arrow from the colimit of V to the colimit of V_1. Let this be the image of the arrow $V \to V_1$ of R. T is easily checked to be a functor, surjective, final, etc.

The following proof consists of nothing but a detailed and precise execution of the above plan. There is only one problem which forces us to make some rather careful definitions, namely: If we understand "subdiagram of" strictly, then $(R, \leqslant)$ is not directed. If "subdiagram of" is meant up to isomorphisms, then $(R, \leqslant)$ is directed, but the uniqueness of the arrow from "a smaller colim to a greater colim" is lost. There is again a way out: restricting ourselves to the so called "coequalised" diagrams (cf. Def. 4). (Roughly: a diagram is *coequalised* if its colimiting cocone contains no parallel arrows. This restores uniqueness of induced maps.)

The following remark (together with Definitions 3–4) is a concise version of the proof, relying heavily on constructions from MacLane. The detailed proof beginning with Definition 3 can be understood without reading it.

Remark. The functor $\mathrm{Colim}\colon (\mathrm{Fin\,Cat} \downarrow I) \to I$ is only partial (cf. MacLane, p. 111, Ex. 5a). Let Ter be the full subcategory of Cat consisting of finite categories with terminal objects. Now, Colim: $(\mathrm{Ter}\downarrow I)\to I$ is a (total) functor. But $(\mathrm{Ter}\downarrow I)$ is not a preorder. Let Monter be the subcategory of Ter consisting of all the isomorphisms of Ter together with all monomorphisms $\mathscr{V}\rightarrowtail\mathscr{V}_1$ of Ter whose images do not contain the terminal object of $\mathscr{V}_1$, and no other morphisms. Now, Colim: $(\mathrm{Monter}\downarrow I)\to I$ is still a functor and $(\mathrm{Monter}\downarrow I)$ is "almost" a preorder. Now, if Coe is the full subcategory of $(\mathrm{Monter}\downarrow I)$ consisting of its coequalised (Def. 4) objects, then Colim: $\mathrm{Coe}\to I$ is the required final functor in the following sense: The functor Colim factors through (admits) the congruence $\sim$ which makes Coe into a preorder: Colim$/\sim$: $\mathrm{Coe}/\sim\;\to I$ is a *functor*. f I is filtered, then this functor is final and $\mathrm{Coe}/\sim$ is directed.

DEFINITION 3. Let $\mathscr{V}$ be a category. The *terminal reflection* $\hat{\mathscr{V}}$ of $\mathscr{V}$ is obtained from $\mathscr{V}$ by adding to $\mathscr{V}$ a new formal terminal object v. I.e.

$$\mathrm{Ob}\,\hat{\mathscr{V}}\stackrel{\mathrm{d}}{=}\mathrm{Ob}\,\mathscr{V}\cup\{v\}\quad\text{where}\quad v\notin\mathrm{Ob}\,\mathscr{V}$$

and

$$\mathrm{Mor}_{\hat{\mathscr{V}}}(i,v)\stackrel{\mathrm{d}}{=}\{h_i\}\quad\text{for every } i\in\mathrm{Ob}\,\hat{\mathscr{V}},$$

$$\mathrm{Mor}_{\hat{\mathscr{V}}}(v,i)\stackrel{\mathrm{d}}{=}\emptyset\quad\text{for every } i\in\mathrm{Ob}\,\mathscr{V},$$

and

$$\mathscr{V}\text{ is a full subcategory of }\hat{\mathscr{V}}.$$

(Note that $\mathscr{V}$ is nothing but the *reflection* of the object $\mathscr{V}$ of Cat in the (non-full) subcategory of Cat consisting of the categories with terminal objects and *terminal object preserving functors*, cf. Herrlich–Strecker, p. 179; see Fig. 1.)

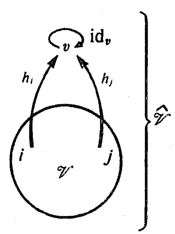

Fig. 1

Conventions. The new (terminal) object in $\hat{\mathscr{V}}$ is denoted by v. The new arrows are denoted by $i\xrightarrow{h_i}v$ (if $i\in\mathrm{Ob}\,\mathscr{V}$).

DEFINITION 4. A functor $\hat{\mathscr{V}}\xrightarrow{V}I$ is *coequalised* iff

$$(\forall i,j\in\mathrm{Ob}\,\mathscr{V})\quad[V(i)=V(j)\Rightarrow V(h_i)=V(h_j)].$$

(I.e. coequalised functors preserve the property of the set $\{h_i\}_{i\in\mathrm{Ob}\mathscr{V}}$ of containing no parallel pairs of arrows.)

Conventions. The restriction of a functor $\mathscr{A}\xrightarrow{F}\mathscr{B}$ to a subcategory $\mathscr{C}\subseteq\mathscr{A}$ is denoted by $F\restriction\mathscr{C}$. (That is, denoting the inclusion functor by $\mathscr{C}\xrightarrow{\mathrm{in}}\mathscr{A}$, the restriction of F is $F\restriction\mathscr{C}\stackrel{\mathrm{d}}{=}\mathrm{in}\circ F$.)

The letters $\mathscr{V}$, V, V' will be used consistently to denote things related to each other in the following way:

$$\hat{\mathscr{V}}\xrightarrow{V}I,\qquad V'=V\restriction\mathscr{V}.$$

Sometimes indices will be used, e.g.

$$\mathscr{V}_1,\ V_1,\ V_1',\ v_1\quad\text{and}\quad h_i^1\ (i\in\mathrm{Ob}\,\mathscr{V}_1).$$

LEMMA. *Let I be a filtered category and let $\mathscr{V}$ be a finite category together with a functor $\mathscr{V}\xrightarrow{V'}I$. Now, there exists a coequalised extension $V\colon\hat{\mathscr{V}}\to I$ of V', i.e.*

$$(\exists V)\quad[V\restriction\mathscr{V}=V'\ \textit{and}\ V\colon\hat{\mathscr{V}}\to I\ \textit{is coequalised}].$$

Proof. V' has an upper bound since it is a finite diagram of a filtered category I. Denote this upper bound by $(V'(i)\xrightarrow{k_i}c)_{i\in\mathrm{Ob}\mathscr{V}}$.

To rule out parallel pairs of arrows from the set $\{k_i\}_{i\in\mathrm{Ob}\mathscr{V}}$ we would like to take an upper bound of it considered as a diagram. But this set of arrows is not necessarily a subcategory of I; moreover, it may generate an infinite subcategory. All the same, it can be considered as a diagram by an appropriate choice of the index category $\mathscr{K}$. (The parallel arrows of $\{k_i\}_{i\in\mathrm{Ob}\mathscr{V}}$ should also be parallel in $\mathscr{K}$. Consider the graph $G=(\{k_i\}_{i\in\mathrm{Ob}\mathscr{V}},\mathrm{do}_I,\mathrm{cod}_I)$ as an object of the category of all graphs. The coreflection of this graph G into the full subcategory consisting of those graphs which are categories (contain no composable edges) is a graph-homomorphism $\mathscr{K}\xrightarrow{K}G$ which is also a functor $\mathscr{K}\xrightarrow{K}I$. This K is the functor we are now going to construct.).

Define a new category $\mathscr{K}$ together with a functor $\mathscr{K}\xrightarrow{K}I$ as follows:

$$\mathrm{Ob}\,\mathscr{K}\stackrel{\mathrm{d}}{=}\{V'(i)\colon i\in\mathrm{Ob}\,\mathscr{V}\}\cup\{w\}$$

where w is a completely new object ($w\notin\mathrm{Ob}\,I$). For every $i\in\mathrm{Ob}\,\mathscr{V}$:

$$\mathrm{Mor}_{\mathscr{K}}(V'(i),w)\stackrel{\mathrm{d}}{=}\{k_j:\ V'(j)=V'(i),\ j\in\mathrm{Ob}\,\mathscr{V}\},$$

and $\mathscr{K}$ contains no other morphisms except the identities. Now, $K(V'(i))\stackrel{\mathrm{d}}{=}V'(i)$, $K(k_i)\stackrel{\mathrm{d}}{=}k_i$ for every $i\in\mathrm{Ob}\,\mathscr{V}$, and $K(w)\stackrel{\mathrm{d}}{=}c$.

K is a functor. $\mathscr{K}$ is finite since $\mathscr{V}$ is such and therefore K has an upper bound (since I is filtered). Denote the upper bound of K by $\langle K(\varkappa) \xrightarrow{g_\varkappa} c'\rangle_{\varkappa\in \mathrm{Ob}\,\mathscr{K}}$. See Figure 2. Now define an extension $V:\hat{\mathscr{V}}\to I$ of V' as follows:

$$V\restriction\mathscr{V} = V' \quad\text{and}\quad V(v)\overset{\mathrm{d}}{=} c',$$

$$V(h_i)\overset{\mathrm{d}}{=} g_{V(i)}\quad \text{for every } i\in\mathrm{Ob}\,\mathscr{V}.$$

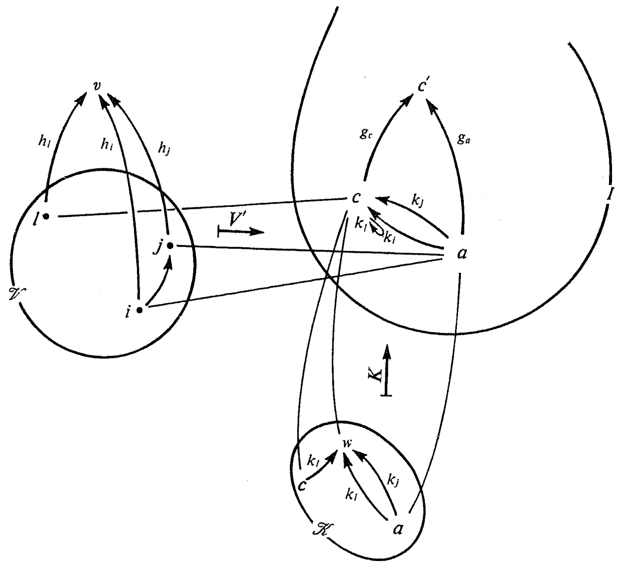

Fig. 2

V is easily seen to be a functor. It is obviously coequalised as the following argument shows: Let $i, j\in\mathrm{Ob}\,\mathscr{V}$. Now, since $V(h_i)$ was defined to be $g_{V(i)}$ if $V(i)=V(j)$, then also $V(h_i)=V(h_j)$. ■

We now turn to defining the directed partial order $(R, \leqslant)$. From now on I is a *fixed filtered category*. First we define a directed preorder $(P, \preceq)$. P consists of the finite coequalised diagrams of I:

$$P\overset{\mathrm{d}}{=}\{\hat{\mathscr{V}}\xrightarrow{V} I:\ \mathscr{V} \text{ is a finite category and } V \text{ is coequalised}\}.$$

Recall the conventions that the letters $\mathscr{V}$, V, V' and v belong to each other. (This means, e.g., that V always denotes a functor whose

domain is a "terminal reflection" of some category which is always denoted by $\mathscr{V}$. Note that $\mathscr{V}$ and v are uniquely determined by V.)

Two elements $V, V_1\in P$ of P are said to be *isomorphic* (in notation $V\cong V_1$) if they are isomorphic objects of the comma category $(\mathrm{Cat}\downarrow I)$. I.e., V, V_1 are isomorphic if there is an isomorphism-functor i such that $V=i\circ V_1$.

The preorder $\preceq$ on P is defined as:

$V\preceq V_1$ iff in the comma category $(\mathrm{Cat}\downarrow I)$ either V is a subobject of V_1' or $V\cong V_1$, i.e.

$$V\preceq V_1 \quad\text{iff}\quad (V\prec V_1 \text{ or } V\cong V_1),$$

where $V\prec V_1$ iff there is a commutative

$$\begin{tikzcd} & \mathscr{V}_1 \arrow[dr, "V_1'"] & \\ \hat{\mathscr{V}} \arrow[ur, tail, "m"] \arrow[rr, "V"] & & I \end{tikzcd}.$$

(Note that in order that $V\prec V_1$ the image of $\hat{\mathscr{V}}$ along the *mono*-functor m should *not* contain v_1. Therefore $\prec$ is antireflexive and antisymmetric.)

Recall that a functor is *mono* in Cat (as well as in $(\mathrm{Cat}\downarrow I)$) iff it is one-one on the objects and on the morphisms.

It is easy to check that $(P, \preceq)$ is a *preorder* indeed.

Since $\cong$ is an equivalence of P, we can form the *factorstructure* (or *quotientstructure*):

$$(R, \leqslant)\overset{\mathrm{d}}{=}(P, \preceq)/\cong.$$

The equivalence-class of an element $V\in P$ is denoted by $[V]$. $((\forall V\in P)\ [V]\in R.)$ Obviously $[V]\leqslant[V_1]$ iff $V\preceq V_1$.

Clearly $(R, \leqslant)$ is a *partial order*.

We now show that it is also *directed*:

R is nonempty (since by the lemma the empty functor $\emptyset:\emptyset\to I$ has an extension in P). Let $[V], [V_1]\in R$ be arbitrary. We construct an upper bound for them (in R). Consider the coproduct $(\hat{\mathscr{V}}\amalg\hat{\mathscr{V}}_1)\xrightarrow{V\amalg V_1} I$. (This is the disjoint union of the categories $\hat{\mathscr{V}}$ and $\hat{\mathscr{V}}_1$ together with the original functors.) Since $\hat{V} \amalg \hat{V}_1$ is finite, we can apply the lemma to obtain a coequalised functor

$$\widehat{(\hat{\mathscr{V}} \amalg \hat{\mathscr{V}}_1)} \xrightarrow{V_3} I$$

such that

$$V_3 \restriction \hat{\mathscr{V}} = V \quad \text{and} \quad V_3 \restriction \hat{\mathscr{V}}_1 = V_1.$$

Now, it is easy to check that $[V_3]$ is an upper bound for $[V_1]$ and $[V]$. So far we have proved that $(R, \leqslant)$ is a *directed partial order*.

We now define a *functor* $(R, \leqslant) \xrightarrow{T} I$, the existence of which was claimed in the theorem.

The peculiarities of the definition of $(R, \leqslant)$ (to consist of *terminal reflections* $\hat{\mathscr{V}}$, coequalised functors, and $[V \prec V_1 \Rightarrow V$ can be embedded *not only* into V_1 but also into $V_1']$) have not been used so far. They will be needed in the construction of T.

Recall that $i \xrightarrow{h_i} v$ denotes the unique arrow of $\hat{\mathscr{V}}$ from the object i to v.

DEFINITION. The *object part* of T:

$$T([V]) \stackrel{\mathrm{d}}{=} V(v) \quad \text{for any } V \in P.$$

The *morphism part* of T: for any $V_1, V \in P$ if $V_1 \prec V$ then

$$T([V_1], [V]) \stackrel{\mathrm{d}}{=} V(h_{m(v_1)}),$$

where $m\colon \hat{\mathscr{V}}_1 \rightarrowtail \mathscr{V}$ is any embedding for which $V_1 = m \circ V'$.

First we have to check that this was a definition.

Note that for any isomorphism $\mathscr{V} \xrightarrow{i} \mathscr{V}_1$ we have

$$i(v) = v_1 \quad \text{and} \quad i(h_a) = h^1_{i(a)} \quad \text{for any } a \in \operatorname{Ob} \mathscr{V}.$$

(a) T is a function on R: Let $[V] = [V_1]$, i.e., $V = i \circ V_1$ for some isomorphism i. Now,

$$T([V]) \stackrel{\mathrm{d}}{=} V(v) = V_1(i(v)) = V_1(v_1) \stackrel{\mathrm{d}}{=} T([V_1]).$$

(b) T is a function on "$\leqslant$" $(= \operatorname{Mor}(R, \leqslant))$: Let $W_1 \cong W \prec V \cong V_1$ be arbitrary elements of P. By definition of $\prec$ there are two monofunctors

$$m\colon \hat{\mathscr{W}} \rightarrowtail \mathscr{V}; \qquad W = m \circ V',$$

$$m_1\colon \hat{\mathscr{W}}_1 \rightarrowtail \mathscr{V}_1; \qquad W_1 = m_1 \circ V_1'.$$

Let m and m_1 be two arbitrary such functors.

We have to show that $V(h_{m(w)}) = V_1(h^1_{m_1(w_1)})$ in order to prove that $T([W], [V])$ is uniquely defined. See Figure 3.

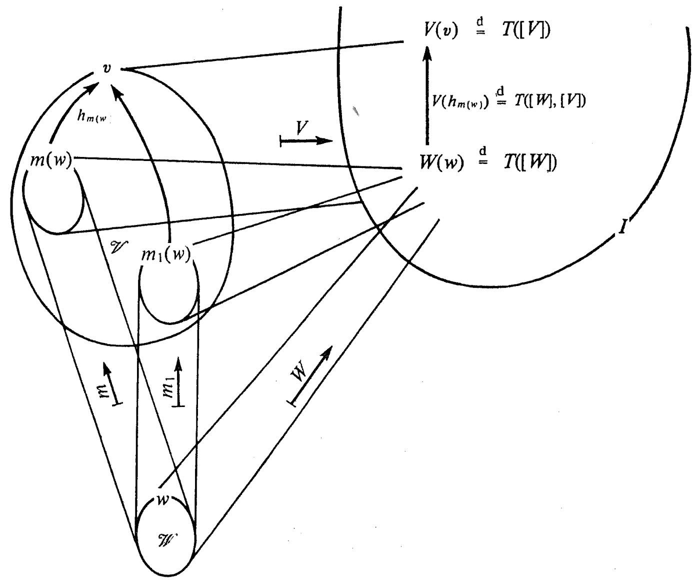

Fig. 3

Let $i\colon \hat{\mathscr{V}} \xrightarrow{\cong} \hat{\mathscr{V}}_1$ be an isomorphism such that $V = i \circ V_1$. (It exists.)

$$V(h_{m(w)}) = V_1(i(h_{m(w)})) = V_1(h^1_{i(m(w))}).$$

Since V_1 is coequalised, to prove $V_1(h^1_{i(m(w))}) = V_1(h^1_{m_1(w_1)})$ it is enough to show that $v_1 \notin \{i(m(w)), m_1(w_1)\}$ and that $V_1(i(m(w))) = V_1(m_1(w_1))$.

Now, $m_1(w_1) \neq v_1$ by definition of m_1. Similarly $m(w) \neq v$, which implies $i(m(w)) \neq v_1$ since i is an isomorphism. (Clearly: $i^{-1}(v_1) = v \neq m(w)$.) Also:

$$V_1(i(m(w))) \underset{V = i \circ V_1}{=} V(m(w)) \underset{\substack{W = m \circ V' \\ m(w) \neq V}}{=} W(w) \underset{W \cong W_1}{=} W_1(w_1) \underset{W_1 = m_1 \circ V_1'}{=} V_1(m_1(w_1)).$$

By this T is indeed a function.

Next we show that $(R, \leqslant) \xrightarrow{T} I$ is a *functor*:

We have to show that

$$T([V], [V_1]) \cdot T([V_1], [V_2]) = T([V], [V_2]),$$

or arbitrary elements $V \prec V_1 \prec V_2$ of P.

Let $m\colon \hat{\mathscr{V}} \rightarrowtail \mathscr{V}_1$ and $m_1\colon \hat{\mathscr{V}}_1 \rightarrowtail \mathscr{V}_2$ be such that

$$V = m \circ V_1' \quad \text{and} \quad V_1 = m_1 \circ V_2'.$$

Clearly $(m \circ m_1)\colon \hat{\mathscr{V}} \rightarrowtail \mathscr{V}_2$ is such that $V = (m \circ m_1) \circ V_2'$.

Therefore

$$T([V],[V_1]) = V_1(h^1_{m(v)}),$$

$$T([V_1],[V_2]) = V_2(h^2_{m_1(v_1)}),$$

$$T([V],[V_2]) = V_2(h^2_{m_1(m(v))}).$$

We have to prove:

$$V_2(h^2_{m_1(m(v))}) = V_1(h^1_{m(v)}) \circ V_2(h^2_{m_1(v_1)}).$$

By $V_1 = m_1 \circ V_2'$ also

$$V_1(h^1_{m(v)}) = V_2\big(m_1(h^1_{m(v)})\big).$$

Since the arrow

$$m_1\big(m(v)\big) \xrightarrow{m_1(h^1_{m(v)})} m_1(v_1)$$

is in $\mathscr{V}_2$ and v_2 is terminal in $\hat{\mathscr{V}}_2$, we have

$$h^2_{m_1(v_1)} m_1(h^1_{m(v)}) = h^2_{m_1(m(v))}.$$

Since V_2 is a functor, this completes the proof.

T obviously preserves domains, codomains and identities; therefore, by the above argument it is a functor.

Now we show that the functor $(R, \leqslant) \xrightarrow{T} I$ is final (in the sense of MacLane, p. 213, which is a generalization of "cofinal").

T is said to be *final* iff for any object k of I the comma category $(k \downarrow T)$ is nonempty and connected. To show this, let $k \in \mathrm{Ob}\, I$; $V, W \in P$ and $k \xrightarrow{p} T([V])$, $k \xrightarrow{q} T([W])$ be arbitrary. We have to prove the existence of a "good" path between V and W in P.

To this end, we shall construct an upper bound Z of V and W (i.e. a coequalised extension of $V \amalg W$, where $\amalg$ is understood in $(\mathrm{Cat}\downarrow I)$) such that p and q will be in the image of Z, i.e. the diagram

$$\begin{tikzcd} & z & \\ v \arrow[ur, "h_v"] & & w \arrow[ul, "h_w"'] \\ & k' \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd} \quad \overset{Z}{\longmapsto} \quad \begin{tikzcd} & Z(z) & \\ Z(v) \arrow[ur, "Z(h_v)"] & & Z(w) \arrow[ul, "Z(h_w)"'] \\ & Z(k')=k \arrow[ul, "Z(p')=p"] \arrow[ur, "Z(q')=q"'] & \end{tikzcd}$$

will exist. This diagram obviously commutes (if it exists), which implies the "goodness" of the path $V \to Z \leftarrow W$ in P.

Construction of Z. Define a new category $\mathscr{X}$ by adding to $(\hat{\mathscr{V}} \amalg \hat{\mathscr{W}})$ a new object k and two new arrows $k \xrightarrow{p} v$ and $k \xrightarrow{q} w$ (Fig. 4).

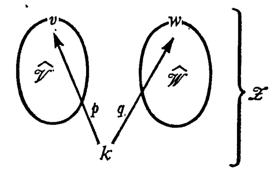

Fig. 4

That is, $\mathscr{X}$ contains $\hat{\mathscr{V}}$ and $\hat{\mathscr{W}}$ as disjoint full subcategories, and one additional object k not contained in either of them.

Define the functor $\mathscr{X} \xrightarrow{Z'} I$ by:

$$Z' \restriction \hat{\mathscr{V}} \overset{\mathrm{d}}{=} V, \quad Z' \restriction \hat{\mathscr{W}} \overset{\mathrm{d}}{=} W \quad \text{and} \quad Z'(k) \overset{\mathrm{d}}{=} k, \quad Z'(p) \overset{\mathrm{d}}{=} p, \quad Z'(q) \overset{\mathrm{d}}{=} q.$$

Since $\mathscr{X}$ is finite, by the lemma this Z' has a coequalised extension $\hat{\mathscr{X}} \xrightarrow{Z} I$ (such that $Z \restriction \mathscr{X} = Z'$).

"z" denotes the terminal object of $\hat{\mathscr{X}}$. Clearly $Z \in P$.

By the definition of Z it is an upper bound of V and W and "$V \to Z \leftarrow W$" is a "good" path because the diagram

$$\begin{tikzcd} & T([Z]) & \\ T([V]) \arrow[ur, "{T([V],[Z])}"] & & T([W]) \arrow[ul, "{T([W],[Z])}"'] \\ & k \arrow[ul, "p"] \arrow[ur, "q"'] & \end{tikzcd}$$

commutes (in I). To check this, observe that $T([V],[Z]) = Z(h_v)$, $T([W],[Z]) = Z(h_w)$, where $v \xrightarrow{h_v} z$ and $w \xrightarrow{h_w} z$ are the unique arrows of $\hat{\mathscr{X}}$ into its terminal object z. ∎

We have seen that T is final.

(It is easy to see that T is, in addition, *surjective*, e.g. for every $i \in \mathrm{Ob}\, I$ the one-element diagram

$$\begin{tikzcd} i \arrow[loop right, "1_i"] \end{tikzcd}$$

is a coequalised extension of the empty diagram $\emptyset$. Similarly for the arrows of I.)

MacLane, p. 213, Th. 1, proves the following:

For any functor $\mathscr{R} \xrightarrow{T} L$, *$T$ is final iff for any diagram* $L \xrightarrow{D} \mathscr{C}$, (i)–(v) *below hold:*

(i) $\operatorname{Colim} D$ *exists iff* $\operatorname{Colim}(T \circ D)$ *exists.*

(ii) $\operatorname{Colim}(T \circ D) = T \circ \operatorname{Colim} D$.

(iii) *For any right inverse T^{-1} of the object part of T (i.e. for any map* T^{-1}: $\operatorname{Ob} L \to \operatorname{Ob} \mathscr{R}$ *such that* $T^{-1} \circ T = \mathrm{Id}_{\operatorname{Ob} L}$) *we have* ([4])

$$\operatorname{Colim} D = T^{-1} \circ \operatorname{Colim}(T \circ D).$$

(iv) *For every upper bound B of D:*

$$\operatorname{Colim} D = B \quad \textit{iff} \quad \operatorname{Colim}(T \circ D) = T \circ B.$$

(v) *B is an upper bound of D iff $T \circ B$ is an upper bound of $T \circ D$.*

The category of upper bounds of D is isomorphic to the category of upper bounds of $T \circ D$, and $(T \circ -)$ is their isomorphism.

In the present case T^{-1} for (iii) can be given constructively, e.g. for every $i \in \operatorname{Ob} I$ let $T^{-1}(i)$ be a one-element diagram

$$\begin{tikzcd} i \arrow[loop right, "1_i"] \end{tikzcd}.$$

Since $\hat{\emptyset}$ is a one-element category, $T^{-1}(i)$ as defined above is in P.

Remark. The above theorem is only partially formulated in Mac Lane but it is actually proved there completely (cf. also p. 214, Ex. 5).

By the finality of T the above theorem completes the proof of Theorem 1. ∎

Theorem 1 states that the structure of the index categories of filtered colimits can be simplified. The question arises: can they be simplified even further?

DEFINITION. A partial order $(R, \leqslant)$ is a *tree* iff no nontrivial lower bounds exist in it. More precisely, it is a *tree* iff for every $a, b \in R$ the set $\{a, b\}$ has a lower bound iff its elements are comparable, i.e. iff $a \leqslant b$ or $b \leqslant a$. (Notice that a tree can be disconnected.)

PROPOSITION. *There is a directed partially ordered set* $(R, \leqslant)$ *such that for any functor* $\mathscr{P} \xrightarrow{T} (R, \leqslant)$, *if T is final in* $(R, \leqslant)$ *then $\mathscr{P}$ is not a tree.*

([4]) Such a T^{-1} exists iff T is surjective on the objects. Therefore the following (iv), (v) are not needed in the present paper.

Proof. For any final functor, if its codomain is connected then its domain is also connected. Any connected tree is also directed. Every directed tree contains a final and totally ordered subclass. (Namely, any right segment $\{q \in P\colon p \preceq q\}$ is such if $(P, \preceq)$ is a directed tree and $p \in P$.) But it is well known that direct limits cannot be reduced to totally ordered ones, i.e.: there is a directed poset $(R, \leqslant)$ such that if $\mathscr{P} \xrightarrow{T} (R, \leqslant)$ is final then $\mathscr{P}$ is not a total order (e.g. (Finite subsets of A, $\subseteq$) is such iff A is not countable.) ∎

Compare this with the fact that for any countable filtered category I there is a final functor $T\colon (\omega, \leqslant) \to I$, where ω is the set of natural numbers and $\leqslant$ has the usual meaning.

Remark. In the present paper stress has been laid on translations of a kind of colimit into another kind in which repeated computation (*iteration*) of the first kind is *not* allowed (i.e., computation of partial results is not allowed).

If we allow iteration, then:

Every filtered colimit can be obtained by iterating *well-ordered colimits*, if the latter exist in the category.

The point of the present paper is that we can do things without "computing partial results". E.g. there are filtered colimits which cannot be obtained by iterating well-ordered colimits. In other words, there is a category $\mathscr{C}$ and a subcategory $\mathscr{A} \subseteq \mathscr{C}$ such that $\mathscr{A}$ is *closed w.r.t. well-ordered limits but is not closed w.r.t. filtered limits.* This is possible because the "partial results" needed to compute a final result may not exist while the final result exists. Namely, $\mathscr{A}$ is closed w.r.t. well-ordered limits because they do not exist, but there is a filtered limit which does. (Consider e.g. a collection of finite and uncountable sets.)

Some consequences

ω-presentable objects of Gabriel–Ulmer [5], p. 63, Def. 6.1, Diers [4] Day [3] coincide with strongly finitary objects of Herrlich–Strecker (22E) and also with strongly small objects of Matthiessen [9], Banaschewski–Herrlich [2], John [7], Németi–Sain [10], Andréka–Németi [1] in *every* category.

α-presentable objects coincide with strongly α-small ones of Banaschewski–Herrlich [2], [12]–[14].

But "ω-presentable" does not coincide with the "finite" of Smyth [11] because in the latter case only limits of chains have to be preserved by the $\hom(a, -)$ functor.

Also, the filtered limit closed subcategories investigated by Diers and Day coincide with the direct limit closed subcategories investigated by Banaschewski, Herrlich and others.

References

[1] H. Andréka and I. Németi, *Formulas and ultraproducts in categories*, Beitrage zur Algebra u. Geom. 8 (1979), 133–151.
[2] B. Banaschewski and H. Herrlich, *Subcategories defined by implications*, Houston Journ. Math. 2 (1976), 149–171.
[3] B. J. Day, *Density representations of functors*, Bull. Austral. Math. Soc. 3 (1977), 321–480.
[4] Y. Diers, *Type de densite d'une sous-categorie pleine*, Ann. Soc. Sci. Bruxelles, I, 90 (1976), 25–47.
[5] P. Gabriel und F. Ulmer, *Local presentierbare Kategorien*, Lecture Notes in Mathematics, Vol. 221, Springer-Verlag, 1971.
[6] H. Herrlich and G. E. Strecker, *Category theory*, Allyn and Bacon, 1973.
[7] R. John, *A note on implicational subcategories*, In: *Contributions to Universal Algebra* (Proc. Coll. Szeged 1975), Colloq. Math. Soc. J. Bolyai, Vol. 17, North-Holland, Amsterdam 1977, 213–222.
[8] S. MacLane, *Categories for the working mathematician*, Springer-Verlag, 1971.
[9] G. Matthiessen, *Regular and strongly finitary structures over regularly finitary categories*, Canadian J. Math. 30 (1978), 250–261.
[10] I. Németi and I. Sain, *Cone-injective subcategories and some Birkhoff type theorems*, In: *Universal Algebra* (Proc. Coll. Esztergom 1977), Colloq. Math. Soc. J. Bolyai, Vol. 29, North-Holland, Amsterdam 1982, 535–578.
[11] M. Smyth, *Power domains*, Journ. of Computer and System Sci. 16 (1) (1978).
[12] H. Andréka and I. Németi, *Łoś lemma holds in every category*, Studia Sci. Math. Hungar. 13 (1978), 361–376.
[13] —, —, *Injectivity in categories to represent all first order formulas*, Demonstratio Mathematica 12 (1979), 717–732.
[14] I. Sain and B. H. Hien, *In which categories are first order axiomatizable hulls characterizable by ultraproducts*, Math. Inst. Hungar. Acad. Sci. Preprint No 56/1981, to appear in Cahiers. Diff. Geom. Top.

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A GENERALIZATION OF ELEMENTARY FORMAL SYSTEMS

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1. Introduction

In [6] Smullyan gave an elegant development of recursion theory based on elementary formal systems. These dealt directly with words over a finite alphabet, and only indirectly with numbers, via "names" for them. We generalize the notion of elementary formal system, by separating "structural properties" from "subject matter." The result provides a natural "recursion theory" for any structure, words and numbers being particular examples.

Our notion of recursion theory over the natural numbers can be turned into hyperarithmetic theory by the addition of a simple infinitary rule (an ω-rule) [1]. We formulate the rule so that it applies to all our recursion theories, turning them into what we call ω-recursion theories. For both recursion and ω-recursion theories we define a natural generalization of enumeration operator. We investigate the structural characteristics of these operators, and prove an analog of the First Recursion Theorem for them.

2. Elementary formal systems

Let $\mathscr{A}$ be an infinite set, and let $\mathscr{R}_1, \ldots, \mathscr{R}_k$ be relations on $\mathscr{A}$. We cal $k+1$ tuple $\langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$ a *structure*. We allow trivial structures $\langle \mathscr{A} \rangle$ We set up a simple logical calculus relative to a particular structure, so for the rest of this section, let $\mathfrak{A} = \langle \mathscr{A}, \mathscr{R}_1, \ldots, \mathscr{R}_k \rangle$ be a fixed structure.

We suppose available an unlimited supply of n-place predicate symbols for each $n > 0$. We informally use P, Q, R, etc. to represent them. The other two symbols of our alphabet are an *arrow* and a *comma*. We will use axiom schemas, so variables are not needed in the language itself, and we need no rule of substitution.