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ON THE THEORY OF BAER LATTICES

M. STERN

Department of Mathematics, Martin Luther University, 402 Halle, G.D.R.

By means of the concept of Baer lattices we shall be able to begin a unified treatment for three classes of lattices, namely AC-lattices, primary lattices and modular lattices satisfying the descending chain condition. We note that R. Baer [1] did the first step in developing a unified theory of projective spaces (the subspace lattices of which are AC-lattices) and finite Abelian groups (the subgroup lattice of which are primary lattices); in view of this pioneering paper we think that the term "Baer lattice" employed by us is justified. Here we sketch some of our results. Detailed proofs will be published elsewhere.

Definition. A lattice L with 0 will be called a *Baer lattice*, if the following three conditions are satisfied:

(i) Every element of L is a join of join-irreducible elements of L;

(ii) For every join-irreducible element u of L the interval $[0, u]$ is a modular sublattice of finite length;

(iii) For an arbitrary join-irreducible element u and for an arbitrary element b of L the intervals $[b \wedge u, u]$ and $[b, b \vee u]$ are isomorphic (an isomorphism being established by the mutually inverse canonical mappings).

In fact, the concept of Baer lattices can be defined in a somewhat more general framework which is, however, too technical to be reproduced here.

From the above definition it is immediate that AC-lattices (see [5]), primary lattices (see [4]) and modular lattices satisfying the descending chain condition are Baer lattices. Moreover, it is easy to construct examples of Baer lattices belonging to none of these three classes.

We have proved, among others, the following results:

(1) In a Baer lattice the following implication holds: if the interval $[x \wedge y, x]$ is a chain of length n, then the interval $[y, x \vee y]$ is also a chain

of length n. From this we obtain, in particular, that a Baer lattice is upper semimodular.

(2) Every interval and every principal dual ideal of a Baer lattice are likewise Baer lattices.

(3) In a Baer lattice the following exchange property holds: if u, v are join-irreducible elements, b an arbitrary element and $v \leqslant b \vee u$ but $v \not\leqslant b \vee u'$ (u' denotes the uniquely determined lower neighbor of u), then $u \leqslant b \vee v$. In the special case of AC-lattices we get from this the Steinitz–MacLane exchange property in its lattice-theoretic form.

(4) In a Baer lattice the Theorem of Kurosh–Ore holds: if an element b can be represented as a join of finitely many join-irreducible elements, then two minimal representations of b as a join of join-irreducible elements have the same number of components.

(5) Calling the number of components in a minimal representation of an element b as a join of join-irreducible elements the rank of b we can show: For a Baer lattice L the subset $F(L)$ of all elements of finite rank is an ideal. Using this notion of rank, a simple necessary and sufficient condition can be given for $F(L)$ to be a standard ideal in the sense of [2]. For the special case of finite-modular AC-lattices we obtain as a corollary that $F(L)$ is always standard (see [3]).

The above-mentioned results suggest that it might be possible to prove many results of [5] on AC-lattices also for Baer lattices.

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DIRECT LIMITS AND FILTERED COLIMITS ARE STRONGLY EQUIVALENT IN ALL CATEGORIES

H. ANDRÉKA and I. NÉMETI

Mathematics Institute, Hungarian Academy of Sciences, Budapest, Hungary

Herrlich–Strecker ([6], p. 151, Def. 22.1), defines "direct limits" as special cases of "filtered colimits" in the sense of MacLane ([8], p. 207). Here we show that the two concepts are strongly equivalent: i.e. every filtered diagram (of any category) can be transformed into a directed diagram, in a rather natural and constructive way, so that the same objects and the same arrows are used and not only the colimit objects but also the co-limiting cocones of the two diagrams coincide (if any of them exists). This implies that the images (1) of the two diagrams coincide. In other words, the two diagrams will be "cofinal" (or, more categorically, final).

We use the word "diagram" as a synonim for "functor". We shall refer to the monographs by Herrlich–Strecker [6] and by MacLane [8] as "Herrlich–Strecker" and "MacLane".

DEFINITION 1 (Herrlich–Strecker, Def. 22.1). A *directed partial order* is a pair $(R, \leqslant)$ where R is a class such that any finite subset of R has an upper bound in $(R, \leqslant)$. (Note that this implies that R is nonempty!) Partial orders are considered to be categories.

A *directed diagram* is a functor $(R, \leqslant) \xrightarrow{D} \mathscr{C}$ from a directed partial order into a category $\mathscr{C}$.

A *direct limit* is a colimit of a directed diagram.

DEFINITION 2 (MacLane, p. 207)., A category I is *filtered* if any finite diagram $\mathscr{V} \xrightarrow{V} I$ (i.e. any functor $V: \mathscr{V} \to I$ such that $\mathscr{V}$ is finite) has an upper bound in I. (By an *upper bound* of V we understand a cocone $(f_i)_{i \in \mathrm{Ob}\,\mathscr{V}}$ compatible with V, i.e. "commuting over V"). (This implies that I is nonempty since $\emptyset \xrightarrow{\emptyset} I$ (cf. MacLane, p. 229) is a finite diagram.)

(1) *Image*: cf. Mac Lane, p. 243, Ex. 4. (The image of a functor need not be a category but only a *partial category*.)