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HOW MANY FOUR-GENERATED SIMPLE LATTICES?

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We call a lattice L *simple* if $|L| > 1$ and L has no nontrivial congruence relations.

For which partially ordered sets P is there a simple lattice generated by P? There is, for instance, precisely one simple lattice generated by the two-element chain, namely, the two-element chain itself. This is the smallest simple lattice. On the other hand, the lattice generated by an n-element chain is not simple if $n \geqslant 3$. Still, a partially ordered set consisting of n elements pairwise noncomparable can generate a simple lattice just as long as $n \geqslant 3$ (for example, the $(n+2)$-element modular lattice of length two).

Let P be the partially ordered set consisting of pairwise noncomparable elements a, b, c and let L be a simple lattice generated by P. If $a \nleqslant b \vee c$, say, then L is the disjoint union of $\{x \in L \mid x \geqslant a\}$ and $\{x \in L \mid x \leqslant b \vee c\}$, whence L has a homomorphism onto the two-element chain. It follows that $a \leqslant b \vee c$. By symmetry and duality L must be the five-element modular lattice of length two. This observation, first recorded by R. Wille [14], shows that there is precisely *one* simple lattice generated by a three-element unordered set (antichain).

Interest in simple lattices generated by an antichain was revitalized by H. Strietz [12] who showed that every lattice of partitions on a finite set with at least four elements is generated by a four-element antichain. There are then at least countably many simple lattices generated by a four-element antichain. Actually there are more.

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THEOREM 1. *There are* $2^{\aleph_0}$ *nonisomorphic simple lattices generated by a four-element antichain.*

The proof of this result relies on an embedding theorem which seems to be of independent interest.

THEOREM 2. *Every countable lattice is embeddable in a simple lattice generated by a four-element antichain.*

While our proofs are almost elementary the results themselves stem from and, in turn, bear upon several important facts.

First of all we shall make use of at least one nontrivial fact: *every countable lattice is embeddable in a lattice generated by a three-element antichain* (R. A. Dean [3], Ju. I. Sorkin [11]). Secondly, we shall require a companion result due to P. Crawley and R. A. Dean [1]: *there are* $2^{\aleph_0}$ *nonisomorphic lattices generated by a three-element antichain.* We shall supply an alternate proof of this fact, indeed, a proof whose simplicity is rather unexpected.

Interest in lattices of partitions originates, of course, in P. M. Whitman's pioneering work in the 1940s [13] (see also [7]). He showed that *every countable lattice is embeddable in the lattice of partitions on a countable set* and conjectured that every *finite* lattice is embeddable in the lattice of partitions on a *finite* set. Since any lattice of partitions on a finite set is simple [7] and generated by a four-element antichain [12] it was of some interest to prove at least that *every finite lattice is embeddable in a finite, simple lattice generated by a four-element antichain.* This result was established by W. Poguntke and I. Rival [8] whose proof, in fact, provides the main ideas for our proof of Theorem 2. Of course, more recently this result, as well as several related ones (cf. [2], pp. 125–131, [5], [6]), has been superseded by the deep results of P. Pudlák and J. Tůma [9] who settled Whitman's conjecture in the affirmative. However, neither the affirmative solution to Whitman's conjecture nor Whitman's original embedding theorem accounts for Theorem 2. While the one is concerned with finite lattices the other is concerned with the lattice of partitions on a countable set which, though simple, is uncountable, whence not even *countably* generated.

Proof of Theorem 2. Let L be a countable lattice. We may assume that L is bounded (for otherwise we would just adjoin universal bounds 0 and 1 to L).

First, we shall embed L into a simple, countable, bounded lattice L'. If L has length at most two then either L is simple, in which case we may choose $L' = L$, or else L consists of at most four elements, in which case we take L' to be the five-element (modular) lattice of length two. Let L have length at least three and let $Q(L)$ denote the set of all quotients of L

which are disjoint from 0 and 1:

$$Q(L) = \{x/y \mid x, y \in L \text{ and } 1 > x > y > 0\}.$$

In all, there are countably many such quotients in L, for $Q(L)$ is itself embeddable in $L \times L$. Hence, the members of $Q(L)$ may be enumerated: $x_1/y_1, x_2/y_2, \ldots$ Let $\{a_1, b_1, c_1, d_1\}, \{a_2, b_2, c_2, d_2\}, \ldots$ be a sequence of quadruples of distinct elements such that $\{a_i, b_i, c_i, d_i\} \cap \{a_j, b_j, c_j, d_j\} = \emptyset$ if $i \neq j$ and, $\{a_i, b_i, c_i, d_i\} \cap L = \emptyset$ for each i. Let

$$L' = L \cup \bigcup_{i \geq 1} \{a_i, b_i, c_i, d_i\}$$

have the partial ordering induced by L and the comparabilities $1 > a_i > y_i$, $1 > b_i > y_i$, $x_i > c_i > 0$, and $x_i > d_i > 0$ for each $x_i/y_i \in Q(L)$. Evidently, L' is a lattice; moreover, it is an easy matter to verify that L' is simple. Hence, we have embedded L in a simple, countable, bounded lattice L'. (1)

In the next step we embed L' in a subdirectly irreducible, bounded lattice L'' generated by three elements. According to the Dean–Sorkin Embedding Theorem, L' is embeddable in a lattice K generated by three elements. Now, K is a subdirect product of subdirectly irreducible lattices K_α, $\alpha \in I$, and each K_α is generated by three elements. In particular, L' is embeddable in the product of the K_α's. As L' is simple, it is, *a fortiori*, subdirectly irreducible, whence there is a K_α such that L' is embeddable in K_α. We set $L'' = K_\alpha$. In summary, we have embedded L in a subdirectly irreducible, bounded lattice L'' generated by three elements.

In the final step we embed L'' in a simple lattice generated by a four-element antichain.

To this end let θ be the minimum nontrivial congruence relation of L''. Unless L'' has length two and is already simple there is $a/b \in Q(L'')$ satisfying $a \equiv b(\theta)$. Let c, d be distinct elements each disjoint from L and let

$$L''' = L'' \cup \{c, d\}$$

be partially ordered by the induced ordering of L'' and the comparabilities $1 > c > b$ and $c > d > 0$. Then L''' is a lattice generated by a four-element antichain: the three generators of L'', and d. Moreover, L''' is still subdirectly irreducible (its minimum nontrivial congruence relation θ satisfies $a \equiv b(\theta)$).

We shall show that L''' is simple. (2) Specifically, we shall prove

(1) Actually, the substance of this construction shows that every lattice is embeddable in a simple lattice (cf. [4]).

(2) Our construction shows, in fact, that every n-generated subdirectly irreducible lattice L is embeddable in a simple $(n+1)$-generated lattice with precisely two more elements than L.

that $1 \equiv b(\theta)$ and $b \equiv 0(\theta)$. For arbitrary quotients x/y and u/v of L''' we write $x/y \nearrow u/v$ if $x \vee v = u$ and $x \wedge v \geqslant y$, and $x/y \searrow u/v$ if $y \wedge u = v$ and $y \vee u \leqslant x$. Note that $u \equiv v(\theta(x, y))$ if either $x/y \nearrow u/v$ or $x/y \searrow u/v$.

Since L'' is subdirectly irreducible and finitely generated there exist noncomparable elements e_1, e_2 of L'' with $e_1 \vee e_2 = 1$. Then

$$a/b \nearrow 1/c \searrow e_1/e_1 \wedge b \nearrow e_1 \vee b/b.$$

It follows that $e_1 \vee b \equiv b(\theta)$ and similarly $e_2 \vee b \equiv b(\theta)$, whence $b \equiv e_1 \vee e_2 = 1(\theta)$.

Now, choose noncomparable elements f_1, f_2 of L'' satisfying $f_1 \wedge f_2 = 0$. Then

$$1/b \searrow d/0 \nearrow f_1 \vee d/f_1$$

so $f_1 \vee d \equiv f_1(\theta)$ and similarly $f_2 \vee d \equiv f_2(\theta)$. Now, $0 = f_1 \wedge f_2 \equiv (f_1 \vee d) \wedge \wedge (f_2 \vee d)(\theta)$ and since both $f_1 \vee d \geqslant c$ and $f_2 \vee d \geqslant c$ we conclude that $0 < b < c \leqslant (f_1 \vee d) \wedge (f_2 \vee d)$. In particular, $b \equiv 0(\theta)$. This completes the proof of Theorem 2.

Proof of Theorem 1. We require the result, due to Crawley and Dean [1], that *there are* $2^{\aleph_0}$ *nonisomorphic lattices generated by a three-element antichain.* Let L be the lattice illustrated in Fig. 1. Evidently, L is generated by the

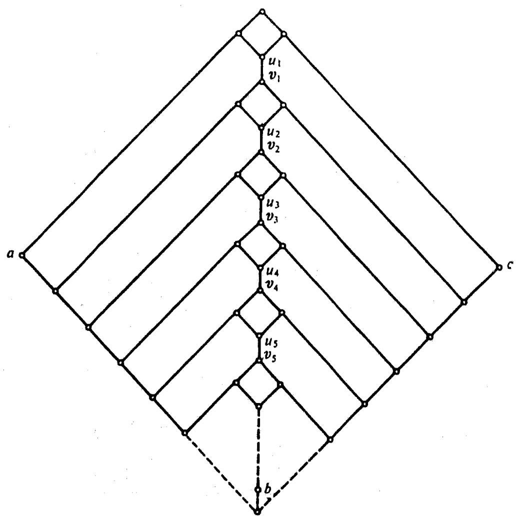

Fig. 1

three-element antichain $\{a, b, c\}$. Note that, for each positive integer i, the smallest congruence relation θ_i prescribed by $u_i \equiv v_i(\theta_i)$ has exactly one nontrivial block, namely, $\{u_i, v_i\}$. Moreover, if I and J are distinct subsets of the set of positive integers then it is easy to see that $L/\theta_I \not\approx L/\theta_J$ where $\theta_I = \bigvee_{i \in I} \theta_i$ and $\theta_J = \bigvee_{j \in J} \theta_j$. In particular, L has $2^{\aleph_0}$ nonisomorphic homomorphic images (!), each, of course, generated by a three-element antichain. (3)

Let (L_a) be a family of $2^{\aleph_0}$ nonisomorphic lattices each generated by a three-element antichain. According to Theorem 2 each L_a is embeddable in a simple lattice generated by a four-element antichain. If there were less than $2^{\aleph_0}$ simple lattices generated by a four-element anitchain then one of them, S, say, contains uncountably many distinct L_a's. As S is countable, S contains only countably many triples whence S can contain only countably many distinct lattices each generated by three elements. It follows that there are $2^{\aleph_0}$ non-isomorphic simple lattices generated by a four-element antichain.

Added in proof. Recently R. Freese (*Some order theoretic questions about free lattices and free modular lattices*, in *Ordered Sets*, D. Reidel, 1982, 355–377) has improved Theorem 1 by showing that there are $2^{\aleph_0}$ nonisomorphic simple *modular* lattices generated by a four-element antichain.

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(3) Parenthetically, a similar exercise can be carried out with certain other partially ordered sets. In fact, there are $2^{\aleph_0}$ nonisomorphic lattices generated by $\mathbf{2}+\mathbf{2}$, the cardinal sum of two two-element chains as well as by $\mathbf{1}+\mathbf{4}$, the cardinal sum of a one-element chain and a four-element chain. Specifically we claim that each of the free lattices $\mathrm{FL}(\mathbf{2}+\mathbf{2})$ and $\mathrm{FL}(\mathbf{1}+\mathbf{4})$ has $2^{\aleph_0}$ nonisomorphic homomorphic images. For the proof of this claim we appeal to the *diagram* of each of these lattices (see H. L. Rolf [10]).

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ON THE THEORY OF BAER LATTICES

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By means of the concept of Baer lattices we shall be able to begin a unified treatment for three classes of lattices, namely AC-lattices, primary lattices and modular lattices satisfying the descending chain condition. We note that R. Baer [1] did the first step in developing a unified theory of projective spaces (the subspace lattices of which are AC-lattices) and finite Abelian groups (the subgroup lattice of which are primary lattices); in view of this pioneering paper we think that the term "Baer lattice" employed by us is justified. Here we sketch some of our results. Detailed proofs will be published elsewhere.

DEFINITION. A lattice L with 0 will be called a *Baer lattice*, if the following three conditions are satisfied:

(i) Every element of L is a join of join-irreducible elements of L;

(ii) For every join-irreducible element u of L the interval $[0, u]$ is a modular sublattice of finite length;

(iii) For an arbitrary join-irreducible element u and for an arbitrary element b of L the intervals $[b \wedge u, u]$ and $[b, b \vee u]$ are isomorphic (an isomorphism being established by the mutually inverse canonical mappings).

In fact, the concept of Baer lattices can be defined in a somewhat more general framework which is, however, too technical to be reproduced here.

From the above definition it is immediate that AC-lattices (see [5]), primary lattices (see [4]) and modular lattices satisfying the descending chain condition are Baer lattices. Moreover, it is easy to construct examples of Baer lattices belonging to none of these three classes.

We have proved, among others, the following results:

(1) In a Baer lattice the following implication holds: if the interval $[x \wedge y, x]$ is a chain of length n, then the interval $[y, x \vee y]$ is also a chain