that $x < b_0 * x < b_1 * x < b_2 * x < \ldots < 1$. Let F be the filter generated by $\{b_i | \ i < \omega\}$, i.e. $F = \{s | \ \exists i < \omega\colon s \geqslant b_i\}$. By (ii) F is comonomial, so let $f \in F$ such that $f * x = \max[x]_F$. Firstly $f \geqslant b_i$ for some $i < \omega$. Then $f * x \leqslant b_i * x < b_{i+1} * x$. On the other hand $b_{i+1} \in F$ and so $b_{i+1} * x \leqslant f * x$. This is a contradiction and so our assumption on x must be false. This proves the theorem.

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PROJECTIONS OF MIXED LIE RINGS

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Introduction

The aim of this article is the study of the lattice isomorphisms (projections) of Lie rings. We will make use of generally accepted terminology (see, for example, [6], [2]).

Notation. $S(\mathscr{L})$ is the lattice of all subrings of $\mathscr{L}$; $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ will denote a lattice isomorphism; $\mathscr{A}^\varphi \subseteq \mathscr{L}^\varphi$ will denote the image of the subalgebra $\mathscr{A} \subseteq \mathscr{L}$ under φ; $N(\mathscr{A})$, $[\mathscr{A}, \mathscr{A}]$ will denote the normalizer and the commutator, respectively, of $\mathscr{A} \subseteq \mathscr{L}$; $Z(\mathscr{L})$ is the centre of $\mathscr{L}$; $C_{\mathscr{A}}(X)$ is the centralizator of X in $\mathscr{A} \subseteq \mathscr{L}$; $\boldsymbol{Z}$ is the ring of real integers; $\{X\}$ denotes the subring generated by X.

An element $a \in \mathscr{L}$ will be called *proper* if $\alpha a \neq 0$ for every $\alpha \in \boldsymbol{Z}$ $(\alpha \neq 0)$; otherwise, it will be called *periodic*. The ring $\mathscr{L}$ is *proper* if all its elements are proper; it will be called *mixed* (or *nonperiodic*) if it contains both the proper and periodic elements, and it will be called *periodic* if all its elements are periodic. The set of all the periodic elements of $\mathscr{L}$ will be denoted by $t(\mathscr{L})$. It is clear that $t(\mathscr{L})$ is an ideal in $\mathscr{L}$. The *dimension* of $\mathscr{L}$, denoted by $\dim\mathscr{L}$, is defined to be the maximal number of linearly independent elements. It is clear that $\dim\big(\mathscr{L}/t(\mathscr{L})\big) = \dim\mathscr{L}$.

We say that the ring $\mathscr{L}$ is *determined* (*strictly determined*) by $S(\mathscr{L})$ if $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ implies $\mathscr{L} \cong \mathscr{L}^\varphi$ (φ is induced by an isomorphism between $\mathscr{L}$ and $\mathscr{L}^\varphi$).

A lattice isomorphism $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ is called *normal* if $N(\mathscr{A}) = N(\mathscr{A}^\varphi)$ for each subring $\mathscr{A} \subseteq \mathscr{L}$.

In Section 1 we prove an analogy of a theorem of A. S. Pekelis [1]. In Sections 2 and 3, with the help of some ideas from [1], [4], we construct examples which give negative answers to natural questions in connection with the theorem of Section 1 and theorems from [3], [4].

1. Projections of mixed 2-nilpotent Lie rings

THEOREM. *Let $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ be a lattice isomorphism between 2-nilpotent Lie rings. If $\mathscr{L}$ contains a proper non-abelian subring, then φ is induced by an isomorphism.*

It is clear that for the proof it is sufficient to consider only the case where $\mathscr{L}$ is finitely generated. The lattice isomorphism φ is induced by the one-to-one mappings φ_1 and $\varphi_2 = -\varphi_1$, which are isomorphisms on any abelian subring $\mathscr{A} \subseteq \mathscr{L}$, and $\mathscr{A}^\varphi \subseteq \mathscr{L}^\varphi$.

The proof of this fact is the same as the proof of a similar fact in the group case (see [5]); we must only remark that φ is normal [3] and that this fact implies that φ preserves the nilpotency class of subrings [1].

Note that from the condition of theorem it follows that there exist elements $x_1, x_2 \in \mathscr{L}$ such that

$$\{x_1\} \cap \{x_2\} = 0, \quad nx_1x_2 \neq 0,$$

for every integer $n \neq 0$. Clearly, $\dim\mathscr{L} \geqslant 3$. On the subring $\{x_1, x_2\}$ the projection φ is induced by only one isomorphism [3].

Of two mappings φ_1 and φ_2 let us take that one which coincides with φ on $\{x_1\}$ and let us denote it by φ. Let $\varphi(x) = y$ in $\mathscr{L}^\varphi$ for each $x \in \mathscr{L}$. It is clear that $\varphi(kx) = k\varphi(x)$ for any $x \in \mathscr{L}$.

Let us show that for each $x \in \mathscr{L}$

$$\text{(A)} \qquad \varphi(x_1+x) = \varphi(x_1)+\varphi(x).$$

Consider the following cases:

1. x is a proper element and $\{x_1\}\cap\{x\} = 0$, $kx_1x \neq 0$ for each $k \in \mathbf{Z}$;
2. x is a proper element and $\{x_1\}\cap\{x\} = 0$, $kx_1x = 0$, $nx_2x \neq 0$ for each $n \in \mathbf{Z}$;
3. x is a proper element and $\{x_1\}\cap\{x\} = 0$, $kx_1x = 0$, $k_0x_2x = 0$;
4. x is a proper element and $\{x_1\}\cap\{x\} \neq 0$;
5. x is a periodic element.

We shall prove (A) for each case.

1. In this case the subring $\{x_1, x\}$ is proper and (A) is evident [3].

2. If $n(x_2+x)x_1 = 0$ for a certain $n \geqslant 1$, then

$$0 = kn(x_2+x)x_1 = knx_2x_1 \neq 0 \Rightarrow n(x_2+x)x_1 \neq 0$$

for each $n \geqslant 1$. On the other hand, if

$$\{x_2\}\cap\{x_1+x\} \neq 0,$$

then

$$k_1(x_1+x) = k_2x_2 \Rightarrow 0 = k_1k(x_1+x)x_1 = k_2kx_2x_1 \neq 0.$$

Consequently, if $n(x_1+x)x_2 \neq 0$, we have

$$\varphi[x_1+(x_2+x)] = \varphi[x_2+(x_1+x)] = y_2+\varphi(x_1+x) = y_1+y_2+y \Rightarrow \text{(A)}.$$

If $k_0(x_1+x)x_2 = 0$ $(k_0 > 1)$, then $n(x+k_0x_2)x_1 \neq 0$ because

$$0 = kn(x+k_0x_2)x_1 = knk_0x_2x_1 \neq 0.$$

Consequently, we have

$$\varphi[(x+x_1)+k_0x_2] = \varphi[(x+k_0x_2)+x_1] = y+k_0y_2+y_1 = \varphi(x_1+x)+k_0y_2 \Rightarrow \text{(A)}.$$

3. In this case $n(x_1+x)k_0x_2 \neq 0$ for each $n \in \mathbf{Z}$ because

$$0 = n(x_1+x)k_0x_2 = nk_0x_1x_2 \neq 0.$$

On the other hand, $n(x+k_0x_2)x_1 \neq 0$ because $knx_1x+knx_2x_1 = 0$ otherwise. As in the previous case, we conclude that (A) is true.

4. Element $z = x_1x_2$ is proper and $z \in Z(\mathscr{L})$. Then

$$\varphi[z+(x_1+x)] = \bar{z}+\varphi(x_1+x) \qquad (\bar{z} \in Z(\bar{\mathscr{L}})).$$

On the other hand, $z+x_1$ is a proper element and

$$\{z+x_1\}\cap\{x_2\} = 0, \quad n(z+x_1)x_2 \neq 0.$$

If $\{z+x_1\}\cap\{x\} = 0$, then using case 3 we get

$$\varphi[(z+x_1)+x] = \bar{z}+y_1+y \Rightarrow \text{(A)}.$$

5. It is clear that there is an integer $k_0 > 1$ such that $k_0x_2 \in C_{\mathscr{L}}(\{x\})$. On the other hand, $\{x_1+x\}\cap\{k_0x_2\} \neq 0$ and $n(x_1+x)k_0x_2 \neq 0$ because $0 = k_0nx_1x_2+nk_0xx_2 \neq 0$ otherwise.

Similarly, $nx_1(x+k_0x_2) \neq 0$. From this we find (as in case 4) that (A) is true.

Proof of the theorem. Suppose that x_4 and x_5 are arbitrary elements of $\mathscr{L}$. From the previous considerations we conclude that it is sufficient to consider the situation where there exists an element $x \in \mathscr{L}$ such that $nx_4x \neq 0$, $mx_5x \neq 0$ for each $m, n \in \mathbf{Z}$.

Suppose that for some $k, k_1, k_2, k_3 \in \mathbf{Z}$ we have

$$kx_4x_5 = 0, \quad k_1x_1x_5 = 0, \quad k_2x_1x_4 = 0, \quad k_3x_4x_2 = 0.$$

Then for $\bar{k} = k_1k_2$ we have $\bar{k}x \in C_{\mathscr{L}}(\{x_4, x_5\})$. If $n(\bar{k}x_1+x_4)x_2 \neq 0$ for each $n \geqslant 1$, then

$$\varphi(\bar{k}x_1+x_4+x_5) = \varphi[(\bar{k}x_1+x_4)+x_5] = \bar{k}y_1+y_4+y_5$$
$$= \bar{k}y_1+\varphi(x_4+x_5) \Rightarrow \varphi(x_4+x_5) = y_4+y_5.$$

To end the proof we must show that

$$\varphi(x_4x_5) = \varphi(x_4)\varphi(x_5), \qquad \forall x_4, x_5 \in \mathscr{L}.$$

If $\mathscr{L}$ is 2-nilpotent and φ is normal, we have

$$\varphi(x_4x_5) = a\varphi(x_4)\varphi(x_5),$$

where a is an integer. The ring $\{x_1, x_4+x_2\}$ is proper.

Consequently, from the previous considerations we have

$$\begin{aligned}\varphi[x_1(x_4+x_2)] &= \varphi(x_1x_4+x_1x_2) = \varphi(x_1)\varphi(x_4+x_2)\\ &= \varphi(x_1)\varphi(x_4)+\varphi(x_1)\varphi(x_2) = \varphi(x_1x_4)+\varphi(x_1)\varphi(x_2)\\ &\Rightarrow \varphi(x_1x_4) = \varphi(x_1)\varphi(x_4) \Rightarrow \varphi[x_4(x_1+x_5)]\\ &= \varphi(x_4x_1+x_4x_5) = \varphi(x_4)\varphi(x_1)+\varphi(x_4x_5) = \varphi(x_4)(x_1+x_5)\\ &= \varphi(x_4)\varphi(x_1)+\varphi(x_4)\varphi(x_5) \Rightarrow \varphi(x_4x_5) = \varphi(x_4)\varphi(x_5).\end{aligned}$$

This completes the proof of the theorem.

2. ΠS-isomorphisms of 2-nilpotent Lie rings

The following questions arise naturally in connection with the theorem of Section 1 and Theorem 6.2 from [3].

1. Is every normal lattice isomorphism of a 2-nilpotent Lie ring $\mathscr{L}$ ($\dim \mathscr{L} \geqslant 2$) induced by an isomorphism?

2. If $\mathscr{L}$ is a mixed n-nilpotent ($n \geqslant 3$) Lie ring which contains a proper n-nilpotent subring, then is every normal lattice isomorphism of $\mathscr{L}$ induced by an isomorphism?

On the other hand, one might consider a more rich lattice than $S(\mathscr{L})$.

A subset $\mathscr{L}_0$ of a ring $\mathscr{L}$ is called *subsemiring* if

$$x_1, x_2 \in \mathscr{L}_0 \Rightarrow x_1+x_2 \in \mathscr{L}_0, \qquad x_1x_2 \in \mathscr{L}_0.$$

It is clear that the collection $\Pi S(\mathscr{L})$ of all subsemirings of $\mathscr{L}$ is a lattice and that $S(\mathscr{L}) \subset \Pi S(\mathscr{L})$. An isomorphism

$$\varphi\colon \Pi S(\mathscr{L}) \to \Pi S(\mathscr{L}^{\varphi})$$

is called a *ΠS-isomorphism*. Isomorphisms of a subsemiring lattice are analogous to ΠS-isomorphisms for groups. From group theory we have the theorem of M. N. Aršinov [1]: Every ΠS-isomorphism of a non-periodic nilpotent group is induced either by an isomorphism or by an anti-isomorphism.

It is therefore natural to pose the question:

3. Is an analogous theorem true for Lie rings?

Below we give examples which answer all these questions in the negative. In constructing these examples we use some ideas from [1], [4].

Example 1. Let the Lie ring $\mathscr{A} = \{x_1, x_2, k\}$ have the defining relations

$$x_1x_2 = k, \quad kx_1 = 0, \quad kx_2 = 0, \quad pk = 0$$

(p is a prime number different from 2). The elements x_1 and x_2 are proper.

It is clear that $\mathscr{A} = \{x_1, x_2\}$, $[\mathscr{A}, \mathscr{A}] = \{k\}$, $\mathscr{A}$ is a 2-nilpotent ring and $\dim\mathscr{A} = 2$. Each element l of $\mathscr{A}$ has a unique expression in the form

$$l = a_1x_1+a_2x_2+\beta k \quad (0 \leqslant \beta < p).$$

Define a one-to-one relation $f\colon \mathscr{A} \to \mathscr{A}$ as follows:

$$l' = f(l) = \begin{cases} l & \text{if } a_1a_2 \equiv 0 \pmod p,\\ l+sk & \text{if } a_1a_2 \not\equiv 0 \pmod p,\end{cases} \tag{1}$$

$$\text{where } 0 \leqslant s < p, \text{ and } s+a_1+a_2 \equiv 0 \pmod p,$$

and let us show that f induces a ΠS-automorphism of $\mathscr{A}$, i.e., that for each $l_1, l_2 \in \mathscr{A}$

$$f(l_1+l_2) = \omega\big(f(l_1), f(l_2)\big),$$

where ω is a two-variable polynomial with positive coefficients. This fact implies that f associates a subsemiring with a subsemiring and f induces a ΠS-automorphism.

There is no need to check the same fact for the product because

$$l_1l_2 \in [\mathscr{A}, \mathscr{A}] = \{k\} \Rightarrow a_1 = a_2 = 0 \Rightarrow f(l_1l_2) = f(l_1)f(l_2) = l_1l_2.$$

The subsemiring generated by the set $X \subseteq \mathscr{A}$ we shall denote by $\{X\}_+$.

Now suppose that

$$\begin{aligned} l_1 &= a_{11}x_1+a_{12}x_2+\beta_1k \quad (0 \leqslant \beta_1 < p),\\ l_2 &= a_{21}x_1+a_{22}x_2+\beta_2k \quad (0 \leqslant \beta_2 < p).\end{aligned}$$

Consider two situations:

(a) Suppose that

$$\Delta = \begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} \not\equiv 0 \pmod p.$$

In this case

$$\begin{aligned} l_1l_2 &= (a_{11}x_1+a_{12}x_2+\beta_1k)(a_{21}x_1+a_{22}x_2+\beta_2k)\\ &= a_{11}a_{22}x_1x_2+a_{12}a_{21}x_2x_1 = \Delta k.\end{aligned}$$

If $\Delta \not\equiv 0 \pmod p$, then it is clear that $k \in \{f(l_1), f(l_2)\}_+$. On the other hand,

$$f(l_1+l_2) = l_1+l_2+sk = (l_1+s_1k)+(l_2+s_2k)+sk-(s_1+s_2)k$$
$$= f(l_1)+f(l_2)+\bar{s}k \in \{f(l_1), f(l_2)\}_+ \quad (\bar{s} = s-s_1-s_2).$$

(b) Suppose that

$$(2) \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv 0 \pmod p.$$

Let us show that

$$(3) \qquad f(l_1+l_2) = f(l_1)+f(l_2).$$

The proof of this fact we shall split into a few steps.

(b_1) If $a_{11}a_{12} \equiv 0 \pmod p$, then either $a_{11} \equiv 0 \pmod p$ or $a_{12} \equiv 0 \pmod p$. If $a_{1i} \equiv 0 \pmod p$ $(i = 1, 2)$ for only one of the a_{1i}, then we have $a_{2i} \equiv 0 \pmod p$ $(i = 1, 2)$ for one of the a_{2i}; then we conclude from (1) that (3) is true.

Now if

$$a_{11} \equiv 0 \pmod p, \quad a_{12} \equiv 0 \pmod p, \quad a_{21}a_{12} \not\equiv 0 \pmod p,$$

then we have

$$f(l_1) = l_1, \quad f(l_2) = l_2+s_2k, \quad a_{21}+a_{22}+s_2 \equiv 0 \pmod p.$$

So $(a_{11}+a_{21})+(a_{12}+a_{22})+s_2 \equiv 0 \pmod p$, and consequently (3) is true.

(b_2) Suppose that $a_{11}a_{12} \not\equiv 0 \pmod p$. Then, if all the considerations of the previous case are true, only l_1 and l_2 change their parts.

(b_3) Now if $a_{11}a_{12} \not\equiv 0 \pmod p$ and $a_{21}a_{22} \not\equiv 0 \pmod p$, then

$$(4) \qquad f(l_1) = l_1+s_1k, \quad a_{11}+a_{12}+s_1 \equiv 0 \pmod p,$$

$$(5) \qquad f(l_2) = l_2+s_2k, \quad a_{21}+a_{22}+s_2 \equiv 0 \pmod p.$$

From (4) and (5) we find that $s_2 = s_1q \pmod p$. So

$$(a_{11}+a_{21})+(a_{12}+a_{22})+(s_1+s_2) \equiv 0 \pmod p.$$

On the other hand,

$$(a_{11}+a_{21})(a_{12}+a_{22}) \equiv 0 \pmod p \Leftrightarrow (1+q)^2a_{11}a_{12} \equiv 0 \pmod p$$
$$\Leftrightarrow 1+\lambda \equiv 0 \pmod p.$$

Thus $s_1+s_2 \equiv 0 \pmod p$. Consequently, (3) is true. We have shown that f induces a ΠS-automorphism on $\mathscr{A}$ and it is clear that f is neither an automorphism nor an anti-automorphism.

Example 1 gives negative answers to questions 1 and 3.

3. Answers to questions 2 and 3

Now we give an example of an n-nilpotent Lie ring which contains a proper n-nilpotent subring and the ΠS-automorphism of which is not induced either by an automorphism or by an anti-automorphism.

EXAMPLE 2. Let the Lie ring

$$\mathscr{B} = \{x_1, x_2, \ldots, x_{n+1}, k_1, k_2, \ldots, k_{n-3}, k\}$$

have the defining relations

$$x_ix_1 = x_{i+1}, \quad x_3x_2 = k_1, \quad pk = 0, \quad i = 1, 2, \ldots, n,$$
$$k_jx_2 = k_{j+1}, \quad k_{n-3}x_2 = k, \quad pk_j = 0, \quad j = 1, 2, \ldots, n-3$$

($n \geqslant 3$ and p is a prime number different from 2). We assume also that the relations which we have not written are trivial. All elements x_i are proper.

It is clear that $\mathscr{B}$ is an n-nilpotent ring, and $\{x_1, px_2\}$ is a proper n-nilpotent subring of $\mathscr{B}$.

Each element l of $\mathscr{B}$ has a unique expression in the form $l = a_1x_1+ + a_2x_2+y$, where $y \in [\mathscr{B}, \mathscr{B}]$. Define a one-to-one mapping $f\colon \mathscr{B} \to \mathscr{B}$ by formula (1), and let us show that f induces a ΠS-automorphism of $\mathscr{B}$, i.e. let us check that for each $l_1, l_2 \in \mathscr{B}$

$$f(l_1+l_2) = \omega(f(l_1), f(l_2)),$$

where ω is a two-wariable polynomial with positive coefficients. This fact implies that f associates a subsemiring with a subsemiring, and f induces a ΠS-automorphism.

As in the previous example, there is no need to check the same for the product, because

$$l_1l_2 \in [\mathscr{B}, \mathscr{B}] \Rightarrow f(l_1l_2) = f(l_1)f(l_2) = l_1l_2.$$

Suppose that

$$l_1 = a_{11}x_1+a_{12}x_2+y_1, \quad y_1 \in [\mathscr{B}, \mathscr{B}],$$
$$l_2 = a_{21}x_1+a_{22}x_2+y_2, \quad y_2 \in [\mathscr{B}, \mathscr{B}].$$

The situation where $\Delta = a_{11}a_{22}-a_{12}a_{21} \equiv 0 \pmod p$ is the same as situation (b) in Example 1, and in this case

$$f(l_1+l_2) = f(l_1)+f(l_2).$$

Now consider the situation where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \not\equiv 0 \pmod p.$$

Let us consider n-products [1]

$$c_1 = l_1 l_2 l_1 \ldots l_1 = (a_{11}x_1 + a_{12}x_2 + y_1)(a_{21}x_1 + a_{22}x_2 + y_2) \times$$
$$\times (a_{11}x_1 + a_{12}x_2 + y_1) \ldots (a_{11}x_1 + a_{12}x_2 + y_1)$$
$$= (a_{11}a_{22}x_1x_2 + a_{12}a_{22}x_2x_1)(a_{11}x_1 + a_{12}x_2)(\ldots (a_{11}x_1 + a_{12}x_2))$$
$$= (-a_{11}^2 a_{22}x_3x_1 - a_{12}a_{22}a_{12}x_3x_2 + a_{12}a_{21}a_{11}x_3x_1 + a_{12}a_{21}a_{12}x_3x_2) \times$$
$$\times (a_{11}x_1 + a_{12}x_2) \ldots (a_{11}x_1 + a_{12}x_2)$$
$$= -a_{11}^{n-1}a_{22}x_{n+1} - a_{11}a_{22}a_{12}^{n-2}k_{n-2} + a_{12}a_{21}a_{11}^{n-2}x_{n+1} + a_{12}a_{21}a_{12}^{n-2}k_{n-2}$$
$$= -a_{11}^{n-2}\Delta x_{n+1} + a_{12}^{n-2}\Delta k_{n-2}.$$

$$c_2 = l_1 l_2 l_2 \ldots l_2$$
$$= (a_{11}x_1 + a_{12}x_2 + y_1)(a_{21}x_1 + a_{22}x_2 + y_2) \ldots (a_{21}x_1 + a_{22}x_2 + y_2)$$
$$= -a_{21}^{n-2}a_{22}a_{11}x_{n+1} - a_{22}^{n-1}a_{11}k_{n-2} + a_{21}^{n-1}a_{12}x_{n+1} + a_{22}^{n-2}a_{21}a_{11}k_{n-2}$$
$$= -a_{21}^{n-2}\Delta x_{n+1} - a_{22}^{n-2}\Delta k_{n-2}.$$

Let us consider the determinant

$$\Delta_1 = \begin{vmatrix} a_{11}^{n-2} & a_{12}^{n-2} \\ a_{21}^{n-2} & a_{22}^{n-2} \end{vmatrix}.$$

If $\Delta_1 \not\equiv 0 \pmod p$, then

$$-c_1 a_{21}^{n-2} = \Delta(a_{11}^{n-2}a_{21}^{n-2}x_{n+1} + a_{12}^{n-2}a_{21}^{n-2}k_{n-2}),$$
$$-c_2 a_{11}^{n-2} = \Delta(a_{21}^{n-2}a_{11}^{n-2}x_{n+1} + a_{22}^{n-2}a_{11}^{n-2}k_{n-2}),$$
$$c_1 a_{21}^{n-2} - c_2 a_{11}^{n-2} = \Delta(a_{11}^{n-2}a_{22}^{n-2} - a_{12}^{n-2}a_{21}^{n-2})k_{n-2} = \Delta\Delta_1 k_{n-2}.$$

Using the anticommutativity of a Lie ring we have $-c_2, -c_1 \in \{l_1', l_2'\}_+$. Consequently,

$$\Delta\Delta_1 k_{n-2} \in \{f(l_1), f(l_2)\}_+.$$

Because the order of k_{n-2} is a prime number, we have

$$k_{n-2} \in \{f(l_1), f(l_2)\}_+.$$

On the other hand,

$$f(l_1 + l_2) = l_1 + l_2 + sk_{n-2}$$
$$= (l_1 + s_1 k_{n-2}) + (l_2 + s_2 k_{n-2}) + [s - (s_1 + s_2)]k_{n-2}$$
$$= f(l_1) + f(l_2) + \bar{s}k_{n-2},$$

where $\bar{s}, s, s_1, s_2 \in \{0, 1, \ldots, p-1\}$, $\bar{s} \equiv [s - (s_1 + s_2)] \pmod p$.

[1] The brackets are omitted.

Now let $\Delta_1 \equiv 0 \pmod p$. Then

$$a_{ik} \not\equiv 0 \pmod p \quad (i = 1, 2;\ k = 1, 2).$$

In fact, let $a_{11} \equiv 0 \pmod p$. From (7) we have

$$a_{22}a_{11} - a_{21}a_{12} \not\equiv 0 \pmod p \Rightarrow a_{21}a_{12} \not\equiv 0 \pmod p \Rightarrow a_{21} \not\equiv 0 \pmod p,$$
$$a_{12} \not\equiv 0 \pmod p.$$

On the other hand,

$$a_{11}^{n-2}a_{22}^{n-2} - a_{21}^{n-2}a_{12}^{n-2} \equiv 0 \pmod p \Rightarrow a_{21}^{n-2}a_{12}^{n-2} \equiv 0 \pmod p.$$

So we get a contradiction.

Let us consider the n-product

$$c_3 = l_1 l_2 l_2 \ldots l_2 l_1$$
$$= (a_{11}x_1 + a_{12}x_2 + y_1)(a_{21}x_1 + a_{22}x_2 + y_2) \times \ldots$$
$$\ldots \times (a_{21}x_1 + a_{22}x_2 + y_2)(a_{11}x_1 + a_{12}x_2 + y_1)$$
$$= -a_{11}a_{22}a_{21}^{n-3}a_{11}x_{n+1} - a_{11}a_{22}^{n-2}a_{12}k_{n-2} + a_{12}a_{21}^{n-2}a_{11}x_{n+1} + a_{12}a_{21}a_{22}^{n-3}a_{12}k_{n-2}$$
$$= a_{11}a_{21}^{n-3}\Delta x_{n+1} + a_{22}^{n-3}a_{11}\Delta k_{n-2}.$$

Let us consider the difference

$$a_{21}^{n-3}c_3 - a_{21}^{n-3}a_{11}c_2$$
$$= a_{21}^{n-2}(a_{11}a_{21}^{n-3}\Delta x_{n+1} + a_{22}^{n-2}a_{12}\Delta k_{n-2}) - a_{21}^{n-3}a_{11}(a_{21}^{n-2}\Delta x_{n+1} + a_{22}^{n-2}\Delta k_{n-2})$$
$$= a_{21}^{n-2}a_{22}^{n-2}a_{12}\Delta k_{n-2} - a_{21}^{n-3}a_{11}a_{22}^{n-2}\Delta^2 k_{n-2}$$
$$= a_{21}^{n-3}a_{22}^{n-3}\Delta k_{n-2}.$$

Consequently, $k_{n-2} \in \{c_2, c_3\}_+$ and moreover

$$k_{n-2} \in \{f(l_1), f(l_2)\}_+.$$

In a similar way we conclude that

$$f(l_1 + l_2) = f(l_1) + f(l_2) + \bar{s}k_{n-2}, \qquad 0 \leqslant \bar{s} < p.$$

So f associates each subsemiring with a subsemiring, i.e., induces a ΠS-automorphism, and it is clear that f is neither an automorphism nor an anti-automorphism.

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HOW MANY FOUR-GENERATED SIMPLE LATTICES?

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We call a lattice L *simple* if $|L| > 1$ and L has no nontrivial congruence relations.

For which partially ordered sets P is there a simple lattice generated by P? There is, for instance, precisely one simple lattice generated by the two-element chain, namely, the two-element chain itself. This is the smallest simple lattice. On the other hand, the lattice generated by an n-element chain is not simple if $n \geqslant 3$. Still, a partially ordered set consisting of n elements pairwise noncomparable can generate a simple lattice just as long as $n \geqslant 3$ (for example, the $(n+2)$-element modular lattice of length two).

Let P be the partially ordered set consisting of pairwise noncomparable elements a, b, c and let L be a simple lattice generated by P. If $a \nleqslant b \vee c$, say, then L is the disjoint union of $\{x \in L | \ x \geqslant a\}$ and $\{x \in L | \ x \leqslant b \vee c\}$, whence L has a homomorphism onto the two-element chain. It follows that $a \leqslant b \vee c$. By symmetry and duality L must be the five-element modular lattice of length two. This observation, first recorded by R. Wille [14], shows that there is precisely *one* simple lattice generated by a three-element unordered set (antichain).

Interest in simple lattices generated by an antichain was revitalized by H. Strietz [12] who showed that every lattice of partitions on a finite set with at least four elements is generated by a four-element antichain. There are then at least countably many simple lattices generated by a four-element antichain. Actually there are more.

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