Extended abstract, presented at the Ninth A. C. M. Symposium on Theory of Computing, Boulder, Colorado, May 2–4, 1977.

[3] G. E. Hughes and M. J. Cresswell, *Introduction to modal logic*, Methuen, London 1968.

[4] E. J. Lemmon, in collaboration with Dana Scott, *Introduction to modal logic*, *American Philosophical Quarterly*, monograph series, vol. 11, Blackwell, Oxford 1977.

[5] Rohit Parikh, *A completeness result for a propositional dynamic logic*, Forthcoming.

[6] V. R. Pratt, *Semantical considerations on Floyd–Hoare logic*, in: *17th I. E. E. E. Symposium on Foundations of Computer Science* (1977), 109–121.

[7] —, *The tableau method for propositional dynamic logic*, Manuscript.

[8] —, *Logic of processes*, Manuscript.

[9] Arthur Prior, *Past, present and future*, Clarendon Press, Oxford 1967.

[10] Krister Segerberg, *Decidability of S4.1*, Theoria 34 (1968), 7–20.

[11] —, *An essay in classical modal logic*, Uppsala: The Philosophical Society and the Department of Philosophy, University of Uppsala, 1971.

[12] —, *von Wright's tense logic*, Completed in 1974 for *The philosophy of Georg Henrik von Wright*, to be edited by P. A. Schilpp.

[13] —, *A completeness theorem in the modal logic of programs*, Notices of the American Mathematical Society 24 (1977), A–552, no. 77T–E69.

*Presented to the Semester*
*Universal Algebra and Applications*
*(February 15 – June 9, 1978)*

UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

# BROUWERIAN SEMILATTICES: THE LATTICE OF TOTAL SUBALGEBRAS

PETER KÖHLER

*Institute of Mathematics, Justus Liebig University, Giessen, F.R.G.*

Any Brouwerian semilattice $S$ can be viewed as a (meet-) semilattice acting on itself, the action being relative pseudo-complementation. This may be formalized by considering $S$ as a (universal) algebra with one binary operation (meet) and for every $a \in S$ a unary operation. The subalgebra lattice of this algebra is the main topic of this paper: It is shown that it is a maximal distributive sublattice of the subalgebra lattice of $S$ considered as an (ordinary) Brouwerian semilattice; Brouwerian semilattices are characterized for which this lattice is Boolean. The question which distributive algebraic lattices can be represented this way is left as an open problem.

## 1. Preliminaries

A *Brouwerian semilattice* is an algebra $\langle S, \wedge, *, 1\rangle$, where $\langle S, \wedge, 1\rangle$ is a meet-semilattice with the greatest element 1, and where the binary operation $*$ is relative pseudocomplementation, i.e. $z \leqslant x*y$ holds for elements $x, y, z \in S$ if and only if $z \wedge x \leqslant y$. Following the usual practice we will mostly identify the Brouwerian semilattice $\langle S, \wedge, *, 1\rangle$ with the underlying set $S$.

For the basic arithmetic of Brouwerian semilattices we refer to [4], [7]. Let us recall the following rules of computation:

For all $x, y, z \in S$:

(1) $x \leqslant y \Leftrightarrow x*y = 1$,

(2) $1*x = x$,

(3) $x*y \geqslant y$,

(4) $x \wedge x*y = x \wedge y$,

(5) $(x \wedge y)*z = x*(y*z)$,

(6) $x*(y\wedge z) = x*y \wedge x*z,$

(7) $(x*y)*y \geqslant x,$

(8) $x \leqslant y \Rightarrow y*z \leqslant x*z$ and $z*x \leqslant z*y,$

(9) $(x*y)*x \leqslant (x*y)*y,$

(10) $((x*y)*y)*y = x*y,$

(11) $(x*y\wedge y*x)*x = (x*y \wedge y*x)*y.$

Rules (1) through (8) may be found in [7]; the proof of (9), (10), (11) is left to the reader. We will often use these rules without special reference.

A *filter* of a Brouwerian semilattice $S$ is a subalgebra, which is also an upper end. The importance of filters in the theory of Brouwerian semilattices rests in the fact that filters of $S$ are in 1-1-correspodence with congruence relations on $S$. To be more precise: If $F$ is a filter of $S$, then the relation $\theta_F$ with

$$x\,\theta_F\,y \Leftrightarrow x*y\wedge y*x \in F$$

is a congruence relation on $S$ and the mapping $F \to \theta_F$ is an isomorphism from $\mathscr{F}(S)$, the lattice of filters of $S$ onto the congruence lattice of $S$, see e.g. [6].

To mention an important class of Brouwerian semilattices — which are not necessarily lattices — we refer to [1], p. 182: Let $P$ be a poset, define $2^{(P)}$ to be the set of all finite antichains in $P$, ordered by

$$A \leqslant B \Leftrightarrow \forall b\in B\ \exists a \in A:\ b \leqslant a.$$

Then $2^{(P)}$ is easily seen to be a Brouwerian semilattice; note that we had to choose the ordering dual to [1].

An element $m$ of a (Brouwerian) semilattice is *meet-irreducible* if $m = x\wedge y$ implies $m = x$ or $m = y$. Obviously an anti-chain $A$ in $2^{(P)}$ is meet-irreducible if and only if $|A| = 1$. Thus the meet-irreducibles of $2^{(P)}$ form a subposet isomorphic to the dual of $P$, and each element of $2^{(P)}$ is a finite meet of meet-irreducibles. Conversely if $S$ is a Brouwerian semilattice such that every element is a finite meet of meet-irreducibles, then $S \cong 2^{(P)}$, where $P$ may be taken to be the dual of the poset of meet-irreducible elements of $S$.

For other notions from Lattice Theory we refer to [1].

## 2. Meet-irreducible elements and total subalgebras

A subalgebra $T$ of a Brouwerian semilattice $S$ is called a *total subalgebra* if $s*t \in T$ for every $t\in T$ and every $s \in S$. This notion was introduced by Nemitz [6]. More recently total subalgebras occurred — under the name Brouwerian subacts — in the theory of quasi-decompositions of



Brouwerian semilattices, as developed by J. Schmidt ([8], [9]) and applied by the author in [2]. This connection will become more transparent in Section 3.

Clearly every filter of a Brouwerian semilattice $S$ is a total subalgebra of $S$. To produce a larger class of examples, we recall the following lemma:

LEMMA 2.1 ([11]). *Let $S$ be a Brouwerian semilattice, let $m \in S$. Then $m$ is meet-irreducible if and only if $m \neq 1$ and for every $s\in S$: $s*m = m$ or $s*m = 1$.*

For a Brouwerian semilattice $S$ let $M(S)$ be the set of its meet-irreducible elements. For any subset $X$ of $M(S)$ let $L_X(S)$ be the subsemilattice generated by $X$:

$$L_X(S) = \{\bigwedge X_f|\ X_f \subseteq X,\ |X_f| < \omega\}.$$

In particular note that $L_\emptyset(S) = \{1\}$.

LEMMA 2.2. *Let $S$ be a Brouwerian semilattice. Then for every $X \subseteq M(S)$ $L_X(S)$ is a total subalgebra of $S$.*

*Proof.* Clearly $L_X(S)$ is meet-closed and $1 \in L_X(S)$. So let $s\in S$, $X_f$ a finite subset of $X$. Then $s*\bigwedge X_f = \bigwedge\{s*x|\ x \in X_f\} = \bigwedge\{x|\ x\in X_f,\ s \nleqslant x\}$ by Lemma 2.1. Hence $s*\bigwedge X_f \in L_X(S)$. Moreover, since $L_X(S) \subseteq S$, we see that $L_X(S)$ is also $*$-closed.

As observed in the proof of this lemma, a subset $T$ of a Brouwerian semilattice $S$ is a total subalgebra of $S$ if and only if $T$ is a subalgebra of $\langle S, \wedge, 1, \langle\varphi_s\rangle_{s\in S}\rangle$, where for each $s \in S$ the unary operation is given by $\varphi_s(x) = s*x$. Consequently, the total subalgebras of $S$ form an algebraic lattice, which we will denote by $\mathscr{T}(S)$.

For any subset $X \subseteq S$ let

$$S*X = \{1\}\cup\{\bigwedge_{i=1}^{n} s_i*x_i|\ 1\leqslant n<\omega,\ s_i\in S,\ x_i \in X\}.$$

In particular we will write $S*x$ instead of $S*\{x\}$. The next lemma shows that $S*X$ is the closure of $X$ in the algebraic closure system $\mathscr{T}(S)$.

LEMMA 2.3. *Let $S$ be a Brouwerian semilattice, let $X \subseteq S$. Then $S*X$ is the smallest total subalgebra containing $X$.*

*Proof.* Without loss of generality we may assume $X \neq \emptyset$. Clearly any total subalgebra containing $X$ must contain $S*X$. Obviously, $X\subseteq S*X$, so it suffices to show that $S*X$ is a total subalgebra of $S$. $S*X$ is meet-closed by definition. So let $s, s_1, \ldots, s_n \in S$, $x_1, \ldots, x_n \in X$. Then

$$s*\bigwedge_{i=1}^{n} s_i*x_i = \bigwedge_{i=1}^{n} s*(s_i*x_i) = \bigwedge_{i=1}^{n}(s\wedge s_i)*x_i \in S*X.$$

Thus $S*X$ is a total subalgebra and the lemma is proven.

As an immediate consequence we have:

COROLLARY 2.4. *Let $S$ be a Brouwerian semilattice. Let $T, U \in \mathscr{T}(S)$. Then $T \vee U = \{t \wedge u \mid t \in T, u \in U\}$.*

This shows that $\mathscr{F}(S)$ is a sublattice of $\mathscr{T}(S)$. Moreover, for total subalgebras $T, U$ of $S$ the join in $\mathscr{S}(S)$ — the subalgebra lattice of $S$ — coincides with the join in $\mathscr{T}(S)$, so that $\mathscr{T}(S)$ is a sublattice of $\mathscr{S}(S)$. But more importantly we have:

THEOREM 2.5. *Let $S$ be a Brouwerian semilattice. Then $\mathscr{T}(S)$ is distributive.*

*Proof*. Let $T, U, V \in \mathscr{T}(S)$. It suffices to show that

$$T \cap (U \vee V) \subseteq (T \cap U) \vee (T \cap V).$$

So let $t \in T \cap (U \vee V)$, i.e. $t = u \wedge v$ for some $u \in U$, $v \in V$. Now $u*t = u*v$, hence $u*t \in T \cap V$; moreover $(u*t)*t = (u*v)*u \wedge (u*v)*v = (u*v)*u$, hence $(u*t)*t \in T \cap U$. Finally $t = (u*t)*t \wedge u*t$, and thus $t \in (T \cap U) \vee (T \cap V)$.

There are three questions which arise quite naturally in this context. Firstly, in general the lattice of subalgebras of a Brouwerian semilattice is highly non-distributive. So one might ask whether $\mathscr{T}(S)$ is a maximal distributive sublattice of $\mathscr{S}(S)$. Secondly, for which Brouwerian semilattices $S$ does the equality $\mathscr{T}(S) = \mathscr{F}(S)$ hold. The third one is much more challenging, and we formulate it as a problem:

PROBLEM. *Characterize distributive lattices which are isomorphic to $\mathscr{T}(S)$ for some Brouwerian semilattice $S$.*

We will answer the first two questions; to deal with the first one we need a simple but useful lemma.

LEMMA 2.6. *Let $S$ be a Brouwerian semilattice, let $a, b \in S$. Then*

$$S*a \cap S*b \subseteq S*\big((a*b)*b \wedge (b*a)*a\big).$$

*Proof*. Let $x \in S*a \cap S*b$. By Lemma 2.3, we conclude

$$x = \bigwedge_{i=1}^{n} s_i*a = \bigwedge_{j=1}^{m} t_j*b$$

for some $1 \leqslant n, m < \omega$; $s_1, \ldots, s_n, t_1, \ldots, t_m \in S$.

Consequently, for any $i, j$ we have:

$$s_i \wedge t_j \wedge x = s_i \wedge t_j \wedge a = s_i \wedge t_j \wedge b$$

and therefore $s_i \wedge t_j \leqslant a*b \wedge b*a$. Moreover $s_i*x = s_i*a$ and $t_j*x = t_j*b$.



This implies:

$$\begin{aligned}
\bigwedge_{(i,j)} (s_i \wedge t_j)*\big((a*b)*b \wedge (b*a)*a\big) &= \bigwedge_{(i,j)} (s_i \wedge t_j)*b \wedge \bigwedge_{(i,j)} (s_i \wedge t_j)*a \\
&= \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} s_i*(t_j*b) \wedge \bigwedge_{j=1}^{m} \bigwedge_{i=1}^{n} t_j*(s_i*a) \\
&= \bigwedge_{i=1}^{n} s_i* \bigwedge_{j=1}^{m} t_j*b \wedge \bigwedge_{j=1}^{m} t_j* \bigwedge_{i=1}^{n} s_i*a \\
&= \bigwedge_{i=1}^{n} s_i*x \wedge \bigwedge_{j=1}^{m} t_j*x = \bigwedge_{i=1}^{n} s_i*a \wedge \bigwedge_{j=1}^{m} t_j*b \\
&= x \wedge x = x.
\end{aligned}$$

Hence $x \in S*\big((a*b)*b \wedge (b*a)*a\big)$ and the lemma is proven.

Note that equality need not hold: To see this let $a, b \in S$, $a < b$, and assume that $a$ is meet-irreducible. Then

$$S*\big((a*b)*b \wedge (b*a)*a\big) = S*b, \quad \text{but } b \notin S*a.$$

However, equality has to hold as soon as $a, b$ are disjoint in the sense that $(a*b)*b \wedge (b*a)*a = 1$. This will be used to prove the maximality of $\mathscr{T}(S)$.

THEOREM 2.7. *Let $S$ be a Brouwerian semilattice. Then $\mathscr{T}(S)$ is a maximal distributive sublattice of $\mathscr{S}(S)$.*

*Proof*. Let $U$ be a subalgebra of $S$ which is not total. Then there exist $u \in U$, $s \in S$ such that $s*u \notin U$. Put $a = s*u$, $b = (s*u)*u$. Then $a \wedge b = u$, $a*u = a*b = b$, $b*u = b*a = a$. In particular, $a, b \notin U$. Moreover, $(a*b)*b \wedge (b*a)*a = 1$, thus, by Lemma 2.6, $S*a \cap S*b = \{1\}$. This implies that $(S*a \cap S*b) \vee U = U$. On the other hand we have $\{a, b\} \subseteq S*a \vee U$ since $a*u = b$, and also $\{a, b\} \subseteq S*b \vee U$ since $b*u = a$. This shows that $\{a, b\} \subseteq (S*a \vee U) \cap (S*b \vee U)$, and hence $(S*a \vee U) \cap \cap (S*b \vee U) \neq U$.

Consequently any sublattice of $\mathscr{S}(S)$ containing $\mathscr{T}(S)$ properly is not distributive.

We add an easy observation.

COROLLARY 2.8. *Let $S$ be a Brouwerian semilattice. Then the following conditions are equivalent*:

(i) *$\mathscr{S}(S)$ is distributive.*

(ii) *$\mathscr{S}(S) = \mathscr{T}(S)$.*

(iii) *$S$ is a chain.*

*Proof.* (i) ⇒ (ii) follows from Theorem 2.7. (iii) ⇒ (i) follows from the fact that $\mathcal{S}(S) \cong \mathcal{P}(S\setminus\{1\})$ in case $S$ is a chain. So it suffices to prove (ii) ⇒ (iii). Let $a, b \in S$ such that $a \not\leqslant b$. Then also $a \not\leqslant a\wedge b$. Moreover $\{b, 1\}$, $\{a\wedge b, 1\} \in \mathcal{T}(S)$ and hence $a\wedge b = a*(a\wedge b) = a*b = b$. This shows that $b \leqslant a$, and so $S$ is a chain.

Note, however, that this could have been proven without referring to Theorem 2.7.

Now we will turn our attention to the second question. A Brouwerian semilattice $S$ will be called a *generalized Boolean algebra* provided the identity

$$(x*y)*y = (y*x)*x \tag{12}$$

holds in $S$. This definition is in accordance with the usual understanding of a generalized Boolean algebra — sometimes the dual notion is used — since Brouwerian semilattices satisfying the identity (12) are exactly the relatively complemented distributive lattices with a greatest element, $a*b$ being the complement of $a$ in the principal filter $[a\wedge b]$, while the join of $a$ and $b$ is given by $(a*b)*b$ $(= (b*a)*a)$. See also [5] for a more general discussion.

THEOREM 2.9. *Let $S$ be a Brouwerian semilattice. Then $\mathcal{T}(S) = \mathcal{F}(S)$ if and only if $S$ is a generalized Boolean algebra.*

*Proof.* Suppose that $S$ is a generalized Boolean algebra. Let $T \in \mathcal{T}(S)$. We show that $T$ is a filter. So let $a \in T$, $b \geqslant a$. Then $b = (a*b)*b = (b*a)*a$, and so $b \in T$. This shows $\mathcal{T}(S) = \mathcal{F}(S)$.

Conversely assume that $\mathcal{T}(S) = \mathcal{F}(S)$. Let $a, b \in S$. Now $S*a$ is a filter, hence $(a*b)*b \in S*a$. Thus there exist $s_1, \ldots, s_n \in S$ such that $(a*b)*b = \bigwedge_{i=1}^{n} s_i*a$. As a consequence,

$$1 = b*\bigwedge_{i=1}^{n} s_i*a = \bigwedge_{i=1}^{n} s_i*(b*a).$$

This implies that $s_i \leqslant b*a$ for $1 \leqslant i \leqslant n$, and hence $s_i*a \geqslant (b*a)*a$. So we have $(a*b)*b \geqslant (b*a)*a$. Interchanging $a$ and $b$ gives the desired equality. Thus $S$ is a generalized Boolean algebra.

Concerning the open problem we can narrow down the class of distributive lattices which occur as lattices of total subalgebras: Every finite member of it must be Boolean. In fact we prove a bit more:

THEOREM 2.10. *Let $S$ be a Brouwerian semilattice such that every element of $S$ is a finite meet of meet-irreducibles. Then $\mathcal{T}(S)$ is Boolean, in fact $\mathcal{T}(S) \cong \mathcal{P}(M(S))$, the power set lattice of $M(S)$.*

*Proof.* Recall that any complete infinitely distributive lattice $L$ such that 1 is a join of atoms is Boolean, even isomorphic to $\mathcal{P}(A)$, where $A$ is the set of atoms of $L$. As an algebraic distributive lattice $\mathcal{T}(S)$ is clearly infinitely distributive. Moreover $\{m, 1\}$ is an atom of $\mathcal{T}(S)$ for each $m \in M(S)$. Since each element of $S$ is a finite meet of meet-irreducibles, we infer from Lemma 2.3 that $S = \bigvee\big\{\{m, 1\} \mid m \in M(S)\big\}$. Moreover any atom of $\mathcal{T}(S)$ must be of the form $\{m, 1\}$ for some $m \in M(S)$. Thus $\mathcal{T}(S) \cong \mathcal{P}\big(M(S)\big)$.



This generalizes a result of Macnab ([3], Thm. 6.8), who showed that $\mathcal{T}(S)$ is Boolean for each finite $S$.

In the following section we will show that the converse of this theorem also holds.

## 3. Total subalgebras which have complements

We start with a lemma on the arithmetic properties of "disjoint" total subalgebras.

LEMMA 3.1. *Let $S$ be a Brouwerian semilattice, let $T$, $U$ be total subalgebras of $S$ such that $T\cap U = \{1\}$. Then for any $t, r \in T$, $u, v \in U$:*

(i) $u\wedge t = t*u \wedge u*t$,

(ii) $u\wedge t \in U \Rightarrow u\wedge t = u$,

(iii) $u\wedge t = v\wedge r \Rightarrow t*u = r*v$.

*Proof.* (i) Due to equation (11) we have

$$(t*u\wedge u*t)*t = (t*u\wedge u*t)*u \in T\cap U = \{1\}.$$

Hence $t*u\wedge u*t \leqslant u\wedge t$. Since $t*u\wedge u*t \geqslant u\wedge t$ anyway, we have equality.

(ii) Let $u\wedge t \in U$, then $u*(u\wedge t) = u*t \in U\cap T = \{1\}$, and hence $u \leqslant t$. This shows $u\wedge t = u$.

(iii) By (i) we have $u\wedge t = t*u\wedge u*t = r*v\wedge v*r = v\wedge r$. Thus $t*u = t*(r*v)\wedge t*(v*r)$. Now $t*(r*v) \in U$ and $t*(v*r) \in T$; by (ii) this implies $t*u = t*(r*v) \geqslant r*v$. The other inequality follows by symmetry.

As an algebraic distributive lattice $\mathcal{T}(S)$ is pseudo-complemented — even relatively pseudo-complemented. It follows from the lemma that for a total subalgebra $T$ of $S$ we must have:

$$T^* \subseteq \{u \mid u\wedge t = t*u\wedge u*t \text{ for all } t \in T\}.$$

A tedious calculation shows that even equality holds. More importantly, however, this lemma allows us to characterize total subalgebras which have complements.

THEOREM 3.2. *Let $S$ be a Brouwerian semilattice, let $T \in \mathcal{T}(S)$. Then $T$ has a complement if and only if for each $x \in S$ the set $\{t*x \mid t \in T\cap[x)\}$ has a greatest element.*

*Proof*. Suppose that $T$ has a complement, say $U$. Let $x \in S$, then there exist $t_0 \in T$, $u_0 \in U$ such that $x = t_0 \wedge u_0$. We claim that $t_0 * x = \max\{t*x \mid t \in T \cap [x)\}$. Clearly $t_0 * x$ belongs to this set. So let $r \in T \cap [x)$. Then $r*x \leqslant (r \wedge t_0)*x$. Moreover $r \wedge t_0 \in T$ and $r \wedge t_0 \wedge u_0 = r \wedge x = x = t_0 \wedge u_0$. Hence by Lemma 3.1 (iii) $(r \wedge t_0)*u_0 = t_0 * u_0$, and so in particular

$$(r \wedge t_0)*x = (r \wedge t_0)*u_0 = t_0 * u_0 = t_0 * x.$$

This shows $r*x \leqslant t_0 * x$.

Conversely let $j(x) = \max\{t*x \mid t \in T \cap [x)\}$. In particular $j(x) = t_0 * x$ for some $t_0 \in T \cap [x)$. Consequently $j(x) \geqslant x$. Moreover $j$ is idempotent. To see this let $r \in T \cap [j(x))$. Then $r \geqslant j(x) \geqslant x$, and so $r*x \leqslant j(x)$. Hence $r*j(x) = r*(t_0*x) = t_0*(r*x) \leqslant t_0*j(x) = j(x)$. This shows $r*j(x) = j(x)$ and thus $j(j(x)) = j(x)$. This means that $j$ is a weak closure operator — in the terminology of J. Schmidt [10]. Let $U = j(S) = \{x \mid j(x) = x\}$. Observe that $T = \{x \mid j(x) = 1\}$. We proceed in a number of steps.

(i) $T \cap U = \{1\}$ follows immediately.

(ii) For each $x \in S$ there exist $t \in T$, $u \in U$ such that $x = t \wedge u$. For if $j(x) = t_0 * x$ then $x = t_0 \wedge j(x)$.

(iii) For every $s \in S$, $u \in U$ we have $s*u \in U$. Let $t \in T$, $t \geqslant s*u$. Then $t \geqslant u$ and so $t*u = u$. Thus

$$t*(s*u) = s*(t*u) = s*u.$$

Hence $j(s*u) = s*u$.

(iv) Let $V$ be the total subalgebra generated by $U$. In view of Lemma 2.3 and (iii) we have

$$V = \{u_1 \wedge \ldots \wedge u_n \mid 1 \leqslant n < \omega,\ u_1, \ldots, u_n \in U\}.$$

Then $V \in \mathscr{T}(S)$ and by (ii) $V \vee T = S$. We claim $V \cap T = \{1\}$. Suppose not, and let $u_1, \ldots, u_n \in U$ such that $u_1 \wedge \ldots \wedge u_n \neq 1$, $u_1 \wedge \ldots \wedge u_n \in V \cap T$. By (i) we have $n \geqslant 2$. Moreover let $n$ be minimally chosen, i.e. $u_1 \wedge \ldots \wedge u_{n-1} \notin V \cap T$. Then by (iii)

$$(u_1 \wedge \ldots \wedge u_{n-1})*(u_1 \wedge \ldots \wedge u_n) = (u_1 \wedge \ldots \wedge u_{n-1})*u_n \in U \cap T.$$

Hence (i) shows $(u_1 \wedge \ldots \wedge u_{n-1})*u_n = 1$, and so $u_1 \wedge \ldots \wedge u_{n-1} \leqslant u_n$. In particular $u_1 \wedge \ldots \wedge u_{n-1} = u_1 \wedge \ldots \wedge u_n$. This is a contradiction. So we have $V \cap T = \{1\}$ and $V$ is the complement of $T$.

*Remark*. Knowing that $V$ is the complement of $T$ we can even infer that $V = U$, i.e. $U$ is also meet-closed, which we have not been able to prove directly. In fact, let $v \in V$, and let $t \in [v) \cap T$. Then $v = t \wedge v = v*t \wedge t*v = t*v$ by Lemma 3.1 (i). Hence $j(v) = v$ and so $v \in U$.



For filters we obtain:

COROLLARY 3.3. *Let $S$ be a Brouwerian semilattice, let $F$ be a filter of $S$. Then $F$ has a complement in $\mathscr{T}(S)$ if and only if for each $x \in S$ the set $\{f*x \mid f \in F\}$ has a greatest element.*

*Proof*. It suffices to show that for a filter $F$ the sets $\{f*x \mid f \in F \cap [x)\}$ and $\{f*x \mid f \in F\}$ coincide. In fact, let $f \in F$, then $(f*x)*x \in F$ and $f*x = ((f*x)*x)*x$.

This result is known; see e.g. [3], Thm. 5.13. We should also point out the connections to [8], [9]: Denote the congruence class of $x$ modulo $F$ by $[x]_F$. Then $f*x \in [x]_F$ for every $f \in F$, and if $y \in [x]_F$ then $y*x \in F$ and $(y*x)*x \geqslant y$. So $\max\{f*x \mid f \in F\} = \max [x]_F$, i.e. each congruence class has a greatest element. Such filters are called *comonomial.* Moreover it follows that the map $j$ as defined in the proof of Theorem 3.2 is also an endomorphism of $S$, so that we have a split exact sequence:

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & F & \longrightarrow & S & \xrightarrow{\ j\ } & U & \longrightarrow & \{1\}. \\ & & & & \nwarrow & & \swarrow & & \\ & & & & & U & & & \end{array}$$

In other words, $F$ and $U$ yield a *quasi-decomposition* of $S$.

Finally, we have collected enough to prove the converse of Theorem 2.10.

THEOREM 3.4. *Let $S$ be a Brouwerian semillattice. Then the following conditions are equivalent:*

(i) *$\mathscr{T}(S)$ is Boolean.*

(ii) *Every filter of $S$ is a comonomial.*

(iii) *$S \cong 2^{(P)}$ for some poset $P$.*

*Proof*. (i) ⇒ (ii) follows from Corollary 3.3, (iii) ⇒ (i) is Theorem 2.10. It remains to prove (ii) ⇒ (iii). We show that (ii) implies that every element of $S$ is a finite meet of meet-irreducibles. By way of contradiction assume that there is some $x \in S$ which is not a finite meet of meet-irreducibles. In particular $x \neq 1$ and $x$ is not meet-irreducible. Hence there exist — by Lemma 2.1 — $a \in S$ such that $x < a*x < 1$. Consequently also $x < (a*x)*x < 1$. Since $x = a*x \wedge (a*x)*x$, at least one of these elements cannot be a finite meet of meet-irreducibles. Summarizing, there exists $b_0 \in S$ such that $x < b_0*x < 1$ and $b_0*x$ is not a finite meet of meet-irreducibles. The same reasoning shows that there exists $c \in S$ such that $b_0*x < c*(b_0*x) < 1$ and $c*(b_0*x) = (c \wedge b_0)*x$ is not a finite meet of meet-irreducibles. Put $b_1 = c \wedge b_0$. Repeating this process — with the use of the axiom of choice — we get a decreasing sequence $b_0 > b_1 > b_2 > \ldots$ such that $x < b_0*x < b_1*x < b_2*x < \ldots < 1$. Let $F$ be the filter generated by $\{b_i|\ i < \omega\}$, i.e. $F = \{s|\ \exists i < \omega\colon s \geqslant b_i\}$. By (ii) $F$ is comonomial, so let $f \in F$ such that $f*x = \max[x]_F$. Firstly $f \geqslant b_i$ for some $i < \omega$. Then $f*x \leqslant b_i*x < b_{i+1}*x$. On the other hand $b_{i+1} \in F$ and so $b_{i+1}*x \leqslant f*x$. This is a contradiction and so our assumption on $x$ must be false. This proves the theorem.

## Acknowledgements

The results of this paper have been presented at the Banach-Semester on Algebra and Applications, Spring 1978 in Warsaw. This visit has been supported by the Stefan Banach International Mathematical Center and by the Deutsche Forschungsgemeinschaft. Thanks are due to the participants of the Semester, in particular to Stan Zachorowski; discussions with him eventually led to the proof of Theorem 2.7.

## References

[1] G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Coll. Publ., Vol. 25, Providence 1967.
[2] P. Köhler, *The semigroup of varieties of Brouwerian semilattices*, Trans. Amer. Math. Soc. 241 (1978), 331–342.
[3] D. S. Macnab, *An algebraic study of modal operators on Heyting algebras with applications to topology and sheafification*, Ph. D. thesis, Univ. of Aberdeen, 1976.
[4] W. C. Nemitz, *Implicative semi-lattices*, Trans. Amer. Math. Soc. 117 (1965), 128–142.
[5] —, *Semi-Boolean lattices*, Notre Dame J. Formal Logic 10 (1969), 235–238.
[6] —, *Implicative homomorphisms with finite ranges*, Proc. Amer. Math. Soc. 33 (1972), 319–322.
[7] H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, 3rd ed., Monografie Matematyczne, Tom 41, PWN, Warszawa 1970.
[8] J. Schmidt, *Quasi-decompositions, exact sequences and triple sums of semigroups I. General Theory*, Contributions to Universal Algebra, Szeged, Colloq. Math. Soc. Jànos Bolyai 17 (1975), 365–398.
[9] —, *Quasi-decompositions, exact sequences and triple sums of semigroups II. Applications*, ibidem 17 (1975), 399–428.
[10] —, *Binomial pairs, semi-Brouwerian and Brouwerian semilattices*, Notre Dame J. Formal Logic 19 (1978), 421–434.
[11] D. Smith, *Meet-irreducible elements in implicative lattices*, Proc. Amer. Math. Soc. 34 (1972), 57–62.

*Presented to the Semester*
*Universal Algebra and Applications*
*(February 15 – June 9, 1978)*

UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

# PROJECTIONS OF MIXED LIE RINGS

ALEXANDER A. LAŠHI

*Institute of Mathematics, Georgian SSR Academy of Sciences, Tbilisi, U.S.S.R.*

## Introduction

The aim of this article is the study of the lattice isomorphisms (projections) of Lie rings. We will make use of generally accepted terminology (see, for example, [6], [2]).

*Notation.* $S(\mathscr{L})$ is the lattice of all subrings of $\mathscr{L}$; $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ will denote a lattice isomorphism; $\mathscr{A}^\varphi \subseteq \mathscr{L}^\varphi$ will denote the image of the subalgebra $\mathscr{A} \subseteq \mathscr{L}$ under $\varphi$; $N(\mathscr{A})$, $[\mathscr{A}, \mathscr{A}]$ will denote the normalizer and the commutator, respectively, of $\mathscr{A} \subseteq \mathscr{L}$; $Z(\mathscr{L})$ is the centre of $\mathscr{L}$; $C_{\mathscr{A}}(X)$ is the centralizator of $X$ in $\mathscr{A} \subseteq \mathscr{L}$; $\boldsymbol{Z}$ is the ring of real integers; $\{X\}$ denotes the subring generated by $X$.

An element $a \in \mathscr{L}$ will be called *proper* if $\alpha a \neq 0$ for every $\alpha \in \boldsymbol{Z}$ $(\alpha \neq 0)$; otherwise, it will be called *periodic*. The ring $\mathscr{L}$ is *proper* if all its elements are proper; it will be called *mixed* (or *nonperiodic*) if it contains both the proper and periodic elements, and it will be called *periodic* if all its elements are periodic. The set of all the periodic elements of $\mathscr{L}$ will be denoted by $t(\mathscr{L})$. It is clear that $t(\mathscr{L})$ is an ideal in $\mathscr{L}$. The *dimension* of $\mathscr{L}$, denoted by $\dim\mathscr{L}$, is defined to be the maximal number of linearly independent elements. It is clear that $\dim\big(\mathscr{L}/t(\mathscr{L})\big) = \dim\mathscr{L}$.

We say that the ring $\mathscr{L}$ is *determined* (*strictly determined*) by $S(\mathscr{L})$ if $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ implies $\mathscr{L} \cong \mathscr{L}^\varphi$ ($\varphi$ is induced by an isomorphism between $\mathscr{L}$ and $\mathscr{L}^\varphi$).

A lattice isomorphism $\varphi\colon S(\mathscr{L}) \to S(\mathscr{L}^\varphi)$ is called *normal* if $N(\mathscr{A}) = N(\mathscr{A}^\varphi)$ for each subring $\mathscr{A} \subseteq \mathscr{L}$.

In Section 1 we prove an analogy of a theorem of A. S. Pekelis [1]. In Sections 2 and 3, with the help of some ideas from [1], [4], we construct examples which give negative answers to natural questions in connection with the theorem of Section 1 and theorems from [3], [4].