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SOME OPEN PROBLEMS AND DIRECTIONS FOR FURTHER RESEARCH IN SEMIGROUPS OF CONTINUOUS SELFMAPS

K. D. MAGILL, JR.

Department of Mathematics, State University of New York at Buffalo, Buffalo, N. Y. 14226, U.S.A.

This paper contains a few new results but, as the title suggests, it is primarily a discussion of various possibilities for further research in the theory of semigroups of continuous selfmaps. Most of the problems discussed here are natural developments from three lectures I gave at the Stefan Banach International Mathematical Center and I would like to take this opportunity to thank everyone there for their kind hospitality.

The problems fall within four different topics: dense subsemigroups of $S(X)$, congruences on $S(X)$, homomorphisms on $S(X)$ and Green's relations for $S(X)$.

1. Dense subsemigroups of $S(X)$

Throughout this paper, $S(X)$ will denote the semigroup, under composition, of all continuous selfmaps of the topological space X. For general information about $S(X)$, one may consult [4] and [8].

It has been known for quite a long time that if a Hausdorff space X is locally compact, then $S(X)$ is a topological semigroup when given the compact-open topology. The converse is not true. That is, there exist Hausdorff spaces X for which $S(X)$ is a topological semigroup under the compact-open topology and yet X is not locally compact. In fact, they are as far from being locally compact as they can be. J. de Groot [6] proved the existence of $2^{\mathfrak{c}}$ 1-dimensional connected locally connected subspaces of the Euclidean plane with the property that the only continuous maps from one space into another are the constant maps and for any such space X, $S(X)$ consists entirely of the constant maps together with the identity map. Thus, $S(X)$ is a left zero semigroup with identity and is therefore a topological semigroup for any topology whatsoever on

$S(X)$ and for the compact-open topology in particular. With the latter topology, $S(X)$ is simply a copy of X together with an isolated point. However, X is not locally compact at a single point. Suppose it is. Then X would contain a nondegenerate Peano continuum and it would readily follow that $S(X)$ would contain a nonconstant map other than the identity. But these examples are pathological and if one requires that $S(X)$ have a sufficiently rich supply of functions, then the converse does hold. S. Subbiah [26] has produced an extensive class of spaces with the property that for any space X of the class, $S(X)$ is a topological semigroup if and only if X is locally compact.

Throughout this section, the topology on $S(X)$ will be the compact-open topology. In 1934, J. Schreier and S. Ulam discovered a remarkable fact about the semigroup of all continuous selfmaps of the Euclidean N-cell I^N. They showed that $S(I^M)$ contains a dense subsemigroup which is generated by five functions. Specifically, the proved the following

THEOREM 1.1 (J. Schreier and S. Ulam [23]). *There exist three continuous selfmaps f_1, f_2, f_3 of I^N and a homeomorphism h from I^N onto I^N such that the subsemigroup generated by $\{f_1, f_2, f_3, h, h^{-1}\}$ is dense in $S(I^N)$.*

Here, of course, the compact-open topology coincides with the topology of uniform convergence. When one thinks about the diversity of the functions in $S(I)$ alone, the result is surprising. There are those functions which are not injective on any nondegenerate subinterval (e.g., the continuous nowhere differentiable functions) on the one hand and the homeomorphisms from I onto I on the other. These are essentially two very different types of functions and yet they can all be approximated to any desired degree of accuracy by finite compositions of five functions.

Once we know that there are finitely generated dense subsemigroups, it is only natural to try to determine the least number of functions needed. Let us note first that if X is locally compact and Hausdorff and has more than one point, then at least two functions are needed. Suppose, to the contrary that $S(X)$ contains a dense subsemigroup A which is generated by one element. Then A is commutative and since $S(X)$ is a topological semigroup this would force $\mathrm{cl}\,A = S(X)$ to be commutative which is a contradiction since X has more than one point.

So we see right away that we need at least two functions in order to generate a dense subsemigroup of $S(I^N)$. The question is, can it be done with two functions? In the same issue of the Fundamenta Mathematicae which carried the paper of Schreier and Ulam there also appeared a paper by W. Sierpiński (in fact, his paper is the one immediately following the paper by Schreier and Ulam) in which he was able to reduce the number of functions in the case of $S(I)$.

THEOREM 1.2 (W. Sierpiński [24]). *$S(I)$ contains a dense subsemigroup which is generated by four functions.*

The following year in 1935, V. Jarník and V. Kníchal completely settled the problem in the case of $S(I)$.

THEOREM 1.3 (V. Jarník and V. Kníchal [7]). *$S(I)$ contains a dense subsemigroup which is generated by two functions.*

Nothing further was done with this problem for the next thirty-four years. Then in 1969, H. Cook and W. Ingram completely settled the problem for a class of spaces which properly includes the Euclidean N-cells. Six years later in 1975, S. Subbiah being unaware of the result by Cook and Ingram proved a similar theorem. What we now state is actually a corollary of either result.

THEOREM 1.4 (H. Cook and W. Ingram [3], S. Subbiah [25]). *Let X represent any Euclidean N-cell the Cantor discontinuum, the countably infinite discrete space, the space of rational numbers or the space of irrational numbers. Then $S(X)$ has a dense subsemigroup which is generated by two functions.*

S. Subbiah also proved

THEOREM 1.5 (S. Subbiah [25]). *$S(R^N)$ contains a dense subsemigroup generated by three functions.*

PROBLEM 1.6. *Determine if $S(R^N)$ has a dense subsemigroup generated by two functions. We conjecture no.*

The results we discussed above suggest several avenues for further research. In order to discuss it further, it is convenient to introduce some terminology.

DEFINITION 1.7. The *density index* $D(S)$ of a topological semigroup S is ∞ if S has no finitely generated dense subsemigroup. If S does have such a semigroup, $D(S)$ is the smallest integer for which S contains dense subsemigroups generated by that number of elements.

In this terminology then, previous theorems tell us that $D\big(S(I^N)\big) = 2$ and $D\big(S(R^N)\big) \leqslant 3$. Problem 1.6 is to determine whether or not $D\big(S(R^N)\big) = 2$.

PROBLEM 1.8. *Characterize those spaces X for which $D\big(S(X)\big)$ is finite.*

Related to this is

PROBLEM 1.9. *For each positive integer N, characterize those spaces X for which $D\big(S(X)\big) = N$. The case where $N = 2$ is of special interest.*

H. Cook and W. Ingram [3] and S. Subbiah [25] actually proved more than the fact that $D(S(I^N)) = 2$. They proved that each countable subset of $S(I^N)$ is contained in a subsemigroup generated by two elements and since $S(I^N)$ is separable it readily follows that $D(S(I^N)) = 2$. This leads us to.

Definition 1.10. If there does not exist a positive integer N such that each countable subset of a semigroup S is contained in a subsemigroup generated by N elements, then define the *countability index of* S, denoted by $C(S)$, to be ∞. Otherwise, define $C(S)$ to be the least positive integer N such that each countable subset of S is contained in a subsemigroup generated by N elements.

Note that the countability index is defined for any semigroup while the density index makes sense only if the semigroup is endowed with a topology. Any time X is separable and metrizable $S(X)$ will be separable so that for such spaces X, we have $D(S(X)) \leqslant C(S(X))$. However, it can happen that $C(S(X))$ is finite while $D(S(X))$ is infinite. For example, if X is discrete and uncountable, then $D(S(X))$ is certainly infinite but it follows from results in [3] and [25] that $C(S(X)) = 2$. There also exist nondiscrete spaces for which this is true. Examples of spaces X for which $D(S(X)) = C(S(X)) = \infty$ are provided by the following

Theorem 1.11. *Let S^N be any Euclidean N-sphere. Then*

$$D(S(S^N)) = C(S(S^N)) = \infty.$$

Proof. Since S^N is separable and metrizable, $D(S(S^N)) \leqslant C(S(S^N))$ so it suffices to show that $D(S(S^N)) = \infty$. Let $\{f_i\}_{i=1}^M$ be any finite collection of functions in $S(S^N)$ and for each f_i, let $n_i = \deg f_i$ (where deg denotes degree). Choose any positive integer t which is not the product of powers of the integers n_i, $i = 1, 2, \ldots, M$ and then choose any function g such that $\deg g = t$. Finally, choose a neighborhood G of g small enough so that $h \in G$ implies $\deg h = t$. Since $\deg(v \circ w) = (\deg v)(\deg w)$ for all $v, w \in S(S^N)$, it readily follows that if k is any function in the subsemigroup generated by $\{f_i\}_{i=1}^M$, then $\deg k \neq t$ and hence $k \notin G$ which in turn implies that the subsemigroup generated by $\{f_i\}_{i=1}^M$ is not dense in $S(S^N)$. Thus, $D(S(S^N)) = \infty$.

Following are some problems and conjectures which involve $C(S(X))$.

Problem 1.12. *Determine $C(S(R^N))$. We conjecture that $C(S(R^N)) = \infty$ for each N.*

We know that $D(S(R^N)) \leqslant 3$ so that if the latter conjecture is valid, we would have spaces with infinite countability index but finite density index.

Problem 1.13. *Characterize those spaces X or which $C(S(X))$ is finite.*

Problem 1.14. *For each positive integer N, characterize those spaces X for which $C(S(X)) = N$. The case where $N = 2$ is of special interest.*

Problem 1.15. *Characterize those spaces X for which $D(S(X)) = C(S(X))$.*

2. Congruences on $S(X)$

This is an important area in the theory of semigroups of continuous selfmaps about which we have very little information. Recall that for any semigroup S, a congruence on S is any equivalence relation θ on S such that for any $(a, b) \in \theta$ and $c \in S$, we have $(ac, bc) \in \theta$ and $(ca, cb) \in S$. One can define a binary operation on S/θ the collection of equivalence classes by $\theta(a)\theta(b) = \theta(ab)$, where for any $c \in S$, $\theta(c)$ is the equivalence class containing c. Then S/θ is a semigroup (called a factor semigroup) which is a homomorphic image of S and, up to isomorphism, one gets all homomorphic images this way. The problem this leads to can be stated quite simply (if somewhat vaguely) but it is likely to be very difficult to solve.

Problem 2.1. *Determine all congruences on $S(X)$.*

This was completely solved in the case X is discrete by A. I. Malcev [21] in 1952 and this turned out to be quite a formidable task. He showed that when X is discrete, the complete lattice of congruences on $S(X)$ is generated by three types of congruences. A very readable account of Malcev's result can be found in the second volume of the well-known treatise *The algebraic theory of semigroups* by A. H. Clifford and G. B. Preston [2]. Among other things, they have filled in some gaps which appear to have been present in Malcev's original paper.

So what about $S(X)$ when X is not discrete? If X has more than one point then X has a proper two-sided ideal (e.g., the collection of all constant functions) and one can get a nontrivial congruence by identifying all the functions in that ideal. Of course, this sort of thing can be done for any semigroup whatsoever. Such a congruence is called a Rees congruence and the corresponding factor semigroup is referred to as a Rees-factor semigroup.

The next reasonable thing to do is to look for congruences on $S(X)$ which are not Rees-factor congruences. We do have one type of congruence which is not a Rees congruence and we describe it. The credit for the basic idea belongs to E. G. Sutov [27] who described the maximal proper congruence on $S(I)$. Define two functions in $S(I)$ to be equivalent if any time one of them is injective on a nondegenerate subinterval then the other function agrees with it on that subinterval. One easily verifies that the equivalence relation just described is a congruence and Sutov proved it is the largest proper congruence on $S(I)$. That congruence identifies all functions which are not injective on any nondegenerate subinterval (this includes all of the continuous nowhere differentiable functions). This collection of functions is the maximal two-sided ideal of $S(I)$ and serves as the zero element for the factor semigroup. Now we expand on Sutov's idea to describe a whole class of congruences.

DEFINITION 2.2. Let X be any topological space. A collection of subsets $\mathscr{U}$ of X is said to be a *unifying family* if for any $A \in \mathscr{U}$ and $f \in S(X)$, $f[A]$ also belongs to $\mathscr{U}$ whenever f is injective on A.

Examples of unifying families are the power set of X, the collection of all finite subsets of X, the collection of all compact subsets of X, the collection of all connected subsets of X, the collection of all compact N-dimensional subsets of X, and so on.

Now for each unifying family $\mathscr{U}$ of X we define a binary relation $\sigma(\mathscr{U})$ on $S(X)$ by requiring that $(f, g) \in \sigma(\mathscr{U})$ if and only if any time one of the functions is injective on some $A \in \mathscr{U}$, then the other function agrees with it there. One easily verifies that $\sigma(\mathscr{U})$ is a *congruence on* $S(X)$. From all these congruences, we choose one and look at it in a bit more detail.

DEFINITION 2.3. Let X be any topological space. The family $\mathscr{U}_R$ of all subsets of X which are images of continuous injections mapping X into X will be referred to as the *family of replicas of* X. $\mathscr{U}_R$ is a unifying family and the corresponding congruence will be referred to as the *replica congruence*.

For a number of X, the replica congruence holds a distinctive position.

THEOREM 2.4 ([4]). *For any clonable space X, the replica congruence is the largest proper congruence on $S(X)$.*

We recall that clonable spaces were defined in [12] and that all Euclidean N-cells as well as the Cantor discontinuum are clonable. The fact that the replica congruence is the largest proper congruence on $S(X)$ when X is clonable plays a crucial role in the proof of Theorem 3.4 of [12].

The replica congruence $\sigma(\mathscr{U}_R)$ is not the largest proper congruence on $S(R^N)$. The replica congruence never identifies distinct homeomorphisms. Consequently, the congruence ω on $S(R^N)$ obtained by identifying all homeomorphisms from $S(R^M)$ onto $S(R^N)$ to a point and identifying all other functions to another point, is a congruence which is strictly larger than the replica congruence. It is quite easy to see that ω is a maximal proper congruence on $S(R^N)$. It turns out that more is true, ω is actually the largest proper congruence on $S(R^N)$. We verify a more general result from which this will follow. But first we need the following

DEFINITION 2.5. A semigroup with identity which is the union of a proper ideal and its group of units is referred to as a separated semigroup.

The ideal referred to above must necessarily be the largest proper ideal of the semigroup. Now let S be any separated semigroup with largest proper ideal M and group G of units. Define a congruence ω on S by

$$\omega = (M \times M) \cup (G \times G).$$

THEOREM 2.6. *ω is the largest proper congruence on S.*

Proof. Suppose δ is a congruence on S which is not smaller than ω. That is, $\delta \nsubseteq \omega$. Then $(a, b) \in \delta - \omega$ for some $a, b \in S$. Since $(a, b) \notin \omega$, one of a, b is in M while the other is in G. We may assume $a \in M$ and $b \in G$. Let φ be the canonical homomorphism from S onto S/δ. Then $\varphi[M]$ is the largest two-sided ideal of S/δ and $\varphi[G]$ is its group of units. Consequently, $\varphi[M] \cap \varphi[G] = \emptyset$. However, $(a, b) \in \delta$ means $\varphi(a) = \varphi(b)$ and this is a contradiction since $a \in M$ and $b \in G$.

COROLLARY 2.7. *The congruence ω on $S(R^N)$ which is obtained by identifying all homeomorphisms mapping R^N onto R^N to a point and then identifying all other functions to another point is the largest proper congruence on $S(R^N)$.*

Proof. $S(R^N)$ is a separated semigroup ([8], p. 244).

We remark that just because a congruence on a semigroup results in only two congruence classes, it need not be the largest proper congruence though it will be a maximal proper congruence. For example, let A and B be two disjoint semigroups and let $S = A \cup B \cup \{0\}$. Define a multiplication on S by letting the product of two elements in A be as before, and also letting the product of two elements in B be as before. For $a \in A$ and $b \in B$ let $ab = ba = 0$ and finally, let $0x = x0 = 0$ for any $x \in S$. Now set

$$\alpha = \big((A \cup \{0\}) \times (A \cup \{0\})\big) \cup (B \times B)$$

and

$$\gamma = (A \times A) \cup \big((B \cup \{0\}) \times (B \cup \{0\})\big).$$

The relations α and γ are congruences on S. Clearly they are not comparable and $S/\alpha \approx S/\gamma$ is the two-element semigroup consisting of zero and unit.

Suppose we return now to our discussion of congruences on $S(X)$. If X is any Euclidean N-cell or any Euclidean N-space, we know what the largest proper congruence is because of Theorem 2.4 and Corollary 2.7. It is quite easy to determine the smallest proper congruence as well.

This is simply the Rees congruence which identifies all constant functions. In fact, this congruence will be the least proper congruence if X is any space with more than one point and $S(X)$ is doubly transitive on X. Let X be such a space and denote the congruence by a. We suppose that γ is any other nontrivial congruence and we show that $a \subseteq \gamma$. Since γ is nontrivial, there exist $(f, g) \in \gamma$ with $f \neq g$. Thus $f(x) \neq g(x)$ for some $x \in X$. Now let any $a, b \in X$ be given and denote the corresponding constant functions by $\langle a\rangle$ and $\langle b\rangle$ respectively. Since $S(X)$ is doubly transitive on X, there exists a $k \in S(X)$ such that $k(f(x)) = a$ and $k(g(x)) = b$. Thus $(f, g) \in \gamma$ implies that $(\langle a\rangle, \langle b\rangle) = (k\circ f\circ \langle x\rangle, k\circ g\circ \langle x\rangle) \in \gamma$. That is, $a \subseteq \gamma$.

Incidentally, the congruence a may be regarded as a congruence arising from a unifying family. Simply take the unifying family to be all two-element subsets of X. At any rate, all this leads us to suggest the following

PROJECT 2.8. *Investigate congruences arising from unifying families.*

With the exception of what is covered in [8], p. 260–264 nothing has been done here. This is an area which is almost certainly laden with results which are interesting, nontrivial, and at the same time, not impossible to prove.

3. Homomorphisms from $S(X)$ into $S(Y)$

We know quite a bit about isomorphisms from $S(X)$ into $S(Y)$. Papers [9], [10], [11], [16] are all concerned with this problem. We know considerably less about homomorphisms but in [12] we did produce two classes of spaces such that if X is from the first and Y is from the second then any nonconstant homomorphism φ from $S(X)$ into $S(Y)$ is necessarily injective and there exists a unique idempotent v in $S(Y)$ and a unique homeomorphism h from X onto the range of v such that $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for all f in $S(X)$. It is likely that theorems of this sort can be proven for other spaces in addition to the ones we considered in [12]. Specifically, we state the following

CONJECTURE 3.1. *Let X be any Euclidean N-cell and let Y be a connected subspace of any Euclidean M-space. We conjecture that a homomorphism from $S(X)$ into $S(Y)$ either maps everything into a single idempotent or it is injective.*

It would then follow from Corollary 3.5 of [16], p. 36, that if the dimension of Y does not exceed that of X then any nonconstant homomorphism φ from $S(X)$ into $S(Y)$ is given by $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for appropriate idempotent v and homeomorphism h. In particular, this would imply

CONJECTURE 3.2. *Let I^N denote the Euclidean N-cell and let φ be any endomorphism of $S(I^N)$. Then either φ maps everything into a single idempotent v or φ is injective and there exists a unique idempotent v in $S(I^N)$ and a unique homeomorphism h from I^N onto the range of v such that $\varphi(f) = h\circ f\circ h^{-1}\circ v$ for each $f \in S(I^N)$.*

It is already known that this conjecture is valid for I^N when $N = 1$. This fact is an immediate consequence of Theorem 3.2 of [12], p. 620. The conjecture is definitely false if one replaces I^N by either the Euclidean N-space or the Euclidean N-sphere. For the Euclidean N-space we take the space R of real numbers. Let v and w be any two idempotents of $S(R)$ such that $v\circ w \; w\circ v = v$. For example, we could define v and w by

$$v(x) = \begin{cases} x & \text{for } -1 \leqslant x \leqslant 1, \\ 1 & \text{for } 1 < x, \\ -1 & \text{for } x < -1, \end{cases}$$

$$w(x) = \begin{cases} x & \text{for } -2 \leqslant x \leqslant 2, \\ 2 & \text{for } 2 < x, \\ -2 & \text{for } x < -2. \end{cases}$$

Then define $\varphi(f) = w$ if f is a homeomorphism from R onto R and $\varphi(f) = v$ otherwise. One shows that φ is an endomorphism of $S(R)$ by appealing to the fact that the composition of two functions in $S(R)$ is a homeomorphism from R onto R if and only if each of the two functions are homeomorphisms.

For still another example, define

$$u(x) = \begin{cases} -x & \text{for } -2 \leqslant x \leqslant 2, \\ 2 & \text{for } x < -2, \\ -2 & \text{for } 2 < x \end{cases}$$

and note that $u\circ w = u$ and $u\circ u = w$. Now define $\theta(f) = w$ if f is an increasing homeomorhism from R onto R, $\theta(f) = u$ if f is a decreasing homeomorphism from R onto R and $\theta(f) = v$ otherwise. One can verify that θ is also an endomorphism of $S(R)$. It is known that for any Euclidean N-space R^N, there is no isomorphism from $S(R^N)$ onto a proper subsemigroup ([16], p. 32) which, we should mention, is in considerable contrast to the case for $S(I^N)$. These considerations all lead us to

CONJECTURE 3.3. *Let φ be any endomorphism of $S(R^N)$. If $\varphi[S(R^N)]$ contains more than three elements, then φ is an automorphism of $S(R^N)$ and there is a homeomorphism h mapping R^N onto R^N such that $\varphi(f) = h\circ f\circ h^{-1}$ for each f in $S(R^N)$.*

If the first part of the conjecture is valid the latter part will follow easily, since it is well known that all automorphisms of $S(R^N)$ (indeed of most $S(X)$) are inner. Conjecture 3.3 fails badly if one replaces R^N by the Euclidean N-sphere S^N. If $N > 1$, the semigroup $S(S^N)$ has many endomorphisms φ such that $\varphi[S(S^N)]$ has more than three elements but is still finite. In fact, we have the following

THEOREM 3.4. *Let $N > 1$ and let T be any finite commutative semigroup with zero and identity. Then there exists an endomorphism φ of $S(S^N)$ such that $\varphi[S(S^N)]$ is isomorphic to T.*

We can actually prove something stronger but we first need some notation. For each positive integer n, let

$$L_n = \{(x, y) \in R^2: y = x/n,\ 0 \leqslant x \leqslant 1,\ x^2+y^2 \leqslant 1\}.$$

Define $W = \bigcup \{L_n\}_{n=1}^{\infty}$ and let it have the topology it inherits from R^2. We now prove a result which has Theorem 3.4 as a special case.

THEOREM 3.5. *Let S^N be any Euclidean N-sphere, let Y be any completely regular Hausdorff space which contains a copy of W and let T be any finite commutative semigroup with zero and identity. Then there exists a homomorphism φ from $S(S^N)$ into $S(Y)$ such that $\varphi[S(S^N)]$ is isomorphic to T.*

Proof. In view of Theorem 3.3 of [11], p. 358, any finite semigroup can be embedded in $S(Y)$ since Y contains a copy of W so we let δ be any isomorphism from T into $S(Y)$. Now let $\mathcal{N}$ the semigroup of all non-negative natural numbers under ordinary multiplication. Define a map σ from $\mathcal{N}$ onto $\delta[T]$ as follows: send 0 into the zero of $\delta[T]$ and send 1 into the unit of $\delta[T]$. Map the primes onto $\delta[T]$ in any manner whatsoever and then extend σ to the composite integers so as to make it a homomorphism. We can now define the homomorphism φ. For any $f \in S(S^N)$, let

$$\varphi(f) = \sigma|\deg f|, \tag{3.5.1}$$

where $\deg f$ is the degree of f. It is well known that D is a homomorphism from $S(S^N)$ onto the multiplicative semigroup of integers so that φ is a homomorphism from $S(S^N)$ onto $\delta[T]$.

4. Green's relations and related topics

In 1951, J. A. Green published a paper [5] in which he defined five equivalence relations on an arbitrary semigroup S. These have subsequently come to be known as Green's relations and they have had considerable impact upon the development of the theory of semigroup. They have

been extremely useful in studying the algebraic structure of semigroups in general. Chapter 2 of [2], Vol. 1, is devoted to a discussion of these and related concepts. We recall briefly the definitions.

Two elements a and b are said to be *$\mathscr{L}$-equivalent* if they both generate the same principal left ideal, *$\mathscr{R}$-equivalent* if they both generate the same principal right ideal and *$\mathscr{J}$-equivalent* if they both generate the same principal two-sided ideal. The relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{J}$ are all clearly equivalence relations. The relation $\mathscr{H}$ is defined by $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. It so happens that the composition of $\mathscr{L}$ with $\mathscr{R}$ commutes. That is, $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ so that $\mathscr{L} \circ \mathscr{R}$ is also an equivalence relation and this relation is denoted by $\mathscr{D}$. One observe that

$$\mathscr{H} \subset \mathscr{L} \subset \mathscr{D} \subset \mathscr{J}$$

and

$$\mathscr{H} \subset \mathscr{R} \subset \mathscr{D} \subset \mathscr{J}.$$

For finite semigroups, $\mathscr{D}$ and $\mathscr{J}$ coincide. In fact, more is true. They coincide for any compact topological semigroup.

Let us recall that an element a of a semigroup S is said to be regular if $axa = a$ for some $x \in S$. This is, of course, just von Neumann's definition for a regular element of a ring [22]. It so happens that if one element of a $\mathscr{D}$-class is regular then all elements in that same $\mathscr{D}$-class will be regular. Accordingly, such a $\mathscr{D}$-class is referred to as a *regular* $\mathscr{D}$-class and those with no regular element are called *irregular* $\mathscr{D}$-classes. Some very familiar topological spaces X are characterized within fairly extensive classes of spaces simply by the number of their regular $\mathscr{D}$-classes. For example, we have

THEOREM 4.1 ([15]). *Let X be any Peano continuum. Then $S(X)$ has exactly two regular $\mathscr{D}$-classes if and only if X is an arc.*

THEOREM 4.2 ([20]). *Let X be any Peano continuum with no cut points. Then $W(X)$ has exactly three regular $\mathscr{D}$-classes if and only if X is a simple closed curve.*

THEOREM 4.3 ([20]). *Let X be any nondegenerate Peano continuum which contains at least one cut point. Then $S(X)$ had exactly three regular $\mathscr{D}$-classes if and only if X is a triode.*

This immediately suggests

PROBLEM 4.4. *For each integer $N > 3$, characterize within the Peano continua those spaces X for which $S(X)$ has exactly N regular $\mathscr{D}$-classes.*

Related to the latter problem is

PROBLEM 4.5. *Characterize within the Peano continua those spaces X for which $S(X)$ has only a finite number of regular $\mathscr{D}$-classes.*

We mention that among all the Euclidean N-cells I^N, the Euclidean N-spaces R^N and Euclidean spheres S^N, only $S(I)$, $S(R)$ and $S(S^1)$ have finitely many regular $\mathscr{D}$-classes. The semigroups of all the other spaces have infinitely many regular $\mathscr{D}$-classes. This is a consequence of Theorem 3.3 of [15], p. 1490.

The problem of describing Green's relations for regular elements of $S(X)$ has been completely solved in a satisfactory manner [15]. In fact, the results in that peper are sufficiently general to apply to all sorts of endomorphism semigroups. As for $S(X)$, two regular elements f and g of $S(X)$ are $\mathscr{L}$-equivalent if and only if the decompositions they induce on X are identical. They are $\mathscr{R}$-equivalent if and only if their ranges coincide, $\mathscr{D}$-equivalent if and only if their ranges are homeomorphic and $\mathscr{J}$-equivalent if and only if range of each contains a retract of X which is homeomorphic to the range of the other.

For elements of $S(X)$ which are not necessarily regular, the problem is far from being solved and we propose as a general

PROJECT 4.6. *Determine Green's relations for elements of $S(X)$ which are not necessarily regular.*

Some work has been done in this direction ([13], [14], [17], [18]) but the surface has barely been scratched and there is much yet to do.

Again, let S denote an arbitrary semigroup and let $\mathscr{L}(S)$ and $\mathscr{R}(S)$ denote respectively the collections of all $\mathscr{L}$-classes and $\mathscr{R}$-classes of S. These collections can be partially ordered in a natural way. The same sort of thing can be done for the $\mathscr{J}$-classes but we won't be concerned with that here. For any two $\mathscr{L}$-classes L_a and L_b define $L_a \leqslant L_b$ if the principle left ideal generated by a is contained in that generated by b. Define $R_a \leqslant R_b$ in an analogous manner. It is natural to try to determine if these things are lattices and if not, how close are they to being lattices. Before we discuss $\mathscr{L}(S(X))$ and $\mathscr{R}(S(X))$ we will say just a few words about $\mathscr{L}(L(V))$ and $\mathscr{R}(L(V))$ where $L(V)$ is the semigroup of all linear operators on the vector space V. The results here are about as nice as one could expect and we state them formally as

THEOREM 4.7. *$\mathscr{L}(L(V))$ is dual isomorphic to the partially ordered family of all subspaces of V and $\mathscr{R}(L(V))$ is isomorphic to that partially ordered family. Consequently, both $\mathscr{L}(L(V))$ and $\mathscr{R}(L(V))$ are complete lattices and if V is finite dimensional over the reals they are isomorphic to each other.*

I have never seen the latter result stated formally before. It follows rather easily from general results in [15] and the fact that any finite dimensional vector space over the reals is selfdual ([1], p. 3). One reason we have stated the result is that it provides considerable contrast to the situation for $\mathscr{L}(S(X))$ and $\mathscr{R}(S(X))$. For one thing, $\mathscr{R}(S(X))$ is not even a lattice if X has more than one point and only in this trivial situation can $\mathscr{L}(S(X))$ and $\mathscr{R}(S(X))$ be isomorphic. We do have a few results about $\mathscr{L}(S(X))$ and $\mathscr{R}(S(X))$ which we will mention and they will lead naturally to some further conjectures. Our first result is concerned not with $\mathscr{R}(S(X))$ but rather with $\mathscr{R}_R(S(X))$ the partially ordered family of all *regular* $\mathscr{R}$-classes of $S(X)$.

THEOREM 4.8 ([19]). *$\mathscr{R}_R(S(X))$ is a complete upper semilattice and a conditionally complete lower semilattice when X is a 0-dimensional metric space.*

CONJECTURE 4.9. *$\mathscr{R}(S(X))$ is a complete upper semilattice and a conditionally complete lower semilattice when X is a 0-dimensional metric space.*

THEOREM 4.10 ([19]). *The following statements are all equivalent for any Peano continuum X:*

(4.10.1) *$\mathscr{R}_R(S(X))$ is an upper semilattice.*

(4.10.2) *$\mathscr{R}_R(S(X))$ is a complete upper semilattice.*

(4.10.3) *$\mathscr{R}_R(S(X))$ is a conditional lower semilattice.*

(4.10.4) *$\mathscr{R}_R(S(X))$ is a conditionally complete lower semilattice.*

(4.10.5) *X is a dendrite.*

PROBLEM 4.11. *Determine if Theorem 4.10 remains true when $\mathscr{R}_R(S(X))$ is replaced by $\mathscr{R}(S(X))$.*

Although $\mathscr{R}(S(X))$ cannot possibly be a lattice if X has more than one point, the partially ordered set $\mathscr{L}(S(X))$ can be. For example we have the following

THEOREM 4.12 ([14]). *Let X be either the Hilbert cube or discrete or a compact 0-dimensional metric space. Then $\mathscr{L}(S(X))$ is a complete lattice.*

PROJECT 4.13. *Try to get more information about precisely what spaces X have the property that $\mathscr{L}(S(X))$ is a complete lattice or even just a lattice.*

CONJECTURE 4.14. *$\mathscr{L}(S(X))$ is not a lattice X is any Euclidean N-space or Euclidean N-cell.*

Our final result and subsequent conjecture relates the $\mathscr{L}$-classes and the $\mathscr{R}$-classes

THEOREM 4.15. *The following statements about any two compact 0-dimensional metric spaces X and Y are equivalent:*

(4.15.1) *$\mathscr{R}_R(S(X))$ and $\mathscr{R}_R(S(Y))$ are order isomorphic.*

(4.15.2) *The complete lattices $\mathscr{L}(S(X))$ and $\mathscr{L}(S(Y))$ are order isomorphic.*

(4.15.3) *The semigroups $S(X)$ and $S(Y)$ are isomorphic.*

(4.15.4) *The space X and Y are homeomorphic.*

Theorem 4.15 is an immediate consequence of Theorem 2.2 in [14] and Theorem 3.4 in [19].

CONJECTURE 4.16. *The four statements in Theorem 4.15 are also equivalent to the statement that $\mathscr{R}(S(X))$ is order isomorphic to $\mathscr{R}(S(X))$.*

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Added in proof.

Since this paper was written, some progress has been made with several of the problems and conjectures. S. Subbiah and the author were able to show that $C(S(R^N)) \geqslant 4$. A proof of this fact can be found in [28], Corollary 1.8. S. Subbiah has settled Problem (1.6) in the case $N = 1$ [32]. Specifically, she has shown that there exist two functions in $S(R)$, the semigroup of all continuous selfmaps of the reals, which generate a dense subsemigroup. Unfortunately, her techniques do not at all carry over to the higher dimensional cases. Nevertheless, in contrast to what we previously supposed, there is now reason to suspect that $D(S(R^N)) = 2$ for all N.

P. R. Misra, U. B. Tewari and the author were able to make a little progress with Project (2.5) [31]. They were able to determine all the congruences on $S(I)$, the semigroup of all continuous selfmaps of the closed unit interval, which arise from unifying families. There are precisely four such congruences. The next step would be to look at dentrites and the triod in particular.

Problem (4.5) has been completely settled in [29]. It turns out that if X is any Peano continuum, then $S(X)$ has only finitely many regular $\mathscr{D}$-classes if and only if X is a local dendrite which satisfies an additional property. In that same paper, another result appears which has some bearing on Problem (4.4) of this paper. Those Peano continua X which have the property that $S(X)$ has exactly four regular $\mathscr{D}$-classes are characterized. Up to homeomorphism, there are two such spaces. Finally, Conjecture (4.16) has been settled in the affirmative [30].

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