10.3. It is easy to generalize the notion of a hyperalgebra and to define a *hyperalgebra in a cartesian closed category*. For example, the Scott model 7.3 (A) gives rise to the following hyperalgebra $\mathfrak{A}$ in the cartesian category Clatt of complete lattices and continuous functions: the underlying object of $\mathfrak{A}$ is D_∞, and the hyperoperations are the following arrows in Clatt:

$ap_n\colon [D_\infty^n \to D_\infty]^2 \to [D_\infty^n \to D_\infty]$ defined by $ap_n(f, g) = \varepsilon^{\Phi}_{\Phi^{-1}} \circ \langle f, g\rangle$,

$ab_n\colon [D_\infty^{n+1} \to D_\infty] \to [D_\infty^n \to D_\infty]$ defined by $ab_n(f) = \Phi^{-1} \circ \lambda_{D_\infty^n, D_\infty}[f]$,

$c_i^n\colon [D_\infty^1 \to D_\infty]^0 \to [D_\infty^n \to D_\infty]$ defined by $c_i^n(\bot) = \mathrm{pr}_i^n(D_\infty)$, where $[A \to B]$ means the lattice of all continuous functions from the lattice A to the lattice B.

References

[1] H. Barendregt, *The type free lambda calculus*, in: *Handbook of Mathematical Logic*, North-Holland, 1977, 1092–1131.

[2] F. W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. 50 (1963), 869–872.

[3] S. MacLane, *Categories for the Working Mathematician*, Springer Verlag, 1972.

[3a] A. Obtułowicz, *Functorial semantics of the type free λ-βη calculus*, in: *Fundamentals of computation theory*, Springer Lecture Notes in Computer Science 56, New York 1977, 302–307.

[4] D. Scott, *Continuous lattices*, in: *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics, Vol. 274, Springer Verlag, 1972, 97–136.

[5] C. P. Wadsworth, *The relation between lambda-expressions and their denotations*, SIAM J. Comput. 5 (3) (1976), 488–521.

Presented to the Semester
Universal Algebra and Applications
(February 15 – June 9, 1978)

UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

SOME PROBLEMS OF BCK-ALGEBRAS AND GRISS TYPE ALGEBRAS

KIYOSHI ISÉKI

Department of Mathematics, Kobe University, Rokko, Kobe 657, Japan

The notions of BCK-algebras and Griss algebras were formulated first in 1966 (see [2], [3]). For example, BCK-algebras are obtained as unified theory generalizing some elementary and common properties of set-difference in set theory and implication in propositional calculi.

We know the following simple relations in set theory:

$$(A-B)-(A-C) \subset C-B,$$

$$A-(A-B) \subset B.$$

In propositional calculi, these relations are denoted by

$$(p \to q) \to \big((q \to r) \to (p \to r)\big),$$

$$p \to \big((p \to q) \to q\big).$$

From these relationships, we have a new class of algebras as follows:

Definition 1. Let X be a set with a binary operation $*$ and a constant 0. X is called a *BCK-algebra* if it satisfies the following conditions:

(1) $(x*y)*(x*z) \leqslant z*y$,

(2) $x*(x*y) \leqslant y$,

(3) $x \leqslant x$,

(4) $0 \leqslant x$,

(5) $x \leqslant y$, $y \leqslant x$ implies $x = y$,

(6) $x \leqslant y$ if and only if $x*y = 0$.

We introduced another class of algebras which are called Griss algebras. The notion is an algebraic formulation of negationless logic considered by G. F. C. Griss [1].

DEFINITION 2. By a *Griss algebra*, we mean an algebra $\langle X, *, \vee, 0\rangle$ of type $(2, 2, 0)$ satisfying the following conditions:

(7) X is a $\vee$-semilattice with 0 as least element,

(8) $(x\vee y)*(y\vee z) \leqslant x*z$,

(9) $x*y \leqslant (x*z)\vee(z*y)$,

(10) $x \leqslant y \rightleftarrows x*y = 0 \rightleftarrows x\vee y = y$.

In this definition, the first condition (7) is strong, as will be shown by Example 3. In this example, X is only a semigroup with zero 0.

In this paper we shall state several unsolved problems on BCK-algebras and Griss type algebras.

In a BCK-algebra, we have the following elementary and basic properties (for proofs, see [7], [8], and [9]):

(11) $x \leqslant y$ implies $z*y \leqslant z*x$, $x*z \leqslant y*z$,

(12) $x \leqslant y$, $y \leqslant z$ implies $x \leqslant z$,

(13) $(x*y)*z = (x*z)*y$,

(14) $x*y \leqslant z$ implies $x*z \leqslant y$,

(15) $x*y \leqslant x$,

(16) $x*0 = x$.

On the other hand, in a BCK-algebra,

(17) $x*(x*y) = y*(y*x)$

is not always true.

DEFINITION 3. If, in a BCK-algebra, (17) holds, then it is called a *commutative BCK-algebra* or *Tanaka algebra*.

This algebra is characterized by the following

THEOREM 1. *A BCK-algebra is commutative, if and only if it is a semilattice with resepct to* $\wedge$, *where* $x\wedge y = y*(y*x)$.

An axiom system for commutative BCK-algebras is given in the following

THEOREM 2 (H. Yutani). *An algebra* $\langle X, *, 0\rangle$ *of type* $(2, 0)$ *is a commutative BCK-algebra iff it satisfies the following conditions*:

(13) $(x*y)*z = (x*z)*y$,

(18) $x*(x*y) = y*(y*x)$,

(3′) $x*x = 0$,

(16) $x*0 = x$.

Therefore, the *class of commutative BCK-algebras is a variety*.

PROBLEM 1. *Is the class of BCK-algebras a variety?*

Recently H. Yutani [10] considered a useful class of BCK-algebras which is a variety. To define this new class, we use the following notations.

A polynomial $Q_{m,n}(x, y)$ of variables x, y of a BCK-algebra X is inductively defined as follows:

$$Q_{0,0}(x, y) = x*(x*y),$$
$$Q_{m+1,n}(x, y) = Q_{m,n}(x, y)*(x*y),$$
$$Q_{m,n+1}(x, y) = Q_{m,n}(x, y)*(y*x).$$

Under these notations, the commutativity is denoted by $Q_{0,0}(x, y) = Q_{0,0}(y, x)$.

DEFINITION 3. A *quasi-commutative BCK-algebra* (*of type* (i, j, k, l)) is a BCK-algebra which satisfies for some i, j, k, and l

(19) $Q_{i,j}(x, y) = Q_{k,l}(y, x)$ for x, y.

As is easily seen, a commutative BCK-algebra, a positive implicative algebra and an implicative algebra are all quasi-commutative.

THEOREM 3. *The class of quasi-commutative BCK-algebras is a variety.*

In fact, a quasi-commutative BCK-algebra of type (i, j, k, l) is defined by the following equations:

(1′) $\big((x*y)*(x*z)\big)*(z*y) = 0$,

(16) $x*0 = x$,

(4′) $0*x = 0$,

(19) $Q_{i,j}(x, y) = Q_{k,l}(y, x)$.

By the definition of a quasi-commutative BCK-algebra, the above equations hold. Conversely, we assume that the above equations hold. Then, if $x*y = y*x = 0$, we have $Q_{i,j}(x, y) = x$, $Q_{k,j}(y, x) = y$. Hence (5) holds. From (1′), we have $\big(x*(x*y)\big)*y = 0$, which implies (2). Hence the algebra is a BCK-algebra.

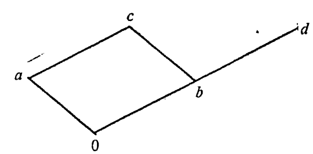
Moreover, we have the following

THEOREM 4. *Any finite BCK-algebra is quasi-commutative.*

EXAMPLE 1 (H. Yutani). Let X be the set $\{0, a, b, c, d\}$. We define the operation $*$ by the following table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	b	b	0

Table 1



Then X is a BCK-algebra. Moreover, it is a quasi-commutative algebra of type $(1,1,1,1)$, i.e. $Q_{1,1}(x,y) = Q_{1,1}(y,x)$ for all x, y.

EXAMPLE 2 (Y. Seto). Let X be a partially ordered set with the least element 0, and a distinguished element e such that $0 \leqslant x$ for all $x \in X$ and $e \leqslant x$ for all nonzero $x \in X$. The operation $*$ is defined as following:

$$x*y = \begin{cases} 0 & \text{if } x \leqslant y, \\ x & \text{if } y = 0, \\ e, & \text{otherwise}. \end{cases}$$

X is a BCK-algebra. Moreover, X is a quasi-commutative algebra of type $(1,2,1,2)$.

PROBLEM 2. *Find a quasi-commutative BCK-algebra of type* (i,j,k,l) *for any natural number* i,j,k, *and* l.

Next we shall define a special class of BCK-algebras which is called a BCK-algebra with condition (S).

Let X be a BCK-algebra, and let x, y be any fixed elements of X. Then the set of u satisfying

(20) $\quad u*x \leqslant y$

is not empty, because $0*x = 0 \leqslant y$. We assume that there is the greatest element u satisfying (20). This greatest element is uniquely determined. The greatest element is denoted by $x\circ y$.

DEFINITION 4. If there exists $x\circ y$ for any elements x, y of a BCK-algebra X, then X is called a *BCK-algebra with condition* (S).

In a BCK-algebra with condition (S), $x*x \leqslant y$ and $y*x \leqslant y$ imply $x, y \leqslant x\circ y$.

THEOREM 5. *Any BCK-algebra X with condition* (S) *is a commutative semigroup with respect to* $\circ$.

Proof. By the definition of $\circ$, we have

$$(x\circ y)*x \leqslant y.$$

Hence $(x\circ y)*y \leqslant x$. Therefore $x\circ y \leqslant y\circ x$, so we obtain $x\circ y = y\circ x$.

Next by the definition of $\circ$,

$$\big((x\circ y)\circ z\big)*z \leqslant x\circ y.$$

Hence

$$\Big(\big((x\circ y)\circ z\big)*z\Big)*y \leqslant x.$$

From (14), we have

$$\Big(\big((x\circ y)\circ z\big)*y\Big)*z \leqslant x.$$

Therefore

$$\big((x\circ y)\circ z\big)*y \leqslant x\circ z,$$

so we have

$$(x\circ y)\circ z \leqslant y\circ(x\circ z) = (x\circ z)\circ y.$$

Since y and z are symmetric, we obtain

$$(x\circ y)\circ z = (x\circ z)\circ y.$$

This equality and the commutative law of $\circ$ imply

$$(x\circ y)\circ z = (x\circ z)\circ y = (z\circ x)\circ y = x\circ(z\circ y) = x\circ(y\circ z),$$

which proves the associativity of $\circ$.

Hence X is a commutative semigroup with respect to $\circ$.

Moreover, $u*x = 0 \rightleftarrows u \leqslant x$. Hence $x\circ 0 = x$.

If X is bounded, then $u \leqslant 1$ for any $u \in X$. Hence for any $u \in X$,

$$u*x \leqslant 1.$$

Therefore $1 = x\circ 1$.

If we put $Nx = 1*x$, we have $x\circ Nx = 1$.

COROLLARY 1. *If X is a BCK-algebra with condition* (S), *then*

$$x\circ 0 = 0\circ x = x.$$

If X has 1, *then*

$$x\circ 1 = 1\circ x = 1, \quad x\circ Nx = 1.$$

THEOREM 6. $x \leqslant y \to x\circ z \leqslant y\circ z$.

Proof. $x \leqslant y$ implies

$$(x\circ z)*y \leqslant (x\circ z)*x \leqslant z,$$

which implies shows $x\circ z \leqslant y\circ z$.

THEOREM 7. *The following propositions* (21), (22) *are equivalent*:

(21) $\quad x \leqslant y \to x\circ y = y$,

(22) $\quad x\circ x = x$ *for any* $x \in X$.

Proof. Assume (21). $x \leqslant x$ implies $x\circ x \leqslant x$ by (21). From Theorem 6, $x = x\circ 0 \leqslant x\circ x$. Hence $x\circ x = x$, which proves (22).

Assume (22). $x \leqslant y$ implies $x\circ y \leqslant y\circ y = y$, by Theorem 6. Hence we have (21).

THEOREM 8. *In a BCK-algebra X with condition* (S), *we have*

$$x*y \leqslant (x*z)\circ(z*y),$$

$$(x\circ y)*(y\circ z) \leqslant x*z \leqslant x\circ z.$$

Proof. $(x*y)*(x*z) \leqslant z*y$ implies

$$x*y \leqslant (x*z)\circ(z*y).$$

On the other hand, $x*(x*y) \leqslant y$ implies

$$x \leqslant (x*y)\circ y.$$

By Theorem 6, we have

$$x\circ z \leqslant (x*y)\circ(y\circ z).$$

Hence

$$(x\circ z)\circ(y\circ z) \leqslant x*y.$$

Since X is a BCK-algebra, $x*y \leqslant x \leqslant x\circ y$. Therefore

$$(x\circ z)*(y\circ z) \leqslant x\circ y,$$

which completes the proof.

PROBLEM 3. *Is the class of BCK-algebras with condition* (S) *a variety?*

In a BCK-algebra with condition (S), $x\circ x = x$ is not always true. We have the following interesting

THEOREM 9. *Let X be a BCK-algebra with condition* (S). *The following propositions are equivalent:*

(I) *X is positive implicative,*

(II) *$x \leqslant y$ implies $x\circ y = y$,*

(III) *$x\circ x = x$ for all $x \in X$,*

(IV) $(x\circ y)*z = (x*z)\circ(y*z)$.

EXAMPLE 3. Let $X = \{0, a, b, 1\}$ with $0 < a < b < 1$. We define $*$ as in Table 2. Then X is a BCK-algebra, but not positive implicative. Moreover, X has condition (S), and the operation $\circ$ is given in Table 3. Therefore $\langle X, *, \circ; 0\rangle$ is not a Griss algebra, since X is not a semilattice with respect to $\circ$. But X satisfies (8), (9).

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	a	0	0
1	1	a	a	0

Table 2

$\circ$	0	a	b	1
0	0	a	b	1
a	a	1	1	1
b	b	1	1	1
1	1	1	1	1

Table 3

DEFINITION 5. Let A be a non-empty subset of a BCK-algebra X. A is called to be an *ideal*, if (1) $0 \in A$ and (2) $y*x$, $x \in A$ implies $y \in A$.

Let X_α $(\alpha \in I)$ be a family of BCK-algebras. The direct product $\prod X_\alpha$ is a BCK-algebra, if we define

$$(f*g)(\alpha) = f(\alpha)*g(\alpha),$$

where $f, g \in \prod X_\alpha$.

Then we have the following

THEOREM 10. *Any ideal I of $X_1 \times X_2$ is represented as a product of ideals I_1, I_2 in X_1, X_2 respectively, i.e., $I = I_1 \times I_2$, where I_i is the projection i of I_i $(i = 1, 2)$.*

PROBLEM 4. *Let I be an ideal of an infinite direct product $\prod X_2$ of BCK-algebras X_α. Is I represented as a product of ideals of X_α?*

PROBLEM 5 (H. Yutani). *Let X be a BCK-algebra. Find a free BCK-algebra Y that X is an ideal of Y.*

PROBLEM 6 (H. Yutani). *Let Y be a subalgebra of a BCK-algebra X and I an ideal of Y. Does an ideal J of X exist with the property: $I = J \cap Y$?*

In a BCK-algebra, we can define some kinds of different homomorphisms. Usually, we consider the following type of homomorphisms.

Let X, Y be two BCK-algebras. A mapping $f\colon X \to Y$ is said to be a *homomorphism* if for all $x, y \in X$,

$$f(x*y) = f(x)*f(y).$$

Then we can consider a category BCK with objects all BCK-algebras, and morphisms all homomorphisms from a BCK-algebra to another. In BCK there exist finite limits.

On the other hand, the class of BCK-algebras is a Jonsson class. There is an interesting problem.

PROBLEM 7. *Does the coproduct of any two objects in BCK exist?*

In this paper we stated important unsolved problems, but some of those problems have mutual relationships.

A Griss algebra on a totally ordered set (this is a semilattice) satisfying $x*0 = x$ is uniquely determined, i.e.

$$x*y = \begin{cases} 0 & \text{if } x \leqslant y, \\ x & \text{if } y < x. \end{cases}$$

On the other hand, there exists at least two Griss structures that $x*0 = x$ is not always true on any totally ordered set with cardinality $\geqslant 4$.

Next, we shall state an unsolved problem related to Griss type algebras.

In our definition of the Griss algebra, we can replace condition (8) by the following equation:

$$(x\vee y)*z = (x*z)\vee(y*z).$$

PROBLEM 8. *Discuss the relationship between a Griss algebra and an algebra satisfying* (7), (9), (10) *and* $(x\vee y)*(x\vee z)\leqslant y\vee z$.

Many results on BCK-algebras and Griss type algebras are found in papers inserted into Mathematics Seminar Notes, Kobe University.

Added in proof. Problem 1 has been solved by Professor A. Wronski. He proved that the class of BCK-algebras is not a variety.

References

[1] G. F. C. Griss, *Logic of negationless intuitionistic mathematics*, Indag. Math. 13 (1951), 3–11.
[2] Y. Imai and K. Iséki, *On Griss algebras I*, Proc. Japan Acad. 42 (1966), 213–216.
[3] J. Iséki, *An algebra related with a propositional calculus*, ibid. 42 (1966), 26–29.
[4] —, *A special class of BCK-algebras*, Math. Seminar Notes 5 (1977), 191–198, Kobe University.
[5] —, *On BCK-algebras with condition* (S), ibid. 5 (1977), 215–222.
[6] —, *On congruence relations on BCK-algebras*, ibid. 5 (1977), 327–346.
[7] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japonica 21 (1976), 351–366.
[8] —, —, *Introduction to the theory of BCK-algebras*, ibid. 23 (1978), 1–26.
[9] —, —, *BCK-algebras*, Rev. Roum. Math. pures et appl. 23 (1978), 61–66.
[10] H. Yutani, *Quasi-commutative BCK-algebra and congruence relations*, Math. Seminar Notes 5 (1977), 469–480.
[11] —, *Congruences on the products of BCK-algebras*, ibid. 6 (1978), 169–172.

Presented to the Semester
Universal Algebra and Applications
(February 15 – June 9, 1978)

UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

ON AN ALGEBRAIC AND KRIPKE SEMANTICS FOR INTERMEDIATE LOGICS

HIROAKIRA ONO

Faculty of Integrated Arts and Sciences, Hiroshima University, Hiroshima 730, Japan

CECYLIA RAUSZER

Institute of Mathematics, Warsaw University, Warsaw, Poland

The topic of this paper is to show that Kripke semantics and algebraic semantics for intermediate predicate logics are incomplete.

Then we prove a weaker version of the converse theorem to the following one:

Let $\boldsymbol{K}$ *be a class of intermediate logics such that for every* $L\in\boldsymbol{K}$ *the formula of the form* $\forall x\big(a(x)\cup\beta\big)\Rightarrow\big(\forall x\, a(x)\cup\beta\big)$, *where* x *does not appear in* β *is provable in* L.

If $L\in\boldsymbol{K}$ *and* L *has a characteristic Kripke model, then* L *has a characteristic set of algebraic models.*

First of all, we describe a language $\mathscr{L}$ in order to define intermediate predicate logics. Let $\mathscr{L}$ consist of a countable infinite set V of individual variables $x, y, z, \ldots$; a countable infinite set of n-ary predicate variables $p^{(n)}, q^{(n)}, r^{(n)}, \ldots$, and a countable infinite set of constants. 0-ary predicate variables are identified with propositional variables. The logical symbols of $\mathscr{L}$ are $\cap, \cup, \Rightarrow, \neg, \forall, \exists$. The set of formulas F is defined in the usual way and elements of F we denote by $a, \beta, \gamma, \ldots$

We will identify a logic with the set of formulas provable in it. Thus, by LK we mean the *set of all formulas provable in the classical predicate calculus* and by LI we mean the *set of all formulas provable in the intuitionistic predicate calculus*. Let a be a formula provable in LK. Then by $\mathrm{LI}+a$ we denote an *intermediate predicate logic* obtained by adding an axiom scheme a to LI.

DEFINITION 1. A set of formulas L is said to be an *intermediate predicate logic* if it satisfies the following conditions:

(1) $\mathrm{LI}\subset L\subset \mathrm{LK}$,