CATEGORICAL, FUNCTIONAL AND ALGEBRAIC ASPECTS OF THE TYPE-FREE LAMBDA CALCULUS

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0. Introduction

From the set-theoretical point of view, the type-free $\lambda$-calculus initiated by A. Church around 1900 may be labelled as a deductive system fit for examining sets with the property

$$A = A^*.$$  \hspace{1cm} (1)

Unfortunately, (1) is satisfied only if $A$ is a one-element set and if we agree to identify the unique element in $A$ with the unique function $A \rightarrow A$. The problem of finding non-trivial models of the type-free $\lambda$-calculus turned out to be difficult and was solved by D. Scott in 1969. Even the question what should be meant by a “model” of the type-free $\lambda$-calculus requires some consideration. In this paper we shall outline a certain new approach to the syntax and the semantics of the type-free $\lambda$-calculus. In Sections 2 and 3 some modifications of the classical syntax of the type-free $\lambda$-calculus are described. In Sections 4, 5 and 6 we “categorize” the syntax of the type-free $\lambda$-calculus: we construct some categories from $\lambda$-terms and we introduce the concept of a Church algebraic theory. In Section 7 “models” in the style of Wadsworth [5] are discussed; we give a new characterization of these “models”, which is independent of the syntax of the type-free $\lambda$-calculus. In Section 8 a method of “functionalizing” the semantics of the type-free $\lambda$-calculus is described; “models” of the type-free $\lambda$-calculus are identified with certain functors defined on Church algebraic theories. In Sections 9, 10, the new concepts of a hyperalgebra and hyperhomomorphism are introduced and discussed; it is shown that certain “models” of the type-free $\lambda$-calculus can be treated as hyperalgebras.
1. Preliminaries

1.1. We shall use the following symbols:

\( \uparrow \) is the symbol of a variable,

\( \mathbb{N} \) is the set of all non-negative integers \( \{0, 1, 2, \ldots\} \),

\( \mathbb{N}^* \) is the set of all positive integers \( \{1, 2, 3, \ldots\} \),

\( \mathfrak{n} \) is the set \( \{1, 2, \ldots, \mathfrak{n}\} \), and \( \mathfrak{n}+1 \) is the set \( \{1, 2, \ldots, \mathfrak{n}+1\} \).

If \( A \) is a set, then \( \text{card}(A) \) is the cardinal number of \( A \). By a family \( (a_i; i \in T) \) of elements of \( A \) we mean the function \( \uparrow \mapsto a_i \) from the set \( T \) into the set \( A \). If \( T = \{\ast\} \) is a one-element set, then we shall identify \( (a_\uparrow; i \in T) \) with \( a_\uparrow \). If \( A \) and \( B \) are sets, then the set of all functions from \( B \) to \( A \) will be denoted by \( A^B \). If \( f \) is a function from \( B \) to \( C \), then

\[ A^C \rightarrow A^B, \quad f^C \colon B^C \rightarrow C^C \]

are functions defined by

\[ A^C(f; C \rightarrow A) = gcf, \quad f^C(k; A \rightarrow B) = f \circ k, \]

respectively, where \( \circ \) is the composition of functions. To avoid superfluous notational complications we shall identify the sets \( A^* \), where \( \mathfrak{n} \) may be considered as a von Neumann number, with sets defined inductively as follows:

\[ A^0 = \{0\}, \quad A^1 = A, \quad A^{n+1} = A^n \times A \quad \text{for} \quad n \in \mathbb{N}^*. \]

If \( f \) is a function from \( A \) to \( B \), then \( f^* \colon A^n \rightarrow B^n \) may be identified with the function given by

\[ f^*(a_1, \ldots, a_n) = f(a_1), \ldots, f(a_n); \]

in particular, \( f^0 \colon A^0 \rightarrow B^0 \) is given by \( f^0(0) = 0 \).

The symbol \( \mathfrak{m}(A) (n \in \mathbb{N}) \) will denote the constant function from \( A^n \) to \( A^0 = \{0\} \).

The symbol \( \mathfrak{pr}^i(A) (n \in \mathbb{N}^*, i \in \mathfrak{n}) \) will denote the \( i \)th projection from \( A^n \) onto \( A \): if \( (a_1, \ldots, a_n) \in A^n \), then \( \mathfrak{pr}^i(A)(a_1, \ldots, a_n) = a_i \); in particular, \( \mathfrak{pr}^0(A) \equiv \mathfrak{m}(A) \) is the identity function on \( A \). If \( (f_i; A^n \rightarrow A; i \in \mathfrak{n}) \) is a family of functions, then \( \langle f_i; i \in \mathfrak{n} \rangle \) will denote the function from \( A^n \) into \( A^* \) defined as

\[ \langle f_i; i \in \mathfrak{n} \rangle(x) = \{f_1(x), \ldots, f_\mathfrak{n}(x)\} \quad \text{for all} \quad x \in A^n. \]

If \( A \) is a set, then \( \lambda_f \{\uparrow\} \) will denote the mapping assigning to any function \( f \colon B \times A \rightarrow A \) the function \( \lambda_f(f) \colon g \colon B \rightarrow A^* \) defined by

\[ g(b)(a) = f(b, a) \quad \text{for all} \quad b \in B \quad \text{and} \quad a \in A, \]

and to any function \( f \colon A \rightarrow A \) the function \( \lambda_f(f) \colon A^0 \rightarrow A^* \) defined by \( g(0) = f \).

1.2. Let \( f \colon A \rightarrow B \) and \( g \colon C \rightarrow D \) be functions. We shall say that the composition \( g \circ f \) is defined iff the set of values of \( f \) is contained in the domain of \( g \); if that is the case, then \( g \circ f \) is a function from \( A \) to \( D \). Let us note that, contrary to our convention, the composition \( g \circ f \) is usually considered only in the case of \( B = C \).

1.3. For all unexplained terms concerning category theory we refer the reader to MacLane [3]. If \( K \) is a category, then \( ObK \) will denote the class of all objects of \( K \), and \( ArK \) will denote the class of all arrows of \( K \). If \( f \colon A \rightarrow B \) is an arrow, then \( \text{dom}(f) \) will denote its domain \( A \), and \( \text{cod}(f) \) will denote its codomain \( B \). If \( A, B \in \text{ObK} \), then \( K(A, B) \) will denote the set ("hom-set") of all arrows with domain \( A \) and codomain \( B \). The composition of arrows \( f \colon A \rightarrow B \) and \( g \colon B \rightarrow C \) will be denoted by \( gf \colon A \rightarrow C \) or sometimes by \( g \cdot f \colon A \rightarrow C \). The opposite category of \( K \) will be denoted by \( K^{op} \).

The category of sets and functions is denoted by Set. The symbol \( \mathcal{K}(\mathfrak{I}_1, \mathfrak{I}_2) \colon K^{op} \times K \rightarrow \text{Set} \) denotes the hom-bifunctor.

If \( (f_i; A \rightarrow B_i; i \in \mathfrak{n}) \) is a family of arrows in \( K \) and \( B \) is a categorical product \( B_1 \times \ldots \times B_\mathfrak{n} \), with product projections \( p_i \colon B \rightarrow B_i \), then the symbol \( \langle f_i; i \in \mathfrak{n} \rangle \) or \( \langle f_1, \ldots, f_\mathfrak{n} \rangle \) will denote the unique arrow \( h \colon A \rightarrow B \) such that \( p_i \circ h = f_i \) for all \( i \in \mathfrak{n} \).

1.4. A congruence on a category \( K \) is an equivalence relation \( R \) on \( ArK \) satisfying the following conditions:

- (c1) if \( f \not\in f' \), then \( \text{dom}(f) = \text{dom}(f') \) and \( \text{cod}(f) = \text{cod}(f') \);
- (c2) if \( f, f' \in K(A, B) \), then \( g \cdot f, g \cdot f' \in K(B, C) \), and \( g \cdot f = g \cdot f' \) whenever \( g \cdot f = g \cdot f' \); and \( g \cdot f' = g \cdot f' \).
- If \( R \) is a congruence on \( K \), then the quotient category \( K/R \) has the same objects as \( K \) and \( (K/R)(A, B) = K(A, B)/R_{A,B} \), where \( R_{A,B} \) is the restriction of \( R \) to \( K(A, B) \); it follows from (c2) that the composition of arrows in \( K \) induces the composition of arrows in \( K/R \).

1.5. An algebraic theory (cf. Lawvere [2]) is a triple \( T = (T, \{1\}, P) \) such that

- \( \{1\} \) is a category, \( \{1\} \) is a bijection \( n \mapsto [n] \) with domain \( N \) and codomain \( \text{Ob} T \), and \( P = \{p^n; n \in \mathbb{N}^*, i \in \mathfrak{n}\} \) is a family of arrows of \( T "); \( \{1\} \) is a family of product projections for all \( n \in \mathbb{N}^* \).

- \( \{1\} \) is the object \([n]\) of \( n \mapsto [n] \) for all \( n \in \mathbb{N}^* \) and all \( i \in \mathfrak{n} \).

- \( \{1\} \) the object \([n]\) is the product of \( n \) copies of \([1]\) for all \( n \in \mathbb{N} \), and \( \{p^n; i \in \mathfrak{n}\} \) is a family of product projections for all \( n \in \mathbb{N}^* \).

It follows from \( \{1\} \) that the object \([0]\) is the product of the empty family of objects, i.e. \([0]\) is a terminal object in \( T \); in other words, for any \( n \in \mathbb{N} \) there is a unique arrow in \( T \) from \([n]\) to \([0]\). This arrow will be denoted by \( !: [n] \rightarrow [0] \).
An algebraic theory $T$ can be considered as a category with selected products $[m] \times [n] = [m + n]$. We denote products $[m] \rightarrow [n]$ by $[m] 	imes [n] ightarrow [m]$, $[m] 	imes [n] ightarrow [n]$. Defined by

$$
\text{pr}_1^{[m] \times [n]} = \begin{cases} m & \text{if } m \in N^+, n \in N; \\ n & \text{if } m = 0, n \in N; \\
\end{cases}
$$

$$
\text{pr}_2^{[m] \times [n]} = \begin{cases} m & \text{if } n \in N^+, m \in N; \\ n & \text{if } n = 0, m \in N.
\end{cases}
$$

In the sequel we shall use the shorter notation:

$$
\text{pr}_1^{[m] \times [n]} \equiv \text{pr}_{m,n}, \quad \text{pr}_2^{[m] \times [n]} \equiv \text{pr}_{m,n}.
$$

The selected products $[m] \times [n]$ give rise to the product-bifunctor $\times : T \times T \rightarrow T$, which is defined on arrows in the following way:

If $f : [m] \rightarrow [n]$, $g : [k] \rightarrow [j]$ are arrows of $T$, then $f \times g : [m + k] \rightarrow [n + j]$ is the unique arrow of $T$ such that

$$
\text{pr}_i^{m\times n} \cdot (f \times g) = f \cdot \text{pr}_i^m \quad \text{and} \quad \text{pr}_{n+j} \cdot (f \times g) = g \cdot \text{pr}_n^k.
$$

In particular, for $g = \text{id}_{[i]}$, we obtain the following formula, which will repeatedly be used in this paper:

$$
f \times \text{id}_{[i]} = \begin{cases} \text{pr}_i^1 \cdot f \cdot \text{pr}_{m,n}, & \text{for } f \in T([m],[n]), m \in N, n \in N^+; \\
\text{pr}_n^i \cdot f \cdot \text{pr}_{m,n} & \text{for } f \in T([m],[n]), m \in N, n = 0;
\end{cases}
$$

1.7. A restricted algebraic congruence on $T = (T, \{1\}, P)$, is an equivalence relation on $\bigcup_{m,n} T([m],[1])$ satisfying the following conditions:

(c1) if $fRf'$, then $\text{dom}(f) = \text{dom}(f')$,

(c2) if $\text{dom}(g) = [m]$, $gRg'$ and $fRf'$ for all $i \in m$, then $g \cdot \text{id}_{[i]} \in E \cdot g' \cdot \text{id}_{[i]}$.

1.8. Let $T = (T, \{1\}, P)$ be an algebraic theory and let $K$ be a category with finite products (including a terminal object). We shall say that a functor $G : T \rightarrow K$ is a p-functor iff $G(\{[n]\})$ is a terminal object in $K$ and $\{G(\text{pr}_i^n) : i \in n\}$ is the family of product projections in $K$ for all $n \in N^+$. It is easy to verify that $G$ is a p-functor iff $G$ preserves finite products.

We shall say that a functor $G : T \rightarrow K$ is an ep-functor iff

$$
G(\{[n]\}) = \{[1]\} \quad \text{for all } n \in N, \quad \text{where } A = G(\{1\})
$$

and

$$
G(\text{pr}_i^n) = \text{pr}_i^n(A) \quad \text{for all } n \in N^+, \quad i \in n
$$

(i.e. $G$ preserves specified finite products).

1.9. A cartesian closed category is a category $K$ equipped with the following adjunctions (cf. MacLane [3], p. 98):

1. there is a right adjoint functor $1 : 1 \rightarrow K$ to the unique functor $K \rightarrow 1$ (the category with one object and one arrow),

2. there is a right adjoint functor $1 \times 1 : K \times K \rightarrow K$ to the diagonal functor $K \rightarrow K \times K$ (the diagonal functor is given by $f \mapsto (f,f)$),

3. for each $A \in K$, there is a right adjoint functor $A : K \rightarrow K$ to the functor $\times A : K \rightarrow K$.

The functor $A$ is called an exponentiation by $A$ and the counit of the adjunction is denoted by $\epsilon A_A$. For any $B \in K$, the arrow $\epsilon A_A : B \times A \rightarrow B$ is the component of the counit $\epsilon A_A$.

If $f : C \times A \rightarrow B$ is an arrow in a cartesian closed category $K$, then $\lambda_{A,B}(f)$ will denote a unique arrow $h : C \rightarrow B \times A$ in $K$ such that the following diagram commutes:

$$
\begin{array}{ccc}
B \times A & \xrightarrow{\epsilon A_A} & B \\
\downarrow \lambda_{A,B} & & \downarrow f \\
C \times A & \xrightarrow{h} & A
\end{array}
$$
Let us note that in the cartesian closed category Set the exponentiation by $A$ is the same as the covariant hom-functor $(A)^j \rightarrow \text{Set}(A, j)$. An example of a cartesian closed category is the category Clatt of all complete lattices and continuous functions (cf. Scott [4]).

2. General lambda-theories

2.1. We shall use $\lambda$-terms in a modified form, using different symbols for free and bound variables. Let $(\overset{\ast}{\xi} : i \in N^*)$ and $(\overset{\ast}{\eta} : i \in N^*)$ be two families such that if $i \neq j$, then $\text{card}(\overset{\ast}{\xi}, \overset{\ast}{\eta}, \overset{\ast}{\xi}, \overset{\ast}{\eta}) = 4$ ($\overset{\ast}{\xi}$ is a free variable, $\overset{\ast}{\eta}$ is a bound variable), and let $C$ be a set of constant symbols (elements of $C$ are different from free and bound variables, $C$ can be empty). The set $\text{Exp}(C)$ of $\lambda$-terms (more precisely $\hat{\lambda}$-terms) is defined by induction as follows:

(i) each constant symbol and each free variable is an element of $\text{Exp}(C)$;

(ii) if $M$ and $N$ are elements of $\text{Exp}(C)$, then $(MN) \in \text{Exp}(C)$;

(iii) if $M$ is an element of $\text{Exp}(C)$ and $j = \min\{k \mid \overset{\ast}{\eta}_k \text{ does not occur in } M\}$, then $\lambda\overset{\ast}{\eta}_j (\overset{\ast}{\xi}_j / \overset{\ast}{\eta}_j) M$ is an element of $\text{Exp}(C)$, where $(\overset{\ast}{\xi}_j / \overset{\ast}{\eta}_j) M$ is the result of substituting $\overset{\ast}{\eta}_j$ for $\overset{\ast}{\xi}_j$ in $M$, for each free variable $\overset{\ast}{\xi}_j$.

Remark. The $\lambda$-terms defined above are in a one-to-one correspondence with equivalence classes of terms defined in the classical way (cf. Barendregt [1], p. 1096), where we identify terms differing only in the names of their bound variables (see Barendregt [1], p. 1097).

The set $\text{Exp}(C)$ will also be denoted by $\text{Exp}$. The set $(\overset{\ast}{\xi}_i : i \in N^*)$ of all free variables will be denoted by $V$. If $M$ is a $\lambda$-term, then $\text{BV}(M)$ will denote the set of all bound variables occurring in $M$, and $\text{FV}(M)$ will denote the set of all free variables occurring in $M$.

2.2. We shall use the following notion of simultaneous substitution for $\lambda$-terms: if $(\overset{\ast}{\xi}_i : i \in n)$ is a family of free variables such that $i \neq k$ implies $\overset{\ast}{\xi}_i \neq \overset{\ast}{\xi}_k$, and if $(N_i : i \in n)$ is a family of $\lambda$-terms, then the result of simultaneous substitution of $N_i$ for $\overset{\ast}{\xi}_i$ in a $\lambda$-term $M$ is the term denoted by

$$(\overset{\ast}{\lambda}_j (\overset{\ast}{\xi}_j / \overset{\ast}{\eta}_j) M) \mid (\overset{\ast}{\sigma}_n \overset{\ast}{\tau}_m / \overset{\ast}{\psi}_p) M,$$

obtained from $M$ by replacing $x_j$ by $N_i$ for each $i \in n$, with a suitable change of bound variables. More precisely, $(\ast)$ is defined by induction as follows:

(i) $(\overset{\ast}{\lambda}_j (\overset{\ast}{\xi}_j / \overset{\ast}{\eta}_j) M) \mid \overset{\ast}{\sigma}_p = \overset{\ast}{\sigma}_p$ for each $c \in C$,

$(\overset{\ast}{\lambda}_j (\overset{\ast}{\xi}_j / \overset{\ast}{\eta}_j) M) \mid \overset{\ast}{\sigma}_p = \overset{\ast}{\sigma}_p$ if $p = j$,

$(\overset{\ast}{\lambda}_j (\overset{\ast}{\xi}_j / \overset{\ast}{\eta}_j) M) \mid \overset{\ast}{\sigma}_p = \overset{\ast}{\sigma}_p$ if $p$ is different from $j_1, \ldots, j_n$.
The general lambda-theory \((\Sigma, \text{con})\) is called the pure type-free \(\lambda\)-calculus and the general lambda-theory \((\Sigma, \text{com})\) is called the pure type-free \(\lambda\)-calculus. Another example of a lambda-theory is the ordered pair \((\Omega, Q)\), where \(\Omega\) is a fixed constant and \(Q\) is the smallest conversion satisfying:

\[(\Omega)\]

\[\lambda z : \Omega \rightarrow \Omega\]

and

\[(\Omega)\]

\[\forall M \in \text{Exp}[\Omega] (Q)\]

(cf. Barendregt [1], p. 1126).

The above examples of lambda-theories give rise to the concept of an equational lambda-theory. An equational lambda-theory is an ordered triple \((\Omega, E, \Theta)\), where \(E \subseteq \text{Exp}[\Omega] \times \text{Exp}[\Omega] \) and \(\Theta\) is the smallest conversion on \(\text{Exp}[\Omega]\) satisfying \(E \subseteq \Theta\).

3. Labelled \(\lambda\)-terms

We shall now introduce the notion of a labelled \(\lambda\)-term, which is needed for the construction of algebraic theories from \(\lambda\)-terms. For any \(\lambda\)-term \(M\), we define the rank of \(M\) to be 0 if \(\text{FV}(M) = \emptyset\) and \(\max \{ i : x_i \in \text{FV}(M) \}\) otherwise. The rank of \(M\) will be denoted by \(\text{rn}(M)\). A labelled \(\lambda\)-term is an ordered pair \((M, n)\), where \(M\) is a \(\lambda\)-term and \(n \geq \text{rn}(M)\). The number \(n\) will be called the index of \((M, n)\). The set \(\{(M, n) : M \in \text{Exp}[\Omega] \}\) and \(\text{rank}(M) \leq n \in \mathbb{N}\) of all labelled \(\lambda\)-terms will be denoted by \(\text{Exp}^n[\Omega]\).

3.1. PROPOSITION. The set \(\text{Exp}^n[\Omega]\) is equal to the set \(\mathfrak{R}[\Omega]\) defined by induction as follows:

(i) \((e, n) \in \mathfrak{R}[\Omega] \) for all \(e \in \Omega\) and all \(n \in \mathbb{N}\),

(ii) \((x_i, n) \in \mathfrak{R}[\Omega] \) for all \(x_i \in X^*\) and all \(i < n\);

(iii) \((\lambda x_i : \Omega)(e, n) \in \mathfrak{R}[\Omega] \) if \(j = \min \{ k : x_k \notin \text{FV}(M) \}\),

where \(\lambda x_i : \Omega\) denotes the result of substituting \(x_i\) for \(x_0\) in \(M\).

Sketch of proof. For any \(\lambda\)-term \(M\), let the degree of \(M\), denoted by \(\text{deg}(M)\), be the total number of occurrences of symbols \(\lambda\) and \(\text{in}(M)\). By induction on the degree of a \(\lambda\)-term we can prove that

(a) \((\lambda x_i : \Omega)(M) \in \text{Exp}[\Omega]\) and \(\text{deg}(\lambda x_i : \Omega)(M) = \text{deg}(M)\) for all \(M \in \text{Exp}[\Omega]\),

and by induction on the length of an expression \(M\) we can prove that

(b) if \(x_i\) does not occur in \(M\), then \((\lambda x_i : \Omega)(M) = (\lambda x_i : \Omega)((x_i : \Omega)(M))\),

where \((x_i : \Omega)(M)\) denotes the result of substituting \(x_i\) for \(x_0\) in \(M\).

Using (a) and (b), we infer by induction on the degree of a \(\lambda\)-term that \(\text{Exp}^n[\Omega] \subseteq \mathfrak{R}[\Omega]\). The inclusion \(\mathfrak{R}[\Omega] \subseteq \text{Exp}^n[\Omega]\) is immediate.

3.2. PROPOSITION. If \((M, n+1)\) is a labelled \(\lambda\)-term and \((XI, n) : i \in \mathbb{N}\) is a family of labelled \(\lambda\)-terms, then

\[
\text{[s_1}_1 : X_{11}, \ldots, \text{s}_n : X_n, \lambda x_i : \Omega] \text{e}_i \text{e}_j = \lambda x_i : \Omega(e_{i+1}) M \text{e}_j \text{e}_j = \lambda x_i : \Omega(e_{i+1}) M, \noindent where \(j = \min \{ k : x_k \notin \text{FV}(M) \}\)

and

\[
k = \min \{ i : x_k \notin \text{FV}(M) \} \}

Using 3.1 and 3.2, we prove by induction on \(\mathfrak{R}[\Omega]\) the following composition rule for simultaneous substitution:

3.3. PROPOSITION. If \((M, n)\) is a labelled \(\lambda\)-term and \((P_i, k) : i \in \mathbb{N}\) and \((XI, n) : i \in \mathbb{N}\) are families of labelled \(\lambda\)-terms, then

\[
\text{[s}_1 : P_1, \ldots, \text{s}_n : P_n, \lambda x_i : \Omega](s_1 : X_{11}, \ldots, \text{s}_n : X_n) = [\text{s}_1 : P_1, \ldots, \text{s}_n : P_n, \lambda x_i : \Omega](s_1 : X_{11}, \ldots, \text{s}_n : X_n); j \in \mathbb{N}. \]

4. Algebraic theories constructed from \(\lambda\)-terms

4.1. Let \(T[\Omega] = (T[\Omega], [1], P[\Omega])\) be the following algebraic theory:

the objects of \(T[\Omega]\) are non-negative integers, i.e. \(\text{Ob} T[\Omega] = \mathbb{N}, [n] = n\) for all \(n \in \mathbb{N}\); the arrows from \([n]\) to \([m]\) (\(m \geq 1\)) are \(m\)-tuples of \(X^*\) with index \(n\), i.e.

\[
T[\Omega][[n], [m]] = \{(M, n) : i \in \mathbb{M} \} \forall [n] \in \mathbb{M} \in \text{Exp}^n[\Omega];
\]

the (unique) arrow from \([n]\) to \([0]\) is \((n, 0)\), i.e. \(!^n = (n, 0)\) (for technical reasons we shall assume that the set \(C\) is disjoint with \(N\), in this case all non-sets \(T[\Omega][[n], [m]]\) are disjoint), the composition of \(f = (M, n)\) and \(g = ([n], m)\) is given by

\[
g \circ f = ([n], m) \rightarrow ([m], j) \rightarrow [x] \quad \text{for} \quad m \geq 1; \]

and the composition of \(f : [n] \rightarrow [0]\) and \(h : ([n], 0) \rightarrow [s] \) is given by

\[
h \circ f = ([n], 0) \rightarrow [s] \); \quad \text{the family of projections} \ P[\Omega] = \{p_0, p_1, \ldots, p_m\} \quad \text{as} \quad p_0^n = (n, n) : [n] \rightarrow [1]
\]

By 3.3 the composition is associative and by the definition of simultaneous substitution the family \(P[\Omega]\) is, in fact, the family of projections, i.e. condition 1.3(2) holds for \(P[\Omega]\).
4.2. Now we shall introduce a concept of a lambda-congruence.

A lambda-congruence on \( \bigcup_{n \in \mathbb{N}} T(n) \) or on \( \text{Exp}^* \) is a restricted algebraic congruence on \( T[\mathcal{C}] \) (cf. 1.7) satisfying the following conditions:

(i) \( (\lambda \xi: M) \sim (\lambda \xi: M) \),

(ii) \( (\lambda \xi: M) \sim (\lambda \xi: M) \) for \( j = \min \{ k : \xi \notin \text{BV}(M) \} \).

By an algebraic theory constructed from \( \lambda \)-terms, we shall mean the algebraic theory \( T[\mathcal{C}] \) and the quotient algebraic theory \( T[\mathcal{C}]/\sim \), where \( \sim \) is a lambda-congruence.

The following two theorems establish the correspondence (equivalence from the proof-theoretical point of view) between quotient algebraic theories \( T[\mathcal{C}]/\sim \) and general lambda-theories.

4.3. Theorem. If \( (\mathcal{C}, E) \) is a general lambda-theory, then the binary relation \( \sim \) defined on \( \text{Exp}^* \) as follows:

(i) \( (M, n) \sim (N, m) \) iff \( M \equiv N \) and \( m = n \geq \max \{ \text{rn}(M), \text{rn}(N) \} \),

is a lambda-congruence on \( \text{Exp}^* \).

Conversely, if \( \sim \) is a lambda-congruence on \( \text{Exp}^* \), then the pair \( (\mathcal{C}, E) \), where \( E \) is the binary relation defined on \( \text{Exp}^* \) as follows:

(ii) \( M \equiv N \) iff there is an \( n \in \mathbb{N} \) such that \( (M, n) \sim (N, n) \),

is a general lambda-theory.

4.4. Theorem. If \( (\mathcal{C}, E, \Theta) \) is an equational lambda-theory, then the lambda-congruence defined by formula 4.3 (i) for \( E = \Theta \) is the smallest lambda-congruence on \( \text{Exp}^* \) satisfying the following condition:

(iii) \( (M, n) \sim (N, n) \) for all \( n \geq \max \{ \text{rn}(M), \text{rn}(N) \} \).

5. Algebraic theories with application and abstraction

We shall now describe properties of algebraic theories \( T[\mathcal{C}] \) and \( T[\mathcal{C}]/\sim \) in a more categorical way.

5.1. An algebraic theory with application and abstraction is an ordered triple \( \mathcal{S} = (\mathcal{T}, e, (\,)^* ) \), where \( \mathcal{T} \) is an algebraic theory, \( e : (2) \rightarrow (1) \) is a distinguished arrow of \( T \), and \((\,)^* \) is a mapping assigning to each arrow \( f : [n+1] 

where

(i) \( e / \sim = \{(w, w), (\bar{x}, 2)\} / \sim \);
(ii) \([(M, n+1) / \sim] / \sim = \{\lambda \bar{x}_j \cdot (\bar{x}_{j+1} / \bar{x}) M, n \} / \sim \quad \text{for} \quad j = \min (k : l_k \notin \mathcal{B} \mathcal{V} (M)) \),

is a Church algebraic theory (here \((N, m) / \sim\) denotes an equivalence class with respect to \(\sim\)).

5.6. The pure type-free \(\lambda\beta\eta\) calculus leads to a stronger version of the concept of a Church algebraic theory. By an algebraic theory of type \(\lambda\beta\eta\) (cf. Oostdijk [3]) we mean an algebraic theory with application and abstraction \(\langle T, \varepsilon, (1)^* \rangle\) satisfying (5) and the following condition:

\( (\varepsilon \cdot (h \times \text{id}_{(1)^*}))^n = h \quad \text{for all} \quad h \in T([w], [1]), \quad n \in \mathbb{N} \) (cf. 1.5).

By virtue of 1.5 (9) condition (\(\tilde{\eta}\)) is equivalent to the conjunction of the following two conditions:

\( (\varepsilon \cdot (h \times \text{id}_{(1)^*}))^n = h \quad \text{for all} \quad h \in T([0], [1]); \)

\( (\varepsilon \cdot (\langle h \cdot (\varphi \rho)^{n+1} \cdot i \in n \rangle, \varphi \rho^{n+1})^o)^n = h \quad \text{for all} \quad n \in \mathbb{N}^+ \) and all \( h \in T([w], [1]); \)

hence the concept of an algebraic theory of type \(\lambda\beta\eta\) characterizes general lambda-theories \((\mathcal{C}, \mathcal{E})\) with \(\mathcal{E}\) satisfying condition (\(\tilde{\eta}\)) in 2.2, where \(\mathcal{E}\) is put instead of \(\mathcal{C}\) and, in a similar way as Church algebraic theories characterize all general lambda-theories (cf. 4.3, 4.4 and 6.4).

5.7. THEOREM. If \(\mathcal{F} = \langle T, \varepsilon, (1)^* \rangle\) is an algebraic theory of type \(\lambda\beta\eta\), then \(T\) is a cartesian closed category with the exponentiation (cf. 1.9) satisfying

\[ [T]^2 = \mathbb{1}. \]

If \(T = \langle T, [1], P \rangle\) is an algebraic theory, where \(T\) is a cartesian closed category with exponentiation satisfying \([1] = [1]^2\), then the structure of \(T\) is cartesian closed category of \(T\) induces application and abstraction such that \(T\) with those data is an algebraic theory of type \(\lambda\beta\eta\).

Remark. For any algebraic theory with application and abstraction satisfying (5), condition (\(\tilde{\eta}\)) implies condition (\(\varphi\)), but the converse is, in general, not true (cf. the remark in 7.7).

6. Interpretation of labelled \(\lambda\)-terms in algebraic theories with application and abstraction

There is a natural question: does the concept of a Church algebraic theory characterize general lambda-theories completely? After Theorems 4.3, 4.4 and Proposition 5.5, this question reduces to the following question:

is each Church algebraic theory isomorphic to some algebraic theory \(T([1]) / \sim\) constructed from \(\lambda\)-terms? A positive answer to this last question is contained in Theorem 6.4.

6.1. PROPOSITION. Let \(\mathcal{F} = \langle T, \varepsilon, (1)^* \rangle\) be an algebraic theory with application and abstraction. For any set \(\mathcal{C}\) and any function \(f:\mathcal{C}\) to \(T([1]), [1])\), there is a unique function \(h\) from \(\text{Exp}^\mathcal{C}([1])\) to \(\bigcup T([w], [1])\) such that the following conditions hold:

(i) \(h(e, n) = f(e) \cdot (1)^n \) for \(e \in \mathcal{C}, \quad n \in \mathbb{N}\), and \(h(a, n) = \text{pr}_1^n\) for \(a \in N^n, \quad i \in \mathbb{N}\); 
(ii) \(h([M], n) = \varepsilon \cdot h(M, n), \quad h(N, n)\); 
(iii) \(h(\lambda \bar{x}_j \cdot (\bar{x}_{j+1} / \bar{x}) M, n) = (h(M, n+1))^n\).

The function \(h\) defined in the above proposition will be called an interpretation of labelled \(\lambda\)-terms in \(\mathcal{F}\) (more precisely: \(h\) will be called the interpretation induced by \(f\)).

6.2. PROPOSITION. Let \(\mathcal{F} = \langle T, \varepsilon, (1)^* \rangle\) be a lambda-algebraic theory and let \(\mathcal{C}\) be a set. If \(h\) is an interpretation of labelled \(\lambda\)-terms in \(\mathcal{F}\), then

\[ h([x, N], i) \in \mathcal{C} \mathcal{V} (M), \quad m = h(M, n) \cdot (\varepsilon h(\bar{x}, m), i \in n) \]

for all \(m \in N^n, \quad (M, n) \in \text{Exp}^\mathcal{C}([1]), \quad (N, m) \in \text{Exp}^\mathcal{C}([1])\).

6.3. PROPOSITION. Let \(\mathcal{F} = \langle T, \varepsilon, (1)^* \rangle\) be a Church algebraic theory, let \(\mathcal{C}\) be a set, and let \(h\) be an interpretation of labelled \(\lambda\)-terms in \(\mathcal{F}\). The binary relation \(\sim\) defined on \(\text{Exp}^\mathcal{C}([1])\) as follows:

\( (M, n) \sim (N, m) \iff h(M, n) = h(N, m) \)

is a lambda-congruence on \(\text{Exp}^\mathcal{C}([1])\).

The proof follows by Proposition 6.2.

6.4. THEOREM. For each Church algebraic theory \(\mathcal{F} = \langle T, \varepsilon, (1)^* \rangle\) there exist a set \(\mathcal{C}\) and a lambda-congruence \(\sim\) on \(\text{Exp}^\mathcal{C}([1])\) such that the category \(T\) is isomorphic with \(T([1]) / \sim\).

Sketch of proof. Let \(\mathcal{C} = T([0], [1])\) and let \(h\) be the interpretation of labelled \(\lambda\)-terms in \(\mathcal{F}\) induced by the identity function on \(T([0], [1])\). We define \(\sim\) by condition 6.3 (i). Since for each arrow \(g: [s] \to [1]\) in \(T\) we have

\[ h((\ldots ((g \cdot s_{n-1}) s_{n-2}) \ldots) s_0, n) = \varepsilon \cdot (\ldots \varepsilon \cdot (\langle h(\bar{x}, m), \text{pr}_1^n \rangle, \text{pr}_2^n \ldots \text{pr}_{n-1}^n) = g, \]

where \(g: [0] \to [1]\) is the result of abstraction \((1)^*\) applied \(n\) times to \(g\), we conclude that \(h\) is a surjection; hence by 6.3 the mapping

\[ I: \bigcup T([0]) / \sim \cup ([s], [n]) \to \bigcup T([w], [s]) \]

for \(m, n \in N\)
given by

\[ I((M_i, n)/\sim; i \in n) = \langle h(M_i, n); i \in n \rangle \]

is the arrow function of the functor from \( T[C]/\sim \) to \( T \) which is an isomorphism of categories.

6.5. We shall introduce the following definitions: Let \( \mathcal{F} = (T, e, (\{i\})^*) \) and \( \mathcal{F}' = (T', e', (\{i\})^*) \) be Church algebraic theories. A morphism of Church algebraic theories is a triple \( (H, \mathcal{F}, \mathcal{F}') \), where \( H: T \to T' \) is a functor satisfying the following conditions:

\[ H([0]) = [0'], \]

\[ H(p_{T}) = p_{T'}^{a} \quad \text{for all } a \in N^* \quad \text{and all } i \in n, \]

\[ H(e) = e', \]

\[ H((f)^*_i) = (H(f)^*_i)^* \quad \text{for all arrows } f: [n+1] \to [1] \text{ of } T \text{ and all } n \in N. \]

The category \( \text{CHT} \), called the category of Church algebraic theories, has as objects all Church algebraic theories and as arrows from \( \mathcal{F} \) to \( \mathcal{F}' \) all morphisms of Church algebraic theories \( (H, \mathcal{F}, \mathcal{F}') \); the composition of arrows in \( \text{CHT} \) is the composition of functors.

6.6. Theorem. The forgetful functor \( U: \text{CHT} \to \text{Set} \) defined by

\[ U(\mathcal{F}) = T([0], [1]) \]

has a left adjoint \( F: \text{Set} \to \text{CHT} \) with an object function defined by

\[ F(C) = [T[C]/\sim; e\sim, (\{i\})^*/\sim], \]

where \( \sim \) is the smallest lambda-congruence on \( \text{Exp}^*[C] \) and \( e\sim, (\{i\})^*/\sim \) are defined as in 5.5.

7. Functional interpretation of \( \lambda \)-terms

7.1. Let \( A \) and \( C \) be sets. An ordered pair \( (\mu: A \to C, v: C \to A) \) of functions is called a normal pair iff the following conditions hold:

\[ C \subseteq A^4 \quad \text{and} \quad \mu \circ v = \text{id}_C. \]

The following notion is implicitly contained in a paper of Wadsworth [3]: An ordered pair \( (\mu: A \to C, v: C \to A) \) of functions will be called an interpretable pair iff it is a normal pair and there is a function \( h: \text{Exp} \to A^{\lambda(A)} \) satisfying the following conditions:

(i) \( h(e_{a_1}) = e_{a_1} \) for all \( v \in A^F \) and all \( i \in N^* \),

(ii) \( h([MN])(v) = \left\{ \mu(h(M))(v) \right\}^* \) for all \( v \in A^F \) and all \( M, N \in \text{Exp} \),

(iii) \( h([\lambda x.(e_{a_1})^{(\{i\})^*}](v)) = \left\{ \lambda x.(h(e_a))^{(\{i\})^*} \right\}(v) \) for all \( v \in A^F \), \( M \in \text{Exp} \), \( i \in N^* \), where \( e_{a_1}: A^4 \times A \to A^F \) is the function defined as follows:

\[ e_{a_1}(a, e_a) = \begin{cases} a & \text{if } n = i, \\ e_a & \text{if } n \neq i. \end{cases} \]

For any interpretable pair \( (\mu, v) \) the function \( h \) from \( \text{Exp} \) to \( A^{\lambda(A)} \), satisfying conditions 7.1 (i)-(iii) is unique: we shall call it the explicit interpretation of \( \lambda \)-terms and denote it by \( [\lambda x.(e_a)]^* \) (the value of \( [\lambda x.(e_a)]^* \) for \( M \) is \( [M]^* \)).

7.2. Theorem. If \( (\mu, v) \) is an interpretable pair and \( M, N \) are elements of \( \text{Exp} \) such that \( M \cong N \), then \( [M]^* = [N]^* \).

This theorem shows that an interpretable pair together with the explicit interpretation of \( \lambda \)-terms may be considered as a model of the pure type-free \( \lambda \)-calculus.

7.3. The examples of interpretable pairs due to D. Scott are

(A) the homeomorphism \( \Phi: D_m \to [D_m \to D_m] \) and its converse (Barendregt [1], p. 1110),

(B) the functions fun: \( [\text{Exp}]^n \to [\text{Exp}]^n \), graph: \( [\text{Exp}]^n \to [\text{Exp}]^n \) (Barendregt [1], p. 1106).

7.4. Let \( (\mu: A \to C, v: C \to A) \) be a normal pair and let \( e_{a_1} \) be the function from \( A^2 \) to \( A \) defined by

\[ e_{a_1}(a, a_1) = \mu(a_1)(a_2) \quad \text{for all } a, a_1, a_2 \in A. \]

We define by induction a family \( (E_{a_1}^n; n \in N; k \in N) \) of sets

\[ E_{a_1}^0 = \emptyset, \quad E_{a_1}^1 = \{[i]_k\}, \quad E_{a_1}^2 = \{[p_{T'}(A), p_{T'}^1(A), e_{a_1}^2] \}, \quad E_{a_1}^n = \{p_{T'}^n(A); i \in N\} \quad \text{for } n > 2, \]

and

\[ E_{a_1}^{n+1} = E_{a_1}^n \cup \{e_{a_1}^n(f_1, f_2); f_1, f_2 \in E_{a_1}^n \} \cup \{g; \exists f \in E_{a_1}^{n+1} \text{ the composition } \nu \circ \lambda_{a_1}(f) \text{ is defined and } g = \nu \circ \lambda_{a_1}(f) \} \]

(cf. convention 1.2).

\( A^n(\mu, v) \) will denote the set \( \bigcup_{a_1 \in A} E_{a_1}^n. \)

The family \( A(\mu, v) = \{A^n(\mu, v); n \in N\} \) gives rise to the following definition: a regular pair is a normal pair \( (\mu: A \to C, v: C \to A) \) satisfying the following condition:

\( (Ab) \) for each \( n \in N^* \) and each \( f \in A^n(\mu, v) \) the composition \( \nu \circ \lambda_{a_1}(f) \) is defined.
7.5. Proposition. For each regular pair \((\mu: A \to C, \nu: C \to A)\) there is a unique function \(J_\nu: \text{Exp}^*(\emptyset) \to \bigcup_{n \in \mathbb{N}} A^n(\mu, \nu)\) satisfying the following conditions:

(i) \(J_\nu(a_1, n) = \text{pr}_1^n(A)\) for all \(a \in A^n\) and all \(i \in n\);

(ii) \(J_\nu([M, N], n) = \varepsilon^0 \circ J_\nu(M, n), J_\nu(N, n)\);

(iii) \(J_\nu([\langle s \rangle_n, \langle t \rangle_n], M, n) = \varepsilon_\mu \circ J_\nu(M, n + 1)\).

The function \(J_\nu\) in Proposition 7.5 will be called a functional interpretation of \(\lambda\)-terms. Immediately from Proposition 7.5 we have the following characterization of interpretable pairs, which does not involve the notion of an interpretation of \(\lambda\)-terms:

7.6. Theorem. A pair \((\mu, \nu)\) is an interpretable pair iff it is a regular pair.

7.7. For any interpretable pair \((\mu: A \to C, \nu: C \to A)\) the family \(A(\mu, \nu)\) gives rise to the category \(T_\mu^n\) having as objects all sets \(A^n\) \((n \in \mathbb{N})\) and as arrows \(f: A^n \to A^m\) all functions of the form \(\langle f_i: i \in n\rangle\), where \(f_i \in A^n(\mu, \nu)\) for all \(i \in n\); the composition of arrows in \(T_\mu^n\) is the composition of functions.

The triple \((\mu, \nu, \text{pr}_1^n(A))\), where \([n] = A^n\) and \(P = \text{pr}_1^n(A): n \in \mathbb{N}, i \in n\), is an algebraic pair (note that \(m \neq n\) implies \(A^m \neq A^n\) even in the case of \(\text{card}A = 1\)).

The triple \((\mu, \nu, \text{pr}_1^n(A))\) is defined by

\[
(f)_n = \nu \circ \lambda_\mu([f]) \quad \text{for all } f \in T_\mu^n([m+1], [1]), \quad m \in \mathbb{N},
\]

is a Church algebraic theory. We shall call \(J_\nu\) the Church algebraic theory constructed from the interpretable pair \((\mu, \nu)\).

Remark. The Church algebraic theory constructed from the interpretable pair \((\mu, \nu)\) is an algebraic theory of type \(\lambda\)-theory.

7.8. There is another characterization of interpretable pairs. Let \((\mu: A \to C, \nu: C \to A)\) be a normal pair, and let \(\varepsilon_\mu\) be the function

\[
\varepsilon_\mu = \langle \varepsilon^0 \circ \langle \text{pr}_1^n(A), \text{pr}_2^n(A)\rangle, \varepsilon^0 \circ \langle \text{pr}_1^n(A), \text{pr}_2^n(A)\rangle \rangle.
\]

We shall call \((\mu, \nu)\) a combinatorial pair if all compositions in the following expressions are defined (cf. convention 1.2):

\[
\nu \circ \lambda_\mu[\text{id}_n], \quad \nu \circ \lambda_\mu[\nu \circ \lambda_\mu[\text{pr}_1^n(A)]], \quad \nu \circ \lambda_\mu[\nu \circ \lambda_\mu[\varepsilon_\mu]].
\]

Let

\[
I_\nu = \nu \circ \lambda_\mu[\text{id}_n], \quad K_\nu = \nu \circ \lambda_\mu[\nu \circ \lambda_\mu[\text{pr}_1^n(A)]],
\]

\[
S_\nu = \nu \circ \lambda_\mu[\nu \circ \lambda_\mu[\nu \circ \lambda_\mu[\varepsilon_\mu]]].
\]

The family \((R^*; n \in \mathbb{N}, k \in \mathbb{N})\) is defined by induction as follows:

\[
R_0^* = (I_\nu, K_\nu, S_\nu), \quad R^*_k = (I_\nu \circ \lambda_\mu[I_\nu], K_\nu \circ \lambda_\mu[I_\nu], S_\nu \circ \lambda_\mu[I_\nu], i_d),
\]

\[
E_1^* = (I_\nu \circ \lambda_\mu[I_\nu], K_\nu \circ \lambda_\mu[I_\nu], S_\nu \circ \lambda_\mu[I_\nu], i_d, p_7[I_\nu], p_7[I_\nu], p_7[I_\nu], p_7[I_\nu]),
\]

\[
E_{n+1}^* = (I_\nu \circ \lambda_\mu[I_\nu], K_\nu \circ \lambda_\mu[I_\nu], S_\nu \circ \lambda_\mu[I_\nu] \cup \langle \text{pr}_1^n(A) \vdash i_d \rangle, n > 2,
\]

\[
E_{n+1}^* = E_n^* \cup \langle \varepsilon_\mu \circ \langle f_1, f_2 \rangle, f_1, f_2 \in E_n^* \rangle.
\]

\(\mathcal{W}^*(\mu, \nu)\) will denote the set \(\bigcup_{n \in \mathbb{N}} E_n^*\).

7.9. Theorem. A normal pair \((\mu, \nu)\) is an interpretable pair iff it is a combinatorial pair; moreover,

\(\mathcal{W}^*(\mu, \nu) = A^n(\mu, \nu)\) for all \(n \in \mathbb{N}\).

7.10. The functional interpretation of \(\lambda\)-terms gives rise to the concept of a homomorphism of interpretable pairs.

Let \((\mu: A \to C, \nu: C \to A)\) and \((\mu': B \to D, \nu': D \to B)\) be two interpretable pairs. A lambda-homomorphism of \((\mu, \nu)\) into \((\mu', \nu')\) is a function \(f: A \to B\) such that for any labelled \(\lambda\)-term \((M, n) \in \text{Exp}^*(\emptyset)\) the following diagram is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
\mathcal{W}^*(\mu, \nu) & \xrightarrow{\mathcal{W}^*(\mu, \nu)} & \mathcal{W}^*(\mu', \nu')
\end{array}
\]

7.11. Theorem. Let \((\mu: A \to C, \nu: C \to A)\) and \((\mu': B \to D, \nu': D \to B)\) be two interpretable pairs. A function \(f: A \to B\) is a lambda-homomorphism of \((\mu, \nu)\) into \((\mu', \nu')\) iff the following conditions hold:

\[
\varepsilon'_\mu \circ f = f \circ \varepsilon_\mu, \quad f \circ I_\nu = I'_\nu, \quad f \circ K_\nu = K'_\nu, \quad f \circ S_\nu = S'_\nu.
\]

8. Functional semantics of the type-free \(\lambda\)-calculus.

8.1. Let \(G: T[\emptyset]/\sim_\emptyset \to \text{Set}\) be an sp-functor (cf. 1.8), where \(\sim_\emptyset\) is the smallest lambda-congruence on \(\text{Exp}^*(\emptyset)\), and let \(G([1]) = A\). We shall say that a normal pair \((\mu: A \to C, \nu: C \to A)\) is associated with the functor \(G\) iff the following conditions are satisfied:

(i) \(G(\varepsilon_\mu) = \varepsilon'_\mu: A^1 \to A,\)

(ii) \(G(\langle f, f' \rangle) = \varepsilon_\mu \circ \lambda_\mu(f, f')\) for all \(f \in (T[\emptyset]/\sim_\emptyset)[[n+1], [1]], n \in \mathbb{N}\) (for \(f/\sim_\emptyset\) and \(\langle f^* \rangle/\sim_\emptyset\) see 5.5 (i), (ii)).
3.2. Proposition. If \( (\mu, v) \) and \( (\mu', v') \) are normal pairs associated with the functor \( G \), then \( \mu = \mu' \), \( v = v' \).

Proof. Since \( e_\alpha = G(1) \sim_0 \sim e_\alpha' \), we have for each \( a \in A \)

\[
\mu(a) = \lambda_\alpha [e_\alpha^*](a) = \lambda_\alpha [G(e) \sim_0](a) = \lambda_\alpha [e_\alpha^*](a) = \mu'(a).
\]

Moreover,

\[
v \circ \mu = v \circ \lambda_\alpha [e_\alpha^*] = G([e] \sim_0)^* \sim_0 = v' \circ \lambda_\alpha [e_\alpha^*] = v' \circ \mu',
\]

and hence \( v = v' \), because \( \mu \) is surjective.

3.3. Proposition. If \( (\mu, v) \) is a normal pair associated with some functor \( G \), then \( (\mu, v) \) is a regular pair.

4. We shall now introduce two categories.

The category Int has as objects all interpretable pairs and as arrows \( f : (\mu, v) \rightarrow (\mu', v') \) all triples \( f' = (f, (\mu, v), (\mu', v')) \), where \( f \) is a lambda-homomorphism of \( (\mu, v) \) into \( (\mu', v') \); the composition of arrows in Int is the composition of functions.

By a \( \lambda \)-functorial model we shall mean a \( \lambda \)-functor \( G : T[\emptyset] \sim_0 \rightarrow \text{Set} \) such that there is a normal pair associated with it.

The category \( \text{Fun}_{\lambda} \) has as objects all \( \lambda \)-functorial models and as arrows \( f : G \rightarrow G' \) all natural transformations \( G \rightarrow G' \); the composition of arrows is the composition of natural transformations.

Using the definition of functional interpretations of \( \lambda \)-terms, we may define the functor \( H : \text{Int} \rightarrow \text{Fun}_{\lambda} \), called the identification functor, in the following way:

The object function of \( H \) assigns to each interpretable pair \( (\mu, v) \)

\[
a functor H_{\mu, v} : T[\emptyset] \sim_0 \rightarrow \text{Set} defined as follows:
\]

\[
H_{\mu, v}([M_n, n] \sim_0 i \in m)] = \langle F_{\mu, v}(M_n, n) : i \in m \rangle.
\]

The arrow function of \( H \) assigns to each \( f' = (f, (\mu, v), (\mu', v')) \) the natural transformation

\[
H_{f'} = \{ H_{\mu, v}([M_n, n]) \rightarrow H_{\mu', v'}([M_n, n]) : n \in N \}.
\]

8.5. Theorem. The identification functor \( H : \text{Int} \rightarrow \text{Fun}_{\lambda} \) is an isomorphism of categories.

The proof follows by Propositions 8.2 and 8.3.

8.6. We shall now consider other “models” of the type-free \( \lambda \)-calculus.

A \( \text{pre-\( \lambda \)-object} \) means an ordered pair \( \mathcal{K} = (Y, g) \), where \( Y \) is a set, called a support of \( \mathcal{K} \), and \( g \) is a function from \( \text{Exp} \) to \( Y^{\mathcal{K}} \), called a structure of \( \mathcal{K} \).

Let \( \mathcal{F} = (T, e, (\cdot)^*) \) be an algebraic theory with application and abstraction, let \( h : \text{Exp}^{\mathcal{K}} \rightarrow T[\{\cdot\}, \{\}^*] \) be the interpretation of labelled \( \lambda \)-terms in \( \mathcal{F} \) (cf. 6.1), and let \( G \) be an \( \lambda \)-functor defined on \( T \). We shall say that a \( \text{pre-\( \lambda \)-object} \ \mathcal{K} = (Y, g) \) is associated with the functor \( G \) iff the following conditions are satisfied:

1. \( G([1]) = Y \),
2. \( g([M]e) = G[h([M], n)](e(n_1), \ldots, e(n_k)) \) if \( n = 0 \),
3. \( g([M]0) = 0 \) if \( n = 0 \)

for all \( e \in T^\gamma \).

Let \( \mathcal{A} = (\mathcal{M}, \mathcal{X}) \) be a \( \lambda \)-algebra [a weakly extensional \( \lambda \)-algebra] (cf. Barendregt [1], pp. 1096, 1990), where \( \mathcal{M} = (\mathcal{X}, -) \) is a combinatorial algebra and \( \mathcal{X}^\gamma \) means an assignment \( A \rightarrow \mathcal{X}^\gamma A \). We shall say that a \( \text{pre-\( \lambda \)-object} \ \mathcal{K} = (Y, g) \) is induced by \( \mathcal{A} \) iff the following conditions are satisfied:

1. \( \mathcal{X} = \mathcal{Y} \),
2. \( \mathcal{M}^\gamma \mathcal{X}^\gamma \mathcal{M} \mathcal{M} \) for all \( e \in T^\gamma \), \( M \in \mathcal{M} \), where \( \mathcal{M}^\gamma \mathcal{M} \mathcal{M} \) is defined in Barendregt [1], p. 1098, and \( \mathcal{M}^\gamma \mathcal{M} \) is a \( \lambda \)-term defined in the classical way (cf. Barendregt [1], p. 1096), chosen from the equivalence class corresponding to \( \mathcal{M} \) (cf. Remark in 8.1).

8.7. Theorem. If a \( \text{pre-\( \lambda \)-object} \ \mathcal{K} \) is induced by some \( \lambda \)-algebra, then there is an \( \mathcal{A} \)-functor \( G \) defined on \( T[\emptyset] \sim_0 \) such that \( \mathcal{K} \) is associated with \( G \), where \( \sim_0 \) is defined in 8.1.

8.8. Let \( \mathcal{F} = (T, e, (\cdot)^*) \) be an algebraic theory with application and abstraction. We introduce the following conditions:

1. a weak functorial model of \( \mathcal{F} \) in Set is an \( \lambda \)-functor \( G \) from \( T \) to Set, such that:
2. an ordinary functorial model of \( \mathcal{F} \) in Set is an \( \lambda \)-functor \( G \) from \( T \) to Set satisfying the following condition:
3. a strong functorial model of \( \mathcal{F} \) in Set is an \( \lambda \)-functor \( G \) from \( T \) to Set satisfying the following conditions:
4. there is a normal pair \( (\mu, v) \) \( \text{such that} \)
5. \( (a) \ G([1]) = \mathcal{A}, \)
6. \( (b) \ G(e) = e_\alpha^*, \)
7. \( (c) \ G([f]e) = v \circ \lambda_\alpha [f] \) for all \( f \in T[\{\cdot\}, \{\}^*], \) \( n \in N. \)

The distinction between “weak”, “ordinary”, and “strong” corresponds to the different definitions of interpretation of \( \lambda \)-terms in a “model” of the type-free \( \lambda \)-calculus (it should be stressed that \( \lambda \)-algebras, weakly extensional \( \lambda \)-algebras and interpretable pairs differ essentially in the interpretation of \( \lambda \)-terms). In fact, the class of all interpretable pairs may be identified...
with the class of all strong functorial models of $\mathcal{F}([\mathbb{B}]) \backslash \sim_{0} \cong (T([\mathbb{B}]) \backslash \sim_{0}, s / \sim_{0}, t_{s}^{*} / \sim_{0}, \mathbb{E})$ in $\mathbb{B}$, and the class of all $\lambda$-objects induced by $\lambda$-algebras may be identified with a subclass of the class of all weak functorial models of $\mathcal{F}([\mathbb{B}]) \backslash \sim_{0}$ in $\mathbb{B}$ (cf. Theorems 8.5 and 8.7). Similarly, the class of all $\lambda$-objects induced by weakly extensional $\lambda$-algebras may be identified with a subclass of the class of all ordinary functorial models of $\mathcal{F}([\mathbb{B}]) \backslash \sim_{0}$ in $\mathbb{B}$.

8.9. In definitions 8.8 (1), (2), (3) one may replace $\mathbb{B}$ by an arbitrary cartesian closed category $\mathbb{C}$. This yields the following notions:

1. a weak functorial model of $\mathcal{F}$ in $\mathbb{C}$ is a functor $G: T \to K$ which preserves finite products,

2. an ordinary functorial model of $\mathcal{F}$ in $\mathbb{C}$ is a weak functorial model $G$ of $\mathcal{F}$ in $\mathbb{C}$ satisfying condition 8.8 (2) (i),

3. a strong functorial model of $\mathcal{F}$ in $\mathbb{C}$ is a weak functorial model $G$ of $\mathcal{F}$ in $\mathbb{C}$ satisfying the following condition:

(i) there is an arrow $\kappa: A \to A'$ in $\mathbb{C}$ such that

(a) $G(1) = A$,

(b) $G(\varepsilon) = \varepsilon_{\lambda, A}(\kappa \cdot G(\pi_{T}), G(\pi_{T}'))$,

(c) $G(f)'' = A_{\lambda, \lambda}(G(f)')$ for all $f \in \mathcal{T}([\mathcal{F}]; \mathcal{T}([\mathcal{F}]; [\mathcal{F}]; [\mathcal{F}]; [\mathcal{F}]; [\mathcal{F}]; [\mathcal{F}]; [\mathcal{F}]; [\mathcal{F}]])$, $\mathcal{T} \in \mathcal{T}$.

For example, the Scott models $D_{n}$ and $\mathcal{T}_{n}$ (cf. 7.3) give rise to a strong functorial models of $\mathcal{F}([\mathbb{B}]) \backslash \sim_{0}$ in the category of all complete lattices and continuous functions.

9. Hyperalgebras and hyperoperations

9.1. We shall consider a certain generalization of the notion of an abstract algebra and a homomorphism of algebras. To simplify the notation we shall omit parentheses in the following way: the set $A^{(\varepsilon)}$ will be denoted by $A^{\varepsilon}$, the function $A^{(\pi)}$ will be denoted by $A^{\pi}$, etc.

Let $A$ be a set and let $p, q, n, q \in A$. A hyperoperation on $A$ of the type $(\pi, n, q)$ is a function of the form

$$\omega: (A^{\pi})^{n} \to A^{\varepsilon},$$

i.e. a function $\omega$ which assigns to each $n$-tuple of functions $\phi_{1}: A^{\pi} \to A, \ldots, \phi_{n}: A^{\pi} \to A$ a function

$$\omega(\phi_{1}, \ldots, \phi_{n}): A^{\pi} \to A.$$

A type of hyperalgebras is a quadruple $\tau = (T, P, Q, Q)$, where $T$ is a set and $P, Q, Q$ are functions from $T$ to $N$. A hyperalgebra of type $\tau$ is a pair

$$\mathfrak{A} = (A, \omega_{\tau}^{A}: t \in A),$$

where $A$ is a set (the underlying set of $A$), and $(\omega_{\tau}^{A}: t \in A)$ is a family of hyperoperations

$$\omega_{\tau}^{A}: (A^{\pi_{\tau}})^{A^{\pi_{\tau}}} \to A^{\pi_{\tau}}.$$

Let $\mathbb{A}$ and $\mathbb{B} = (B_{\tau}, \omega_{\tau}^{B}: t \in A)$ be hyperalgebras of the same type $\tau$. A hyperhomomorphism of $\mathbb{A}$ into $\mathbb{B}$ is a function $f: A \to B$ such that for every $t$ in $T$ and for $p = P(t), n = N(t), q = Q(t)$ the following diagram is commutative:

$$\begin{array}{ccc}
(A^{\pi_{\tau}})^{n} & \xrightarrow{f_{n}} & A^{\pi_{\tau}} \\
\downarrow & & \downarrow \\
(B^{\pi_{\tau}})^{n} & \xrightarrow{g_{n}} & B^{\pi_{\tau}}
\end{array}$$

In other words, $f: A \to B$ is a hyperhomomorphism of $\mathbb{A}$ into $\mathbb{B}$ if and only if the following diagram is commutative:

$$\begin{array}{ccc}
(A^{\pi_{\tau}})^{n} & \xrightarrow{f_{n}} & A^{\pi_{\tau}} \\
\downarrow & & \downarrow \\
(B^{\pi_{\tau}})^{n} & \xrightarrow{g_{n}} & B^{\pi_{\tau}}
\end{array}$$

The following diagram is commutative:

$$\begin{array}{ccc}
A^{n} & \xrightarrow{\omega_{\tau}^{A}} & A \\
\downarrow & & \downarrow \\
B^{n} & \xrightarrow{\omega_{\tau}^{B}} & B
\end{array}$$

i.e.

$$\omega_{\tau}^{A}(f_{1}^{\pi_{\tau}}, \ldots, f_{n}^{\pi_{\tau}}) = \omega_{\tau}^{B}(f_{1}^{\pi_{\tau}}, \ldots, f_{n}^{\pi_{\tau}}).$$

Let $\mathbb{A}, \mathbb{B}, \mathbb{C} = (C, \omega_{\tau}^{C}: t \in A)$ be hyperalgebras of the same type $\tau$. It is easy to verify that if $f: A \to B$ is a hyperhomomorphism of $\mathbb{A}$ into $\mathbb{B}$ and $g: B \to C$ is a hyperhomomorphism of $\mathbb{B}$ into $\mathbb{C}$, then $g \circ f$: $A \to C$ is a hyperhomomorphism of $\mathbb{A}$ into $\mathbb{C}$.

By $\text{Hyp}_{\tau}$ we shall mean the category whose objects are all hyperalgebras of type $\tau$, and whose arrows are all hyperhomomorphisms between hyperalgebras.

We have the forgetful functor

$$U: \text{Hyp}_{\tau} \to \text{Set}$$

with the object function given by

$$U(A, \omega_{\tau}^{A}: t \in A) = A.$$
9.2. The following observation explains the meaning of the definition of a hyperhomomorphism. Let \( \tau = (T, P, N, Q) \) and let \( t \) be a fixed element in \( T \). Consider the bifunctors

\[ E_\tau: \text{Hyp}_P \times \text{Hyp}_N \to \text{Set}, \quad G_\tau: \text{Hyp}_P \times \text{Hyp}_N \to \text{Set} \]
defined as follows:

\[ E_\tau(t_1, t_2) = \{ (U(t_1))^n, U(t_2)^n \}; \]

\[ G_\tau(t_1, t_2) = \{ (U(t_1))^n, U(t_2)^n \}; \]

where \( p = P(t), \quad n = N(t), \quad q = Q(t) \). The family \( \alpha^\tau: \text{Hyp}_P \to \text{Ob} \text{Hyp}_N \) is a dinatural transformation (in the sense of MacLane [3], p. 214) of \( E_\tau \) into \( G_\tau \).

9.3. Examples. (A) Let \( T \) be a set and let \( N: T \to N \) be a function. An algebra of type \( (T, N) \) is a pair \( \mathcal{A} = (A, (\alpha^\tau_1: t \in T)) \), where \( A \) is a set and \( \alpha^\tau_1 \) is a \( (t,n) \)-ary operation on \( A \). An algebra of type \( (T, N) \) is (under the identification \( A^\tau_1 = A \)) the same as a hyperalgebra of type \( (T, 0, N, 0) \), where \( 0: T \to N \) is the function defined by \( 0(t) = 0 \) for all \( t \in T \). It is easy to verify that a function \( f: A \to B \) is a hyperhomomorphism between two hyperalgebras of type \( (T, 0, N, 0) \) (iff it is a homomorphism between corresponding algebras). In other words, the category \( \text{Alg}_{(T,N)} \) of all algebras of the type \( (T, N) \) is isomorphic with the category \( \text{Hyp}_{(T,0,N,0)} \).

(B) Let \( (\mu, \nu) \) be a regular pair with the underlying set \( A \) (for the definition of a regular pair see 7.4). The function \( \mu \) is a hyperoperation

\[ \mu: (A^\tau_1)^\mu \to A^\tau_1 \]

of type \( (0, 1, 1) \) on \( A \). For any \( (M, n) \in \text{Exp}^\tau_0 \) (see 3.0) let

\[ \phi^\tau_{(M,n)}(0) = J_\tau(M, n, 0) \]

be a hyperoperation of type \( (1, 0, n) \) on \( A \) defined by

\[ \phi^\tau_{(M,n)}(0) = J_\tau(M, n, 0) \]

where \( J_\tau^e \) is the functional interpretation of \( \lambda \)-terms defined in 7.5. Let \( T = \{ 0 \} \cup \text{Exp}^\tau_0 \) and let \( P: T \to N, N: T \to N, Q: T \to N \) be functions defined as follows:

\[ P(t) = \begin{cases} 0 & \text{for } t = 0, \\ 1 & \text{for } t \in \text{Exp}^\tau_0, \end{cases} \quad N(t) = \begin{cases} 1 & \text{for } t = 0, \\ 0 & \text{for } t \in \text{Exp}^\tau_0, \end{cases} \]

\[ Q(t) = \begin{cases} 1 & \text{for } t = 0, \\ n & \text{for } t = (M, n) \in \text{Exp}^\tau_0, \end{cases} \]

The pair \( (A, (\omega^\tau_1: t \in T)) \), where

\[ \omega^\tau_1(M, n) = \begin{cases} \mu & \text{for } t = 0, \\ 0 & \text{for } t = (M, n), \end{cases} \]

is a hyperalgebra of type \( (T, P, N, Q) \).

Let \( (\mu^*, \nu^*) \) be another regular pair with the underlying set \( B \). It is easy to verify that a function \( f: A \to B \) is a hyperhomomorphism of a hyperalgebra \( (A, (\omega^\tau_1: t \in T)) \) into a hyperalgebra \( (B, (\omega^\tau_1^*: t \in T)) \) iff \( f: A \to B \) is a lambda-homomorphism in the sense of the definition in 7.10.

10. Partial hyperalgebras and hyperalgebras in cartesian closed categories

10.1. By a partial hyperoperation of type \( (p, n, q) \) on a set \( A \) we shall mean a partial function

\[ \omega: (A^\tau_1)^p \to A^\tau_q, \]

i.e., a function \( \omega: X \to A^\tau_q \) defined on some subset \( X \) of \( (A^\tau_1)^p \). If \( \tau = (T, P, N, Q) \) is a type of hyperalgebras, then a partial hyperalgebra of type \( \tau \) is a pair \( \mathcal{A} = (A, (\alpha^\tau_1: t \in T)) \), where \( A \) is a set and, for any \( t \) in \( T \), \( \alpha^\tau_1 \) is a partial hyperoperation of type \( (P(t), N(t), Q(t)) \) on \( A \). The notion of a hyperhomomorphism of hyperalgebras may also be generalized to the case of partial hyperalgebras. Just as in the case of homomorphisms and partial algebras, we obtain some non-equivalent variants of the notion of a hyperhomomorphism.

10.2. Examples. (C) If \( (\mu, \nu) \) is a regular pair with the underlying set \( A \), then the function \( \nu \) is a partial hyperoperation

\[ \nu: (A^\tau_1)^\nu \to A^\tau_0 \]

of type \( (1, 1, 0) \) on \( A \).

(D) If \( (\mu, \nu) \) is a normal pair with the underlying set \( A \) (for the definition of a normal pair see 7.1), then the construction of sets \( S^\tau(M, n, v) \) described in 7.4 gives rise to a partial hyperalgebra with the underlying set \( A \) and partial hyperoperations

\[ \text{ap}_p: (A^\tau_1)^p \to A^\tau_q \text{ of type } (n, 2, n) \text{ defined by } \text{ap}_p(f, g) = \nu^* \circ (f, g), \]

\[ \text{ab}_p: (A^\tau_1)^p \to A^\tau_0 \text{ of type } (n+1, 1, n) \text{ defined by } \text{ab}_p(f, g) = \mu^* \circ J_\tau(f), \]

\[ \text{c}^\tau_1: (A^\tau_1)^3 \to A^\tau_0 \text{ of type } (1, 0, n) \text{ defined by } \text{c}^\tau_1(0) = \nu^* \circ J_\tau(A), \]

where \( n \in N^*, i \in i. \)
10.3. It is easy to generalize the notion of a hyperalgebra and to define a hyperalgebra in a Cartesian closed category. For example, the Scott model 7.3 (A) gives rise to the following hyperalgebra \( H \) in the Cartesian category \( \mathcal{C} \) of complete lattices and continuous functions: the underlying object of \( H \) is \( D_m \), and the hyperoperations are the following arrows in \( \mathcal{C} \):

- \( a_\cdot \colon [D_m^m \to D_m] \to [D_m^m \to D_m] \) defined by \( a_\cdot (f, g) = \pi_2 \circ (f, g) \),
- \( a_\cdot \colon [D_m^{m+1} \to D_m] \to [D_m^m \to D_m] \) defined by \( a_{\cdot \cdot} (f) = \varphi^{-1} \circ a_{\cdot \cdot} (f) \),
- \( c_\cdot \colon [D_m \times D_m \to D_m] \to [D_m^m \to D_m] \) defined by \( c_\cdot (f) = \pi_2 \circ (D_m) \), where \( [A \to B] \) means the lattice of all continuous functions from the lattice \( A \) to the lattice \( B \).

References


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SOME PROBLEMS OF BCK-ALGEBRAS AND GRISS TYPE ALGEBRAS

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The notions of BCK-algebras and Gruss algebras were formulated first in 1966 (see [2], [3]). For example, BCK-algebras are obtained as unified theory generalizing some elementary and common properties of set-difference in set theory and implication in propositional calculus.

We know the following simple relations in set theory:

\[(A \setminus B) \subseteq (A \setminus C) \subseteq C \setminus B,\]
\[A \setminus (A \setminus B) = B.\]

In propositional calculi, these relations are denoted by

\[(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r)),\]
\[p \rightarrow [(p \rightarrow q) \rightarrow q].\]

From these relationships, we have a new class of algebras as follows:

DEFINITION 1. Let \( X \) be a set with a binary operation \( \ast \) and a constant \( 0 \). \( X \) is called a BCK-algebra if it satisfies the following conditions:

1. \( (x \ast y) \ast (z \ast w) \leq x \ast y \),
2. \( x \ast (x \ast y) \leq y \),
3. \( x \leq x \),
4. \( 0 \leq x \),
5. \( x \leq y \) if and only if \( x \ast y = 0 \).

We introduced another class of algebras which are called Gruss algebras. The notion is an algebraic formulation of negationless logic considered by G. F. C. Gruss [1].

[433]