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WEAK HOMOMORPHISMS OF DISTRIBUTIVE p-ALGEBRAS

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The aim of this note is a description of weak isomorphisms and weak homomorphisms of distributive p-algebras and of double Stone algebras. We shall use the notation and the terminology from [7] and [8].

1. Weak homomorphisms of universal algebras

Let $\langle A; F\rangle$ and $\langle B; G\rangle$ be two universal algebras. Let $P^{(n)}(A)$ and $P^{(n)}(B)$ denote the sets of n-ary polynomials of $\langle A; F\rangle$ and $\langle B; G\rangle$, respectively (for details see G. Grätzer [7]). Following A. Goetz and E. Marczewski (see [6]) we can define the concept of a weak homomorphism as follows.

Let $\varphi\colon A \to B$ be a mapping. If for every $n \geqslant 0$ and every $f \in P^{(n)}(A)$ there exists $g \in P^{(n)}(B)$ such that

$$g(x_1\varphi, \ldots, x_n\varphi) = \big(f(x_1, \ldots, x_n)\big)\varphi \tag{1}$$

and if for every $n \geqslant 0$ and every $g \in P^{(n)}(B)$ there exists $f \in P^{(n)}(A)$ such that (1) is true, then $\varphi\colon A \to B$ is called a *weak homomorphism.*

If φ satisfying the above condition is a bijection, we have a weak isomorphism. It can easily be seen that g satisfying (1) is uniquely determined by f if φ is a bijection. Thus we get a mapping $f \to g$ from $P^{(n)}(A)$ into $P^{(n)}(B)$. (The related notions *polymorphism* and *cryptoisomorphism* are discussed by G. Birkhoff ([0], Chapter VI, and also [1].)

Since, by [4], Theorem 7, any weak epimorphism $\varphi\colon A \to B$ can be decomposed into a homomorphism $\mu\colon A \to A/\theta_\varphi$ and a weak isomorphism $\eta\colon A/\theta_\varphi \to B$, the study of weak homomorphisms can be confined to the weak isomorphism. In order to show that a bijection $\varphi\colon A \to B$ is a weak isomorphism it is enough to prove that there exists a bijection $f \to g$ between $P^{(n)}(A)$ and $P^{(n)}(B)$ for all $n \geqslant 0$ satisfying (1).

We recall some more facts which can easily be derived from the definition of a weak isomorphism.

PROPOSITION 1. *If $\varphi\colon A \to B$ is a weak isomorphism of algebras, then $\varphi^{-1}\colon B \to A$ is also a weak isomorphism. Furthermore, if $\varphi\colon A \to B$ is a weak isomorphism and $\theta \in \mathrm{Con}(A)$, then the relation θ' defined by*

$$x\varphi \equiv y\varphi(\theta') \quad \textit{iff} \quad x \equiv y(\theta)$$

is a congruence relation on B. Moreover, the mapping $\varphi_c\colon \mathrm{Con}(A) \to \mathrm{Con}(B)$ defined by $\theta\varphi_c = \theta'$ is a lattice isomorphism.

COROLLARY 1. $\big(\theta(x, y)\big)\varphi_c = \theta(x\varphi, y\varphi)$.

COROLLARY 2. *Let $\varphi\colon A \to B$ be a weak isomorphism. Then A is subdirectly irreducible if and only if B is subdirectly irreducible.*

PROPOSITION 2. *Let $\varphi\colon A \to B$ be a weak isomorphism. Let $\theta \in \mathrm{Con}(A)$ and let $\theta' = \theta\varphi_c$* (see Proposition 1). *Then the mapping*

$$\bar{\varphi}\colon A/\theta \to B/\theta'$$

defined by $([x]\theta)\bar{\varphi} = [x\varphi]\theta'$ is a weak isomorphism and the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\ \varphi\ } & B \\ \downarrow & & \downarrow \\ A/\theta & \xrightarrow{\ \bar{\varphi}\ } & B/\theta'. \end{array}$$

In order to recognize two different weak isomorphisms we need the following observation. Let us consider a weak isomorphism $\varphi\colon A \to B$ between similar algebras, i.e. $F = G$. We can suppose that the set F of fundamental operations is well ordered and that the algebras in question are of type τ, i.e. $A, B \in K(\tau)$.

Let $P^{(n)}(\tau)$ denote the set of all n-ary polynomial symbols of type τ. It is well known (see [7]) that for $A \in K(\tau)$ every polynomial symbol $\tilde{f} \in P^{(n)}(\tau)$ induces a polynomial $f \in P^{(n)}(A)$. Now, having a weak isomorphism $\varphi\colon A \to B$ for $A, B \in K(\tau)$, we can determine a binary relation $R_\varphi^{(n)} \subseteq P^{(n)}(\tau) \times P^{(n)}(\tau)$ $(n \geqslant 0)$ as follows:

$(\tilde{f}, \tilde{g}) \in R_\varphi^{(n)}$ iff the induced polynomials f and g satisfy (1). Conversely, let $R^{(n)}$ (for every $n \geqslant 0$) be a binary relation on $P^{(n)}(\tau)$ such that for any $\tilde{f} \in P^{(n)}(\tau)$ there exists $\tilde{g}$ with $(\tilde{f}, \tilde{g}) \in R^{(n)}$ and for every $\tilde{q} \in P^{(n)}(\tau)$ there exists $\tilde{p} \in P^{(n)}(\tau)$ with $(\tilde{p}, \tilde{q}) \in R^{(n)}$. We say that a bijection $\varphi\colon A \to B$ for $A, B \in K(\tau)$ is an *R-weak isomorphism* if $(\tilde{f}, \tilde{g}) \in R^{(n)}$ implies that the induced polynomials f and g satisfy (1).

Clearly, any R-weak isomorphism $\varphi\colon A \to B$ is a weak isomorphism and $R^{(n)} \subseteq R_\varphi^{(n)}$. Note that an R-weak isomorphism $\varphi\colon A \to B$ is an *isomorphism* if and only if $R^{(n)}$ $(n \geqslant 0)$ is reflexive.

It is also easily seen that Proposition 2 can be reformulated: *If $\varphi\colon A \to B$ is an R-weak isomorphism, then $\bar{\varphi}\colon A/\theta \to B/\theta'$ is an R-weak isomorphism.*

Now we will show that any weak isomorphism between algebras in an equational class is determined by weak isomorphisms between subdirectly irreducibles. If $\theta \in \mathrm{Con}(A)$, we say that θ is *subdirectly irreducible* if A/θ is a subdirectly irreducible algebra.

THEOREM 1. *Let K be an equational class. Let $A, B \in K$ and let $\varphi\colon A \to B$ be a bijection. Then φ is an R-weak isomorphism if and only if*

(i) *$\theta\varphi_c \in \mathrm{Con}(B)$ for every subdirectly irreducible $\theta \in \mathrm{Con}(B)$;*

(ii) *$\bar{\varphi}\colon A/\theta \to B/\theta\varphi_c$ is an R-weak isomorphism for every subdirectly irreducible $\theta \in \mathrm{Con}(A)$.*

Proof. The necessity of (i) and (ii) follows from Propositions 1 and 2. Let us suppose conditions (i) and (ii). Let $A \subseteq \prod(A_i;\ i \in I)$ be a subdirect representation of A (see [7], Theorem 2.3), i.e. A_i are subdirectly irreducible. Moreover (see [7], Theorem 20.1), there exists a system $\{\theta_i \in \mathrm{Con}(A);$ $i \in I\}$ such that $\bigwedge(\theta_i;\ i \in I) = \Delta$ and $A/\theta_i \cong A_i$, i.e. θ_i are subdirectly irreducible. Therefore, $\theta_i' = \theta_i\varphi_c \in \mathrm{Con}(B)$ by (i), and

$$A_i \cong A/\theta_i \cong B/\theta_i' := B_i$$

applying (ii) for every $i \in I$. Using Corollary 2, we see that B_i is subdirectly irreducible. It is not difficult to prove that $\bigwedge(\theta_i';\ i \in I) = \Delta$. Therefore, $B \subseteq \prod(B_i;\ i \in I)$ is a subdirect representation of B (see [7]). By (ii),

$$\bar{\varphi}_i\colon A_i \to B_i \quad (i \in I)$$

is an R-weak isomorphism. There exists an R-weak isomorphism

$$\bar{\varphi}\colon \prod(A_i;\ i \in I) \to \prod(B_i;\ i \in I)$$

because the operations are defined componentwise. Therefore

$$\varphi\colon A \to B$$

is an R-weak isomorphism since φ is a restriction of $\bar{\varphi}$ to A, and the proof is complete.

COROLLARY 3. *Any weak isomorphism between algebras of an equational class K is an isomorphism if and only if every weak isomorphism of subdirectly irreducible algebras from K is an isomorphism.*

EXAMPLES. 1. It is well known that in the class of all lattices there are at least two kinds of weak isomorphisms, namely the isomorphisms and the dual ones. But in the class of all distributive lattices we have only these two kinds of weak isomorphisms. This follows from the fact that only the two-element lattice is a non-trivial subdirectly irreducible one in this class.

2. A trivial application of Theorem 1 shows also that there are only two kinds of weak isomorphisms between Boolean algebras, which was established by other methods in [10], Theorem 1. (We recall that the only non-trivial subdirectly irreducible Boolean algebra is the two-element one.)

3. It is known that the class of all semilattices has no non-trivial proper equational subclasses. It is not difficult to prove that the only non-trivial subdirectly irreducible semilattice is the two-element one. Let $S = \{a, b\}$ and $a = a \wedge b \neq b$. Then it is easily shown that the bijection $\varphi\colon S \to S$ defined by $a\varphi = b$ and $b\varphi = a$ cannot be a weak isomorphism. Hence any weak isomorphism of semilattices is an isomorphism.

2. Weak homomorphisms of distributive p-algebras

A universal algebra $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a *distributive p-algebra* if $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and

$$x \wedge a = 0 \quad \text{iff} \quad x \leqslant a^*.$$

The standard results can be found in [8].

It is known that the distributive p-algebras form an equational class which has countably many equational subclasses. More precisely, we have an infinite chain

$$\mathscr{B}_{-1} \subset \mathscr{B}_0 \subset \mathscr{B}_1 \subset \ldots \subset \mathscr{B}_n \subset \ldots \subset \mathscr{B}_\infty,$$

where $\mathscr{B}_{-1}$ is the trivial subclass, $\mathscr{B}_0$ is the class of all Boolean algebras defined by

$$x \vee x^* = 1$$

and $\mathscr{B}_1$ is the subclass of *Stone algebras* defined by

$$x^{**} \vee x^* = 1.$$

The subdirectly irreducibles can be described as follows. Let B be an arbitrary Boolean algebra. Let e denote the largest element of B. Now add a new element 1 to B. We get $L = B \cup \{1\}$. We define $1 > x$ for all $x \in B$ and $x \leqslant y$ for $x, y \in B$ if the same is true in the given Boolean algebra B. Now L turns into a bounded distributive lattice $L = B \oplus 1$ and, finally, L is also a p-algebra if we define $1^* = e^* = 0$, $0^* = 1$ and $x^* = x'$ for $x \in B$ (for details see [8]).

PROPOSITION 3. *Let L be a subdirectly irreducible distributive p-algebra. Let $|L| \geqslant 3$. Then every weak isomorphism of L is an isomorphism.*

Proof. Let $\varphi\colon L \to L'$ be a weak isomorphism in the class of distributive p-algebras. By Corollary 2, L' is also subdirectly irreducible. Hence there exist Boolean algebras B and B' such that $L = B \oplus 1$ and $L' = B' \oplus 1'$ (cf. [8]). If e and e' denote the largest elements of B and B', respectively, then $\theta(e, 1)$ $(\theta(e', 1'))$ is the smallest non-trivial congruence relation on L (L', resp.). By Corollary 1, $(\theta(e, 1))\varphi_c = \theta(e\varphi, 1\varphi)$. Therefore, $\{e\varphi, 1\varphi\} = \{e', 1'\}$. Since $L/\theta(e, 1) \cong B$ and $L'/\theta(e', 1') \cong B'$, we get a weak isomorphism

$$\bar{\varphi}\colon B \to B'$$

between Boolean algebras by Proposition 2. By [10], Theorem 1 (see also Example 2), $\bar{\varphi}$ is either an isomorphism or $\bar{\varphi}$ is a dual isomorphism, i.e. $(x \vee y)\bar{\varphi} = x\bar{\varphi} \wedge y\bar{\varphi}$, $(x \wedge y)\bar{\varphi} = x\bar{\varphi} \vee y\bar{\varphi}$, $x^*\bar{\varphi} = x^*$, $0\bar{\varphi} = e'$ and $e\bar{\varphi} = 0'$. But we have seen above that $\{e\varphi, 1\varphi\} = \{e', 1'\}$, which means $e\bar{\varphi} = e'$ and, consequently, $0\bar{\varphi} = 0'$, because 0 and 1 are the only nullary operations of distributive p-algebras. Thus $\bar{\varphi}$ is an isomorphism. Since for every congruence class $[x]\theta(e, 1) = \{x\}$ for $x \notin \{e, 1\}$, $\bar{\varphi}$ determines uniquely φ. Thus φ is an isomorphism.

THEOREM 2. *Let $\varphi\colon L \to L'$ be a weak epimorphism of distributive p-algebras. Let L' be not a Boolean algebra. Then φ is an isomorphism.*

Proof. By [4], Theorem 7, $\varphi = \mu \cdot \eta$, where $\mu\colon L \to L/\theta_\varphi = L_1$ is a (natural) epimorphism and $\eta\colon L_1 \to L'$ is an R-weak isomorphism (for some relations $R^{(n)}$, $n \geqslant 0$). Since L' is not a Boolean algebra, there is in subdirect representation of L' a subdirectly irreducible p-algebra having at least three elements. By Proposition 2, the same is true for L_1. Let A be such a p-algebra, i.e. $|A| \geqslant 3$, $A = L_1/\theta$ and θ is subdirectly irreducible. Therefore,

$$\bar{\eta}\colon L_1/\theta \to L'/\theta\eta_c$$

is an R-weak isomorphism by Theorem 1. But $\bar{\eta}$ is an isomorphism by Proposition 3. That means, $(\tilde{f}, \tilde{f}) \in R^{(n)}$ for every $n \geqslant 0$ and every n-ary polynomial symbol $\tilde{f}$. The last result implies that η is an isomorphism, and finally φ is a homomorphism.

Remark. Proposition 3 and Theorem 2 have been proved in [5], Theorem 3 and Corollary 5, for the class of Stone algebras.

3. Weak homomorphisms of double Stone algebras

A universal algebra $\langle L; \vee, \wedge, {}^*, {}^+, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ is called a *double Stone algebra* if $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ is a Stone algebra and $\langle L; \vee, \wedge, {}^+, 0, 1\rangle$ is a dual Stone algebra.

Double Stone algebras form an equational class $\mathscr{S}$, which contains four equational subclasses (see [9], Theorem 1)

$$\mathscr{S}_0 \subset \mathscr{S}_1 \subset \mathscr{S}_2 \subset \mathscr{S}.$$

$\mathscr{S}_0$ are the trivial algebras, $\mathscr{S}_1$ is the class of Boolean algebras defined by

$$x^* = x^+$$

and $\mathscr{S}_2$ is the subclass of *regular double Stone algebras* defined by

$$x \vee x^* \geqslant y \wedge y^+.$$

The non-trivial subdirectly irreducibles are the two-, three- and four-element chains, considered as double Stone algebras.

It can be seen at first glance that in the class $\mathscr{S}$ there are at least two weak isomorphisms, namely the isomorphism and the dual one defined by

(2) $\quad (0,1), (1,0) \in R^{(0)}, \quad ({}^*, {}^+), ({}^+, {}^*) \in R^{(1)}, \quad (\vee, \wedge), (\wedge, \vee) \in R^{(2)}.$

Since every polynomial is a composition of the fundamental operations, we see that conditions (2) generate a relation corresponding to a weak isomorphism.

PROPOSITION 4. *Any weak homomorphism in the class $\mathscr{S}_2$ of all regular double Stone algebras is either a homomorphism or a dual homomorphism.*

Proof. By [4], Theorem 7, we have only to prove the statement for weak isomorphisms. Following Theorem 1 this can be reduced to weak isomorphisms of subdirectly irreducible algebras. By [9], Theorem 1, the only non-trivial subdirectly irreducibles from $\mathscr{S}_2$ are the two- and the three-element chains. Evidently, there are two weak automorphisms of a two-element chain. Let $\varphi\colon L \to L$ be a weak automorphism of a three-element algebra from $\mathscr{S}_2$. Clearly, L is a chain $0 < a < 1$. Since 0 and 1 are the only nullary polynomials, we get $\{0\varphi, 1\varphi\} = \{0, 1\}$. Hence $a\varphi = a$. Thus there exist two bijections which correspond to an automorphism or to a dual one, and the proof is complete.

Remark. Proposition 4 has been proved first in [10], Theorem 1, for the class of Boolean algebras.

PROPOSITION 5. *Let $L = \{0, c, d, 1\}$ be a chain $0 < c < d < 1$ considered as a double Stone algebra (i.e. $c^* = d^* = 1^* = 0$ and $0^+ = c^+ = d^+ = 1$). Then for any weak automorphism $\varphi\colon L \to L$ we have $\varphi = \varphi_i$ for $i \in \{1, 2, 3, 4\}$ and*

(i) *$\varphi_1\colon L \to L$ is the identity automorphism;*

(ii) *$\varphi_2\colon L \to L$ is a dual automorphism, i.e. $0\varphi_2 = 1$, $1\varphi_2 = 0$, $c\varphi_2 = d$ and $d\varphi_2 = c$, and the bijection of polynomials is generated by $0 \to 1$, $1 \to 0$, $x^* \to x^+$, $x^+ \to x^*$, $x \vee y \to x \wedge y$ and $x \wedge y \to x \vee y$;*

(iii) *$\varphi_3\colon L \to L$ is given by $0\varphi_3 = 0$, $1\varphi_3 = 1$, $c\varphi_3 = d$, $d\varphi_3 = c$ and the bijection of polynomials is generated by $0 \to 0$, $1 \to 1$, $x^* \to x^*$, $x^+ \to x^+$,*

$$x \vee y \to h(x, y) = (x \wedge y) \vee (x \wedge y^*) \vee (x^* \wedge y) \vee (x^{**} \wedge y^{++}) \vee (x^{++} \wedge y^{**}),$$

$$x \wedge y \to \bar{h}(x, y) = (x \vee y) \wedge (x \vee y^+) \wedge (x^+ \vee y) \wedge (x^{**} \vee y^{++}) \wedge (x^{++} \vee y^{**});$$

(iv) *$\varphi_4\colon L \to L$ is given by $0\varphi_4 = 1$, $1\varphi_4 = 0$, $c\varphi_4 = c$, $d\varphi_4 = d$ and the bijection of polynomials is generated by $0 \to 1$, $1 \to 0$, $x^* \to x^+$, $x^+ \to x^*$, $x \vee y \to \bar{h}(x, y)$ and $x \wedge y \to h(x, y)$.*

Proof. If $\varphi\colon L \to L$ is a weak automorphism, then $\{0\varphi, 1\varphi\} = \{0, 1\}$, because 0 and 1 are the nullary polynomials. Therefore we have four possibilities (i)–(iv) for the mapping φ. It is straightforward to prove that φ_1 and φ_2 are weak automorphisms.

(iii) It is easy to check that $h(x, y) = h(y, x)$, $\bar{h}(x, y) = \bar{h}(y, x)$ and

$$h(x, y) = \begin{cases} x \wedge y & \text{for } x, y \in \{c, d\}, \\ x \vee y & \text{otherwise}, \end{cases}$$

$$\bar{h}(x, y) = \begin{cases} x \vee y & \text{for } x, y \in \{c, d\}, \\ x \wedge y & \text{otherwise}. \end{cases}$$

By direct computation we get $0\varphi_3 = 0$, $1\varphi_3 = 1$, $x^*\varphi_3 = (x\varphi_3)^*$, $x^+\varphi = (x\varphi_3)^+$, $(x \vee y)\varphi_3 = h(x\varphi_3, y\varphi_3)$ and $(x \wedge y)\varphi_3 = \bar{h}(x\varphi_3, y\varphi_3)$ for all $x, y \in L_3$. It follows that there exists a 1-1 mapping $f \to g$ from $P^{(n)}(L)$ into $P^{(n)}(L)$ $(n \geqslant 0)$ such that the polynomials f and g satisfy (1). It remains to show that this mapping is onto. But this follows from

$$x \vee y = \bar{h}\big(\bar{h}(r, s),\ h(x, y)\big), \qquad x \wedge y = h\big(h(p, q), \bar{h}(x, y)\big)$$

and

$$\begin{aligned} r &= h\Big(h\big(h(x, y^*),\ h(x, y)^{++}\big),\ h(x, x^*)\Big), \\ s &= h\Big(h\big(h(x^*, y), h(x^{++}, y)\big),\ h(y, y^*)\Big), \\ p &= \bar{h}\Big(\bar{h}\big(\bar{h}(x, y^+),\ \bar{h}(x, y^{**})\big),\ \bar{h}(x, x^+)\Big), \\ q &= h\Big(h\big(h(x^+, y), h(x^{**}, y)\big),\ h(y, y^+)\Big) \end{aligned}$$

for all $x, y \in L$.

(iv) The last case can be handled similarly as (iii), which we leave to the reader. The proof is complete.

It should be noted that $h(x, y) = x \vee y$ and $\bar{h}(x, y) = x \wedge y$ in any regular double Stone algebra. Summarizing, we obtain from Theorem 7 of [4], Theorem 1 and Proposition 5 of this paper, and from Theorem 1 of [9] the following

THEOREM 3. *Let A, B be double Stone algebras and let $\varphi: A \to B$ be a weak homomorphism. Then one of the following statement holds:*

(i) *φ is a homomorphism;*

(ii) *φ is a dual homomorphism, i.e. $0\varphi = 1$, $1\varphi = 0$, $x^*\varphi = (x\varphi)^+$, $x^+\varphi = (x\varphi)^*$, $(x \vee y)\varphi = x\varphi \wedge y\varphi$ and $(x \wedge y)\varphi = x\varphi \vee y\varphi$ for every $x, y \in A$;*

(iii) *$0\varphi = 0$, $1\varphi = 1$, $x^*\varphi = (x\varphi)^*$, $x^+\varphi = (x\varphi)^+$, $(x \vee y)\varphi = h(x\varphi, y\varphi)$ and $(x \wedge y)\varphi = \bar{h}(x\varphi, y\varphi)$ for every $x, y \in A$;*

(iv) *$0\varphi = 1$, $1\varphi = 0$, $x^*\varphi = (x\varphi)^+$, $x^+\varphi = (x\varphi)^*$, $(x \vee y)\varphi = \bar{h}(x\varphi, y\varphi)$ and $(x \wedge y)\varphi = h(x\varphi, y\varphi)$ for every $x, y \in A$, where $h(x, y)$ and $\bar{h}(x, y)$ are defined in Proposition 5.*

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PRÄPRIMALE ALGEBREN, DIE ARITHMETISCHE VARIETÄTEN ERZEUGEN

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Diese Arbeit beschäftigt sich mit endlichen Algebren, bei denen die Menge aller abgeleiteten Operationen, der Operationenklon, maximal in der Klasse aller Funktionen über der Trägermenge der Algebra ist, den präprimalen Algebren. Maximale Klassen der Klasse aller Funktionen über einer endlichen Menge spielen bei der Lösung des Vollständigkeitsproblems in den mehrwertigen Logiken eine Rolle. Hier werden präprimale Algebren betrachtet, die arithmetische Varietäten erzeugen. In Theorem 3.2 wird eine algebraische Charakterisierung dieser präprimalen Algebren gegeben und damit teilweise ein für beliebige präprimale Algebren noch offenes Problem gelöst. Ausgangspunkt ist einerseits die von Rosenberg [9] vorgenommene Klassifizierung der maximalen Klassen von Funktionen über einer endlichen Menge, andererseits die Behandlung von Vollständigkeitsfragen in universalen Algebren in Arbeiten von Foster und Pixley.

1. Grundbegriffe

Sei $E_k = \{0, 1, \ldots, k-1\}$ und $\boldsymbol{F}(E_k)$ die Menge aller Funktionen, die auf E_k definiert sind, d.h. $\boldsymbol{F}(E_k) = \bigcup_{n=0}^{\infty} F_n(E_k)$ mit $F_n(E_k) = \{f: E_k^n \to E_k\}$.

$\boldsymbol{A} = \langle E_k, F\rangle$ sei eine endliche Algebra mit $F \subseteq \boldsymbol{F}(E_k)$ als Menge der Fundamentaloperationen. $\bar{F}$ bezeichne die aus F durch Superposition von Funktionen entstehende Klasse von Funktionen aus $\boldsymbol{F}(E_k)$, die alle Projektionen $e_i^n \in F_n(E_k)$ mit $e_i^n(x_1, \ldots, x_n) = x_i$ $(i = 1, \ldots, n)$, enthalten soll. Dann ist $\bar{F}$ Polynomialklasse oder Operationenklon von $\boldsymbol{A}$.

Zwei Algebren heißen *äquivalent*, wenn sie den gleichen Operationenklon erzeugen.

In den Arbeiten von Foster wird der Begriff der primalen Algebra definiert: