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THE CREATIVE SUBJECT AND HEYTING'S ARITHMETIC

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In intuitionistic analysis the effect of Brouwer's 'historic' or 'epistemic' arguments can be satisfactorily captured by means of Kripke's Schema; which itself can be derived straight-forwardly from Kreisel's axioms for the creative subject. In [1] it is pointed out that Analysis + "the creative subject" is conservative over Analysis + KS.

In the same paper it is conjectured that the addition of "the creative subject" to Heyting's Arithmetic presents a conservative extension.

We will prove this conjecture here.

For completeness we repeat the relevant facts. Kripke's Schema is the following schema:

KS $\quad \exists \xi [A \Leftrightarrow \exists x\, \xi x \neq 0]$.

The axioms for the creative subject are:

CS_1 $\quad \forall x (\vdash_{\overline{x}} A \vee \neg \vdash_{\overline{x}} A)$,

CS_2 $\quad \forall xy (\vdash_{\overline{x}} A \to \vdash_{\overline{x+y}} A)$,

CS_3 $\quad \exists x \vdash_{\overline{x}} A \leftrightarrow A$,

where $\vdash$ is a new connective such that $\vdash_{\overline{t}} A$ is a formula if t is a numerical term and A a formula. CS will denote the conjunction of CS_1, CS_2, CS_3.

HA is the first-order theory of intuitionistic arithmetic.

LEMMA. *Let A be a sentence of* **HA**; *then* $\mathbf{HA} + A \Leftrightarrow \exists x\, fx \neq 0$ *is conservative over* **HA**, *where f is a unary function symbol.*

Proof. We will show that a Kripke model for **HA** can be expanded to a model for $\mathbf{HA} + A \leftrightarrow \exists x\, fx \neq 0$.

It is no restriction to consider only Kripke models with an underlying tree and with the property that for each α and β, with $\alpha < \beta$, $D(\beta) - D(\alpha)$ is non-empty (even denumerable), cf. any of the Henkin–Type completeness proofs, e.g. [2]. Let a Kripke model for **HA** be given. We define a function as follows: Suppose we have defined interpretations $f^{[\alpha]}$ for all $\alpha < \beta$; if $\neg \beta \Vdash A$, then we put $f^{[\beta]}(u) = 0$ for all $u \in D(\beta)$; if $\beta \Vdash A$, then we put

$$f^{[\beta]}(u) = \begin{cases} f^{[\alpha]}(u) & \text{for } u \in D(\alpha), \\ 1 & \text{for } u \notin D(\alpha), \end{cases}$$

where α is the immediate predecessor of β.

It is clear that $A \leftrightarrow \exists x fx \neq 0$ holds in the model.

COROLLARY. **HA** + CS *is conservative over* **HA**.

Proof. We add a function f_A, as in the previous lemma, for each sentence A, and we add all the axioms $A \leftrightarrow \exists x f_A x \neq 0$ to **HA**. Now we define $\vdash_{\overline{x}} A$ by

$$\forall x \left(\vdash_{\overline{x}} A \leftrightarrow \exists y \leqslant x f_A(y) \neq 0 \right).$$

We observe that: (i) axioms CS_1, CS_2, CS_3 hold in this definitional extension; (ii) by the lemma, **HA** + CS is conservative over **HA**.

Note that induction is arithmetic.

This proof was found after A. Joyal mentioned the following result at the Mons conference, paraphrased in out terminology: Let T be a first order theory containing **HA**; then T + "for each formula $A(x)$ with 'decidable equality' (i.e. $A(x) \wedge A(y) \to x = y \vee x \neq y$) there is a decidable (removable) subset S_A of N and bijection f_A from S_A to the extension of $A(x)$" is conservative over **HA**. Joyal's theorem can also be proved by the same techniques. Consider a denumerable Kripke model for T with an underlying tree such that the domain of each α contains denumerably many new numerical elements compared with its immediate predecessor. Now define the bijection node-wise: assume that it has been defined in the immediate predecessor of α, then we extend it to α by using the 'new' numerical elements to establish the bijection with the 'new' elements of the extension of $A(x)$. In this way we guarantee that the domain of the bijection is decidable. It is not customary to use partial functions; one can just as well use a relation in the above proof.

The conservative extension result concerning **HA** at least suggests that, as far as arithmetic is concerned, the creative subject must be able to use a form of reflection, e.g.

$$\mathrm{Prov}_{\mathbf{HA}}(n, \ulcorner A \urcorner) \to \vdash_{\overline{n}} A,$$

or even the free variable version.

2. Addenda and Corrigenda to "An interpretation of intuitionistic analysis", Ann. Math. Logic 13 (1978), 1–43

4, + 9 replace '26' by '27'.
9, − 7 replace '[ξ_i]' by '[β_i]'.
17, −13 replace line by '$S^{[\alpha]} = \{n\}$ iff $\alpha \in 0^n * \langle k \rangle$ and $k \neq 0$'.
18, + 2 add 'or $\langle 0 \rangle \leqslant \alpha$'.
Remark in the models the left most branch should play a special role. In this proof the branch $\lambda x \cdot 1$ has (unfortunately) been chosen.
18, +14, +15 $r \neq 0 \to r \# 0$ (twice).
18, +21 replace 'n' by 't'.
19, −10 '$v \neg \vdash_{\overline{x}} A$'.
20, +10 replace '$\vdash$' by '$\vdash_{\overline{n}}$'.
Remark: on second thought these principles are not that plausible at all. After all, the evidence at stages need not be closed under such rules, it may require some actual work to get $\exists x \vdash_{\overline{m}} A(x)$, given $\vdash_{\overline{n}} \exists x A(x)$.
Of course we have $\vdash_{\overline{n}} A \vee B \to \exists m \geqslant n (\vdash_{\overline{m}} A \vee \vdash_{\overline{m}} B)$, but that is a trivial consequence of the axioms.
29, 6.3.2. The conditions, of course, have to be consistent and 'functional', i.e. if $\varphi(n) = m \in P$ and $\varphi(n) = m' \in P$, then $m = m'$.
30 After (ii) (+5) add: "(iii) φ is total".
30, +11 add "or $\varphi(m) = n' \in P$ for $n' \neq n$".
31, + 8 After 'language' add 'and P_{n+1} forces all $(\varphi)_i$, $i \leqslant n$, to take values for arguments in S_{n+1}. Then φ is total'.
32, − 6 Replace 'm' by '$m+1$' and '$(\varphi)_i \beta$' by '$((\varphi)_i \beta)_m$'.
33, +13 Replace '10' by '11'.
+14, +15 replace '$\vdash$' by '$\Vdash$'.
+16 replace '$\Vdash$' by '$\models$'.
+18, replace '12' by '13'.
−1 replace 'Q'' by 'Q'.
−8, −7 The argument that Q contains only conditions on $(\varphi)_{n_1}, \ldots, (\varphi)_{n_k}$ fails. The proof must be amended as follows: Let Q contain, apart from $(\varphi)_{n_1}, \ldots, (\varphi)_{n_k}$, also $(\varphi)_{m_1}, \ldots, (\varphi)_{m_p}$. The latter occur in the condition because of the presence of quantifiers, and they can be replaced by similar $(\varphi)_{u_i}$'s which coincide with the original ones on Q. We now replace the $(\varphi)_{m_i}$'s by $(\varphi)_{u_i}$'s such that $(\varphi)_1$ becomes distinct from all the $(\varphi)_{u_i}$'s (if necessary we also add extra conditions for $(\varphi)_{m_i}$ to extend Q). Call the new condition Q^*. Clearly $Q^* \Vdash A^*(\ldots)$.
We now replace $(\varphi)_{n_1}$ by a $(\varphi)_m$, such that $m \neq n_2, \ldots, n_k$ and $(\varphi)_m$ coincides with $(\varphi)_{n_1}$ on Q^* (i.e. if $(\varphi)_{n_1}(a) = b \in Q^*$, then $(\varphi)_m(a) = b$). Hence $m \neq u_1, \ldots, u_p$.
Now we can follow the original argument, replacing Q by Q^*, since the extra conditions on $(\varphi)_{u_1}, \ldots, (\varphi)_{u_p}$ are in variant under $\pi_m^{n_1}$.
I am indebted to Josje Lodder for pointing out the above lacuna and its remedy.
34, − 4 Replace 'ξ' by 'ξ_i'.
36, −10 Replace 'ξ' by 'a'.

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WEAK HOMOMORPHISMS OF DISTRIBUTIVE p-ALGEBRAS

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The aim of this note is a description of weak isomorphisms and weak homomorphisms of distributive p-algebras and of double Stone algebras. We shall use the notation and the terminology from [7] and [8].

1. Weak homomorphisms of universal algebras

Let $\langle A; F\rangle$ and $\langle B; G\rangle$ be two universal algebras. Let $P^{(n)}(A)$ and $P^{(n)}(B)$ denote the sets of n-ary polynomials of $\langle A; F\rangle$ and $\langle B; G\rangle$, respectively (for details see G. Grätzer [7]). Following A. Goetz and E. Marczewski (see [6]) we can define the concept of a weak homomorphism as follows.

Let φ: $A \to B$ be a mapping. If for every $n \geqslant 0$ and every $f \in P^{(n)}(A)$ there exists $g \in P^{(n)}(B)$ such that

$$g(x_1\varphi, \ldots, x_n\varphi) = \big(f(x_1, \ldots, x_n)\big)\varphi \tag{1}$$

and if for every $n \geqslant 0$ and every $g \in P^{(n)}(B)$ there exists $f \in P^{(n)}(A)$ such that (1) is true, then φ: $A \to B$ is called a *weak homomorphism.*

If φ satisfying the above condition is a bijection, we have a weak isomorphism. It can easily be seen that g satisfying (1) is uniquely determined by f if φ is a bijection. Thus we get a mapping $f \to g$ from $P^{(n)}(A)$ into $P^{(n)}(B)$. (The related notions *polymorphism* and *cryptoisomorphism* are discussed by G. Birkhoff ([0], Chapter VI, and also [1].)

Since, by [4], Theorem 7, any weak epimorphism φ: $A \to B$ can be decomposed into a homomorphism μ: $A \to A/\theta_\varphi$ and a weak isomorphism η: $A/\theta_\varphi \to B$, the study of weak homomorphisms can be confined to the