are said to be equal, if

(3) $$[\underline{A}] = [\underline{A}'], \quad [\underline{B}] = [\underline{B}'],$$

(4) condition 4. (4) is satisfied, where

$$\times \boldsymbol{R} \xrightarrow[l,m]{S} \times \boldsymbol{B} = G^{-1}(\times \boldsymbol{R} \xrightarrow{\times \boldsymbol{B}, S} \times \boldsymbol{B}),$$

$$\times \boldsymbol{R}' \xrightarrow[l',m']{S'} \times \boldsymbol{B}' = G^{-1}(\times \boldsymbol{R}' \xrightarrow{\times \boldsymbol{B}', S'} \times \boldsymbol{B}')$$

are the homogeneous monoid automata which by Theorem 2.1 correspond to the homogeneous admissible bisystems occurring as the arguments of G^{-1}. Composition and parallel connection of h.a.b. can be defined in accordance with composition and parallel connection of h.m.a. (applying G to h.m.a. occurring in the sequences (7) and (8)). In analogy with Theorems 2.1, 2.2, 2.3 we get

LEMMA 5.1. *With respect to the composition just defined the h.a.b. in* Set *form a category* Set-Abih$^{\rightarrow}$. *There is a functorial isomorphy*

$$\text{Gh}: \text{Set-Mauth}^{\rightarrow} \to \text{Set-Abih}^{\rightarrow}$$

given by

$$\langle [\underline{A}], \underline{A}, \underline{R}, \times \boldsymbol{R} \xrightarrow[l,m]{S} \times \boldsymbol{B}, \underline{B}, [\underline{B}] \rangle$$
$$\mapsto \langle [\underline{A}], \underline{A}, \underline{R}, G(\times \boldsymbol{R} \xrightarrow[l,m]{S} \times \boldsymbol{B}), \underline{B}, [\underline{B}] \rangle$$

which is compatible with parallel connection.

We are now in the position to establish the instruction

$$\text{Set-Auth}^{\rightarrow} \overset{\text{Fh}}{\rightarrow} \text{Set-Mauth}^{\rightarrow} \overset{\text{Gh}}{\rightarrow} \text{Set-Abih}^{\rightarrow}$$

and call $(\text{Fh}\,\text{Gh})(\langle \boldsymbol{X}, \times \boldsymbol{X} \xrightarrow[l,m]{S} \times \boldsymbol{Y}, \boldsymbol{Y} \rangle)$ the heterogeneous characteristic bisystem of the h.s.a. $\langle \boldsymbol{X}, \times \boldsymbol{X} \xrightarrow[l,m]{S} \times \boldsymbol{Y}, \boldsymbol{Y} \rangle$. Its behaviour at least with respect to a suitable defined modified parallel connection (e.g. paralleloidal connection) seems to be a little more convenient than the ordinary (= homogeneous) characteristic bisystem.

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REGULARITY IN p-ALGEBRAS AND p-SEMILATTICES

J. C. VARLET

University of Liège, Liège, Belgium

1. Introduction

An algebra is called *regular* if any two of its congruences having a class in common are equal. It is well known that a p-algebra is regular if and only if it is Boolean. A similar property holds for p-semilattices, implicative semilattices and Heyting algebras. Therefore it might seem uninteresting to focus our attention on the concept of regularity when dealing with the aforementioned classes of algebras. However, if one defines regularity not only for classes but also for subsets, congruences and algebras, it appears that p-algebras and related structures constitute a nice domain for investigating the notion of regularity. We consider the following three problems:

PROBLEM 1. How to detect a regular congruence?

PROBLEM 2. Where are the regular congruences located in the congruence lattice?

PROBLEM 3. What are the extremely irregular algebras, that is, the algebras in which the universal congruence ι is the only regular one?

We have a very satisfactory solution of the first problem when the p-semilattices and p-algebras L are distributive: a congruence Θ of L is *regular* if and only if $I_* \supseteq D(L)$, where $I = \ker\Theta$, $I_* = \{x \in L\colon x \geqslant i^* \text{ for an } i \in I\}$ and $D(L)$ is the dense set of L.

The second problem is answered without any restriction: for all the algebras we deal with, the regular congruences form an increasing subset of $\text{Con}(L)$ contained in $[\Theta_{\mathscr{G}})$, where $\Theta_{\mathscr{G}}$ denotes the Glivenko congruence.

Finally, under an assumption of distributivity somewhat slighter than in Problem 1, we show that extreme irregularity is equivalent to any of the following conditions: no *-maximal filter is maximal; every minimal prime ideal I satisfies $I_* \subset L - I$; every 2-class congruence covers a 3-class congruence.

2. Preliminaries

A *p-semilattice* $S = \langle S; \wedge, {}^*, 0, 1\rangle$ is a meet-semilattice in which every $a \in S$ has a pseudocomplement a^* defined by $a \wedge x = 0$ iff $x \leqslant a^*$. A *bounded implicative semilattice* $S = \langle S; \wedge, {}^*, 0, 1\rangle$ is a bounded meet-semilattice in which for any two elements $a, b \in S$ a relative pseudocomplement $a*b$ exists ($a \wedge x \leqslant b$ iff $x \leqslant a*b$).

A *p-algebra* $L = \langle L; \vee, \wedge, {}^*, 0, 1\rangle$ is a p-lattice. A *Heyting algebra* $L = \langle L; \vee, \wedge, {}^*, 0, 1\rangle$ is a bounded implicative lattice. It is known that implicative semilattices as well as Heyting algebras are distributive.

We shall denote the equational classes of p-semilattices, distributive p-semilattices, implicative semilattices, p-algebras, distributive p-algebras and Heyting algebras by $\boldsymbol{S}, \boldsymbol{DS}, \boldsymbol{I}, \boldsymbol{P}, \boldsymbol{DP}$ and $\boldsymbol{H}$ respectively. As a reference for all the results on these classes the reader is referred to [1], [3], [5] and [7]. Clearly, one has $\boldsymbol{H} \subset \boldsymbol{I} \subset \boldsymbol{DS} \subset \boldsymbol{S}$, $\boldsymbol{H} \subset \boldsymbol{DP} \subset \boldsymbol{P} \subset \boldsymbol{S}$ and $\boldsymbol{DP} \subset \boldsymbol{DS}$, where $\subset$ stands for "is a proper mention subclass of"; hence those of our statements which are applicable to $\boldsymbol{H}$ and $\boldsymbol{I}$ are obvious and we will not them, explicitly.

The words ideal and filter will retain their usual meaning when L is a lattice, but we warn the reader that we adopt the following convention when S is a meet-semilattice: a *filter* F is a non-empty increasing subset of S which is closed for finite meets (just as in a lattice), whereas an *ideal* I is a non-empty decreasing subset of S such that for all $a, b \in I$ there is a $c \in I$ satisfying $c \geqslant a$ and $c \geqslant b$. For any ideal I of $L \in \boldsymbol{S}$ or $\boldsymbol{P}$, let $I_* = \{x \in L\colon x \geqslant i^* \text{ for an } i \in I\}$ (see [2]); clearly, I_* is a filter. We shall use the notion of *-maximal filter we introduced in [9], a notion which makes sense in $\boldsymbol{S}$: a filter F of $L \in \boldsymbol{S}$ is **-maximal* if it is generated in L by a maximal filter of $S(L) = \{x \in L\colon x = x^{**}\}$. Various characterizations of *-maximal filters can be found in [9].

Two subsets of the carrier L (lattice or meet-semilattice) play a crucial role: the *skeleton* $S(L)$ and the *dense filter* $D(L) = \{x \in L\colon x^* = 0\}$.

The word congruence will have its usual meaning. If the unary operation* is deleted, we sometimes use the expression semilattice or lattice-congruence. The congruence lattice of L is denoted by $\mathrm{Con}(L)$; its least and greatest elements are ω and ι respectively. For every $\Theta \in \mathrm{Con}(L)$ let $\ker\Theta = [0]\Theta$ and $\mathrm{cok}\,\Theta = [1]\Theta$; they are an ideal and a filter respectively. For every $L \in \boldsymbol{S}$ or $\boldsymbol{P}$, the *Glivenko congruence* $\Theta_{\mathscr{B}}$ is defined by $(x, y) \in \Theta_{\mathscr{B}}$ if $x^* = y^*$; the quotient-algebra $L/\Theta_{\mathscr{B}}$ is Boolean; $\ker\Theta_{\mathscr{B}} = \{0\}$ and $\mathrm{cok}\,\Theta_{\mathscr{B}} = D(L)$.

A subset C of the carrier of an algebra A will be called a *regular* congruence-class if C is the class of exactly one congruence (such subsets are said to be *well-behaved* in [6]). A congruence Θ is regular if all Θ-classes are regular. An algebra is regular if all its congruences are regular. Finally, a class of algebras is regular if all its members are regular.

We use the symbols $\subseteq$, $\subset$ and $-$ for set inclusion, proper inclusion and set difference respectively; moreover, $a \| b$ means a non-comparable with b (with respect to the ordering $\leqslant$).

3. Characterization of the regular congruences

In a lattice bounded below $L = \langle L; \vee, \wedge, 0\rangle$, two congruences having a class in common may have different kernels. The same holds true in a p-semilattice, despite the unary operation of pseudocomplementation. An easy example is provided by the pentagon, i.e., the non-modular 5-element poset $\{0, a, b, c, 1\}$ in which $0 < a < b < 1$, $0 < c < 1$, $a \| c$ and $b \| c$, considered as an algebra of $\boldsymbol{S}$: the congruences ω and $\Theta = \{\{0, c\}, \{a\}, \{b, 1\}\}$ have $\{a\}$ as a common class but have different kernels. Such a situation is not allowed in an algebra of $\boldsymbol{P}$ or $\boldsymbol{DS}$, as shown by

THEOREM 3.1. *In an algebra of $\boldsymbol{P}$ or $\boldsymbol{DS}$ the kernel of a congruence is determined by any of its classes.*

Proof. Let C be a congruence-class of the algebra $L \in \boldsymbol{P}$ (resp. of the algebra $L \in \boldsymbol{DS}$) and let Θ be any congruence having C as a class. We shall show that the kernel I of Θ is given by the following rule: an element a of L belongs to I iff there are elements c_1, c_2 in C satisfying $c_1 \wedge a = 0$ and $c_1 \vee a = c_2$ (resp. $c_2 \geqslant a, c_1$), thus proving that I is independent of Θ.

The condition is obviously sufficient since $(c_1, c_2) \in \Theta$ implies $(c_1 \wedge a, c_2 \wedge a) \in \Theta$, that is $(0, a) \in \Theta$. Conversely, $(0, a) \in \Theta$ implies $(1, a^*) \in \Theta$ and, for any $c \in C$, $(c, a^* \wedge c) \in \Theta$. If $L \in \boldsymbol{P}$, the elements $a^* \wedge c$ and $(a^* \wedge c) \vee a$ both belong to C and, since $a^* \wedge c \wedge a = 0$, they can play the roles of c_1 and c_2 respectively. If $L \in \boldsymbol{DS}$, since $c \geqslant a^* \wedge a$, there are elements d, e in L such that $d \geqslant a^*$, $e \geqslant a$ and $d \wedge e = c$. From $(d, 1) \in \Theta$ follows $(d \wedge e, 1 \wedge e) \in \Theta$, that is $(c, e) \in \Theta$, and $e \in C$. The elements $a^* \wedge c$ and e can be chosen for c_1 and c_2 respectively.

COROLLARY 3.2. *In an algebra of $\boldsymbol{P}$ or $\boldsymbol{DS}$, if two congruences have a class in common, then they have the same kernel.*

COROLLARY 3.3. *Let L be an algebra of $\boldsymbol{P}$ or $\boldsymbol{DS}$. For any $\Theta \in \mathrm{Con}(L)$, $\mathrm{cok}\,\Theta$ determines $\ker\Theta$ but not conversely. More precisely, if $\mathrm{cok}\,\Theta = F$, then $\ker\Theta = \{x \in L\colon x \leqslant f^* \text{ for some } f \in F\}$.*

COROLLARY 3.4. *In an algebra of $\boldsymbol{P}$ or $\boldsymbol{DS}$, a congruence is regular if and only if its kernel is regular.*

The last corollary shows that in $\boldsymbol{P}$ and $\boldsymbol{DS}$ the regular congruences are in a 1-1 correspondence with the regular **-closed ideals. We remind the reader that an ideal I of an algebra of $\boldsymbol{P}$ or $\boldsymbol{S}$ is said to be ***-closed* if $x \in I$ implies $x^{**} \in I$. The latter condition is obviously necessary for I to be a congruence-kernel. But it is also a sufficient one. This was proved by W. H. Cornish, [2], Theorem 1.5, but only for $L \in \boldsymbol{DP}$. Taking into account that, for any algebra L of $\boldsymbol{S}$ or $\boldsymbol{P}$, $L/\Theta_{\mathscr{B}}$ is Boolean and the **-closed ideals of L are in a 1-1 correspondence with the ideals of $L/\Theta_{\mathscr{B}}$, one readily sees that in such algebras an ideal is a congruence-kernel iff it is **-closed. Consequently, it should be interesting to characterize those **-closed ideals which are regular. The general problem seems rather difficult but we have a good solution in the distributive case. We first recall two results of [2].

For any **-closed ideal I, let us denote by $\Theta(I)$ and $\Phi(I)$ respectively, the least and the greatest congruences possessing I as a kernel. If $L \in \boldsymbol{DP}$, then by Theorems 1.5 and 1.6 of [2]

(i) $(x, y) \in \Theta(I)$ iff $x \wedge i^* = y \wedge i^*$ for an $i \in I$;
iff $x \wedge f = y \wedge f$ for an $f \in I_*$;

(ii) $(x, y) \in \Phi(I)$ iff $x \wedge d \wedge i^* = y \wedge d \wedge i^*$ for a $d \in D(L)$ and an $i \in I$;
iff $x \wedge g = y \wedge g$ for a $g \in D(L) \vee I_*$.

We are going to show that equivalences (i) and (ii) remain valid in $\boldsymbol{S}$ and $\boldsymbol{DS}$, respectively.

Lemma 3.5. *Let I be a **-closed ideal of $L \in \boldsymbol{S}$. Then $\Theta(I)$ is defined by* (i). *If, moreover, $L \in \boldsymbol{DS}$, then $\Phi(I)$ is defined by* (ii).

Proof. Firstly, let $L \in \boldsymbol{S}$. For every $\Theta \in \operatorname{Con}(L)$ having the ideal I as a kernel, $\operatorname{cok}\Theta \supseteq I_*$. Moreover, for every filter F of L, the least congruence Ψ with cokernel F is defined by $(x, y) \in \Psi$ iff $x \wedge f = y \wedge f$ for an $f \in F$ (see [8], Lemma 2.10). Recalling that I is **-closed and taking $F = I_*$, one sees that $\ker\Psi = I$; hence $\Theta(I) = \Psi$ is defined by (i).

Secondly, let $L \in \boldsymbol{DS}$. The relation $\Phi(I)$ defined by (ii) is a congruence possessing I as a kernel since $(x, 0) \in \Phi(I)$ iff $x \wedge d \wedge i^* = 0$ iff $x^{**} \wedge d^{**} \wedge i^* = x^{**} \wedge i^* = 0$ iff $x^{**} \leqslant i^{**}$ iff $x \leqslant i^{**}$, that is $x \in I$. It is also the only congruence with cokernel $D(L) \vee I_*$ since every filter containing $D(L)$ is the cokernel of exactly one congruence. It remains to show that $\Phi(I)$ is the greatest congruence with I as a kernel. Otherwise we have $x \notin D(L) \vee I_*$, $(x, 1) \in \Theta$ for some congruence Θ having I as a kernel. Then $x^* \in I$ and $x^{**} \in I_*$. Since L is distributive, there is a prime filter P separating $(x]$ and $D(L) \vee I_*$. The filter P is maximal because it contains $D(L)$. Then $x \notin P$ implies $x^{**} \notin P$, in contradiction with $x^{**} \in I_*$. Therefore we may conclude that the largest congruence with kernel I has $D(L) \vee I_*$ as a cokernel and so is $\Phi(I)$.

Theorem 3.6. *If L is an algebra of $\boldsymbol{DP}$ or $\boldsymbol{DS}$, then a **-closed ideal I of L is regular if and only if $I_* \supseteq D(L)$.*

Proof. 1) *if*: let us suppose $I_* \supseteq D(L)$. Then $D(L) \vee I_* = I_*$, $\Theta(I) = \Phi(I)$ and I is regular.

2) *only if*: let I be a regular **-closed ideal. By Corollary 3.3 any congruence Θ having I_* as a cokernel is such that $\ker\Theta = \{x \in L\colon x \leqslant f^*$ for an $f \in I_*\} = \{x \in L\colon x \leqslant i^{**}$ for an $i \in I\} = I$. We have already proved that the (unique) congruence having $I_* \vee D(L)$ as a cokernel has also I as a kernel. Since I is regular, $I_* \vee D(L) = I_*$ and $I_* \supseteq D(L)$.

Remark. The pentagon described at the beginning of this section shows that $I_* \supseteq D(L)$ is not sufficient for I to be regular, either in $\boldsymbol{P}$ or in $\boldsymbol{S}$ (take $I = \{0\}$); on the contrary, the same condition is necessary in $\boldsymbol{S}$ (the proof uses only the fact that every filter is the cokernel of at least one congruence!) but not in $\boldsymbol{P}$, as shown by the lattice depicted in Figure 1 (take $I = (d]$, $I_* = [e) \nsupseteq D(L) = [c)$ and yet I is regular).

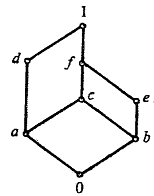

Fig. 1

Corollary 3.7. *Let L be an algebra of $\boldsymbol{DP}$ or $\boldsymbol{DS}$. If a **-closed ideal I is regular, then every **-closed ideal containing I is also regular.*

Let $L \in \boldsymbol{DP}$. A **-closed ideal I of L can be regular in $\langle L; \vee, \wedge, ^*\rangle$ but not in $\langle L; \vee, \wedge\rangle$, as shown by the ideal $(a]$ of the lattice $L = \{0, a, b, c, 1\}$ in which $0 < a < c < 1$, $0 < b < c < 1$ and $a \| b$. This cannot occur in a Stone lattice, i.e., in a lattice $L \in \boldsymbol{DP}$ satisfying $x^* \vee x^{**} = 1$ identically.

Theorem 3.8. *Let L be a Stone lattice. If I is a regular ideal in $\langle L; \vee, \wedge, ^*\rangle$, then it is also a regular ideal in $\langle L; \vee, \wedge\rangle$.*

Proof. Let I be a regular (**-closed) ideal of the Stone algebra $L = \langle L; \vee, \wedge, ^*\rangle$. The least lattice-congruence having I as a kernel will be denoted by $\Theta_{\mathrm{Lat}}(I)$ and is defined by $(x, y) \in \Theta_{\mathrm{Lat}}(I)$ iff $x \vee i = y \vee i$ for an $i \in I$. Hence $x \in \operatorname{cok}\Theta_{\mathrm{Lat}}(I)$ iff $x \vee i = 1$ for some $i \in I$. Since $i^* \vee i^{**} = 1$ and $i^{**} \in I$ for all $i \in I$, we have $\operatorname{cok}\Theta_{\mathrm{Lat}}(I) \supseteq I_*$. By Theorem 3.6, $I_* \supseteq D(L)$. It follows that $\operatorname{cok}\Theta_{\mathrm{Lat}}(I) \supseteq D(L)$ and $\Theta_{\mathrm{Lat}}(I)$ is a $\{\vee, \wedge, ^*\}$-congruence. Consequently, all lattice-congruences with kernel I are $\{\vee, \wedge, ^*\}$-congruences and I is a regular ideal in $\langle L; \vee, \wedge\rangle$.

4. Localization of the regular congruences in Con(L)

Generally, the regular congruences of an algebra A do not constitute a convex subset of Con(A). For instance, the algebra $L = \langle L; \vee, \wedge\rangle$ depicted in Figure 2 is subdirectly irreducible, has two regular congruences, ω and ι, whereas the other three are not regular.

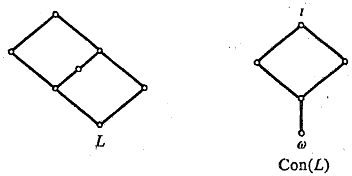

Fig. 2

We shall now define a condition which is sufficient for the subset of regular congruences of the algebra $A \in \boldsymbol{A}$, $\boldsymbol{A}$ a variety, to be convex (even increasing, since it contains ι), and we shall prove that it is satisfied in $\boldsymbol{P}$ and $\boldsymbol{S}$. We point out that we only require that all homomorphic images of A should belong to $\boldsymbol{A}$.

CONDITION (C): $\exists \Psi \in \mathrm{Con}(A)$ and $\exists a \in A$ such that A/Ψ is a regular algebra and $[a]\Theta = \bigcup \{[x]\Psi\colon x \in [a]\Theta\}$ for every $\Theta \in \mathrm{Con}(A)$.

Of course, for Condition (C) to be satisfied it is necessary that $|[a]\Psi| = 1$.

LEMMA 4.1. *Let A be an algebra of the variety $\boldsymbol{A}$. If A satisfies Condition* (C) *and if Θ_1, Θ_2 are two distinct congruences of A such that $\Theta_1 \leqslant \Psi < \Theta_2$ or $\Theta_1 \geqslant \Psi$, $\Theta_2 > \Psi$, then Θ_1 and Θ_2 have no class in common.*

Proof. In the first case, if Θ_1 and Θ_2 have a class in common, then Ψ and θ_2 enjoy the same property and, on A/Ψ, ω and Θ_2/Ψ have a class in common, which contradicts the regularity of A/Ψ. In the second case, the same reasoning can be applied to Θ_1/Ψ and Θ_2/Ψ.

LEMMA 4.2. *Let A be an algebra of the variety $\boldsymbol{A}$. If A satisfies Condition* (C) *and if $\Theta \not\geqslant \Psi$, then neither Θ nor $\Theta \vee \Psi$ are regular.*

Proof. If $\Theta \not\geqslant \Psi$, then $\Theta \neq \Theta \vee \Psi$. These two congruences have a class in common, namely $[a]\Theta$.

THEOREM 4.3. *Let A be an algebra of the variety $\boldsymbol{A}$. If A satisfies Condition* (C), *then the regular congruences of A form an increasing subset R of* Con(A) *contained in $[\Psi)$. More precisely, $R \subset [\Psi)$ if and only if $\Psi \neq \omega$.*

Proof. Let Θ be a regular congruence of A. By Lemma 4.2, $\Theta \geqslant \Psi$. Let us suppose $\Phi > \Theta$ is not regular. Then there is a $\Gamma \in \mathrm{Con}(A)$ having with Φ a class in common. By Lemma 4.1, $\Gamma \| \Psi$. Without loss of generality, we may take $\Gamma < \Phi$. Clearly, $\Gamma \neq \Theta$. Since Γ and Φ have a class in common, $\Gamma \wedge \Theta$ $(\neq \Theta)$ and $\Phi \wedge \Theta$ $(= \Theta)$ enjoy the same property, which contradicts the regularity of Θ.

The following proposition generalizes Corollary 3.7 and shows that it is true without any restriction of distributivity.

THEOREM 4.4. *Let L be an algebra of $\boldsymbol{P}$ or $\boldsymbol{S}$. The regular congruences of L form an increasing subset R of* Con(L) *contained in $[\Theta_{\mathscr{B}})$. More precisely, $R \subset [\Theta_{\mathscr{B}})$ if and only if $\Theta_{\mathscr{B}} \neq \omega$.*

Proof. It suffices to observe that Condition (C) is satisfied in L by $\Psi = \Theta_{\mathscr{B}}$ and $a = 0$ since $x \in [0]\Theta$, $\Theta \in \mathrm{Con}(L)$ implies $(x^{**}, 0) \in \Theta$, that is $x^{**} \in [0]\Theta$; hence $[0]\Theta = \bigcup\{[x]\Theta_{\mathscr{B}}\colon x \in [0]\Theta\}$.

COROLLARY 4.5. *Let L be an algebra of $\boldsymbol{P}$ or $\boldsymbol{S}$. All $\Theta > \Theta_{\mathscr{B}}$ are regular if and only if $\Theta_{\mathscr{B}}$ is a node of* Con(L). *Moreover, L is regular if and only if $\Theta_{\mathscr{B}} = \omega$.*

Proof. If $\Theta_{\mathscr{B}}$ is a node of Con(L) and $\Theta > \Theta_{\mathscr{B}}$, then, by Lemma 4.1, Θ is regular. Conversely, if $\Theta \| \Theta_{\mathscr{B}}$, then, by Lemma 4.2, $\Theta \vee \Theta_{\mathscr{B}}$ $(> \Theta_{\mathscr{B}})$ is not regular. The last part of the statement is obvious and well known.

5. Extremely irregular algebras

Clearly the only regular algebras of $\boldsymbol{P}$ and $\boldsymbol{S}$ are the Boolean algebras. The dense p-algebras of cardinality at least 3 (and particularly the bounded chains) are, in a sense, the p-algebras which are "most unlike" the Boolean algebras with respect to regularity: they are *extremely irregular*, i.e., every congruence other than ι shares a class with another congruence. A nice example of an extremely irregular algebra is provided by the lower semilattices S of cardinality at least 3. Indeed, for any congruence $\Theta \neq \iota$ with no class reduced to a singleton, consider a proper filter of S/Θ; if F is the filter of S which is associated, define Ψ by $(x, y) \in \Psi$ iff $(x = y)$ or (x and y are in $S - F$ and $(x, y) \in \Theta$): Θ and Ψ have at least one class in common.

Let us go back to the algebras of $\boldsymbol{P}$ and $\boldsymbol{S}$. For every L of $\boldsymbol{P}$ or $\boldsymbol{S}$, Con(L) is coatomic, the coatoms are the Θ_2's (that is, the congruences with two classes only) and have a minimal prime ideal as a kernel. Hence by Theorem 4.4 and Corollary 3.4 it is clear that an algebra of $\boldsymbol{P}$ or $\boldsymbol{DS}$ is extremely irregular iff every minimal prime ideal is the kernel of at least two congruences. Figure 3 presents an extremely irregular p-algebra and its congruence lattice.

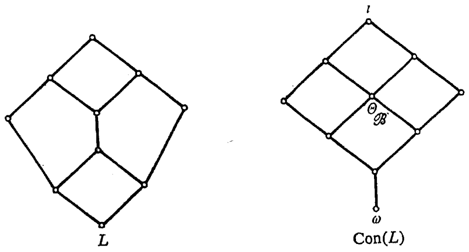

Fig. 3

The preceding remark shows the interest of investigating the congruences whose kernel is a minimal prime ideal.

Lemma 5.1. *Let L be an algebra of* $\boldsymbol{P}$ *or* $\boldsymbol{S}$. *If P is a prime ideal of L, then*:

(i) $P \cup P_* \supseteq S(L)$;

(ii) *P_* is a *-maximal filter if and only if P is a minimal prime ideal.*

Proof. (i) If $x^* \notin P$, then $x \in P$ and $x^* \in P_*$.

(ii) If P is a minimal prime ideal, then P_* is contained in a maximal filter and for every $x \in P_*$ there is a $y^* \in P_*$ such that $y^* \leqslant x$. Hence P_* is *-maximal. Conversely, let us assume that P_* is a *-maximal filter and $P \supset Q$, Q a minimal prime ideal. For any ideal I of L, if $I \cap I_* \neq \emptyset$, then $I_* = L$. Hence we have $P \cap P_* = \emptyset$. If $a \in P - Q$, then $a^* \in P_*$, $a^* \notin Q$ and $a \wedge a^* = 0 \in Q$, a contradiction.

Lemma 5.2. *Let L be an algebra of* $\boldsymbol{P}$ *or* $\boldsymbol{S}$. *For a minimal prime ideal P, the following are equivalent*:

(1) *P_* is a maximal filter*;

(2) $P_* = L - P$;

(3) *$L - P$ is a *-maximal filter.*

Lemma 5.3. *Let L be an algebra of* $\boldsymbol{P}$ *or* $\boldsymbol{S}$. *If Θ is a 2-class congruence of L and if* $\ker\Theta = J$, *then each of the following conditions implies the next one*:

(1) *$L - J$ is a *-maximal filter*;

(2) *Θ is regular*;

(3) *in* $\mathrm{Con}(L)$, *Θ does not cover a Θ_3 (i.e., a 3-class congruence).*

*If $L \in$ **DP** or **S**, then the three conditions are equivatent.*

Proof. (1) ⇒ (2). Since $L - J$ is a maximal filter, it is the cokernel of exactly one congruence, namely Θ. It remains to show that J is the kernel of no congruence distinct from Θ. Let $x \in L - J$. By the very definition of a *-maximal filter, there is a $y^* \in L - J$ such that $y^* \leqslant x$; hence $y^{**} \in J$. Then, for every congruence Ψ with kernel J, we have $(y^{**}, 0) \in \Psi$; hence $(y^*, 1) \in \Psi$, $(x, 1) \in \Psi$ and $\Psi = \Theta$.

(2) ⇒ (3). Let us suppose that Θ covers $\Psi \in \{\Theta_3\}$. Clearly, $[0]\Theta = [0]\Psi$ and Θ is not regular.

If $L \in \boldsymbol{DP}$ or $L \in \boldsymbol{S}$, then (3) ⇒ (1). Let us suppose that $L - J$ is not *-maximal. Then $J_* \subset L - J$. By the definition of $\Theta(J)$ in $\boldsymbol{DP}$ and $\boldsymbol{S}$ (Lemma 3.5) we have $\operatorname{cok}\Theta(J) = J_*$. Consequently, $\Theta(J)$ has at least 3 classes. The ideal J is an $\wedge$-irreducible element of $L/\Theta(J)$, whence $L/\Theta(J)$ is a dense algebra of $\boldsymbol{DP}$ or $\boldsymbol{S}$ whose cardinality is greater than 2 and $L/\Theta(J)$ has at least a 3-class congruence with kernel $\{J\}$. Coming back to L itself, we exhibit a Θ_3 which is covered by Θ, in contradiction with the hypothesis.

Remark. In the last part of Lemma 5.3 the assumption of distributivity cannot be deleted. In fact, the pentagon shows that in $\boldsymbol{P}$ (2) does not imply (1); the p-algebra of Figure 4 has no Θ_3 and nevertheless the unique Θ_2 with kernel $\{0\}$ is not regular, showing that in $\boldsymbol{P}$ (3) does not imply (2).

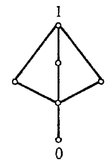

Fig. 4

The following characterization of the extremely irregular algebras of $\boldsymbol{DP}$ and $\boldsymbol{S}$ is then immediate:

Theorem 5.4. *Let L be an algebra of* $\boldsymbol{DP}$ *or* $\boldsymbol{S}$. *Then the following are equivalent*:

(1) *L is extremely irregular*;

(2) *no *-maximal filter of L is maximal*;

(3) *every minimal prime ideal J of L satisfies $J_* \subset L - J$*;

(4) *in* $\mathrm{Con}(L)$, *every Θ_2 covers a Θ_3.*

Corollary 5.5. *Let L be an algebra of* $\boldsymbol{DP}$ *or* $\boldsymbol{S}$ *and let L be atomic. Then L is extremely irregular if and only if no atom of L belongs to $S(L)$.*

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THE CREATIVE SUBJECT AND HEYTING'S ARITHMETIC

D. VAN DALEN

Institute of Mathematics, University of Utrecht, Utrecht, Holland

1

In intuitionistic analysis the effect of Brouwer's 'historic' or 'epistemic' arguments can be satisfactorily captured by means of Kripke's Schema; which itself can be derived straight-forwardly from Kreisel's axioms for the creative subject. In [1] it is pointed out that Analysis + "the creative subject" is conservative over Analysis + KS.

In the same paper it is conjectured that the addition of "the creative subject" to Heyting's Arithmetic presents a conservative extension.

We will prove this conjecture here.

For completeness we repeat the relevant facts. Kripke's Schema is the following schema:

KS $\quad \exists \xi [A \leftrightarrow \exists x\, \xi x \neq 0].$

The axioms for the creative subject are:

$CS_1 \quad \forall x (\vdash_{\overline{x}} A \vee \neg \vdash_{\overline{x}} A),$

$CS_2 \quad \forall xy (\vdash_{\overline{x}} A \to \vdash_{\overline{x+y}} A),$

$CS_3 \quad \exists x \vdash_{\overline{x}} A \leftrightarrow A,$

where $\vdash$ is a new connective such that $\vdash_{\overline{t}} A$ is a formula if t is a numerical term and A a formula. CS will denote the conjunction of CS_1, CS_2, CS_3.

HA is the first-order theory of intuitionistic arithmetic.

LEMMA. *Let A be a sentence of* **HA**; *then* $\mathbf{HA} + A \leftrightarrow \exists x\, fx \neq 0$ *is conservative over* **HA**, *where f is a unary function symbol.*

Proof. We will show that a Kripke model for **HA** can be expanded to a model for $\mathbf{HA} + A \leftrightarrow \exists x\, fx \neq 0$.