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INTERVALS, CONVEX SUBLATTICES AND SUBDIRECT REPRESENTATIONS OF LATTICES

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Preliminaries

Let $\mathfrak{L} = (L; \wedge, \vee)$ be a lattice. The set of all convex sublattices of $\mathfrak{L}$ (including the empty set) under the set-theoretic inclusion forms a complete lattice $C(\mathfrak{L})$. The intervals, i.e. the sets $[a, b] = \{x \in L|\ a \leqslant x \leqslant b\}$ form a sublattice $I(\mathfrak{L})$ of $C(\mathfrak{L})$ ($[a, b]$ is empty whenever $a \leqslant b$ fails to be true). Several authors studied the lattices $C(\mathfrak{L})$ and $I(\mathfrak{L})$ and their relations to $\mathfrak{L}$ (see e.g. [3], [4], [8], [2]). Recently such a study was done by G. Birkhoff [1]. In [2] there is proved that for two finite lattices $\mathfrak{L}$ and $\mathfrak{L}'$, $C(\mathfrak{L})$ is isomorphic to $C(\mathfrak{L}')$ if and only if $\mathfrak{L}$ and $\mathfrak{L}'$ are direct products of sublattices that are themselves isomorphic or dually isomorphic. In this note there is shown that this assertion is true for arbitrary lattices. Moreover, some other equivalent conditions for $\mathfrak{L}$ and $\mathfrak{L}'$, to have corresponding lattices $C(\mathfrak{L})$ and $C(\mathfrak{L}')$ isomorphic, are given.

Given $a, b \in L$, $\langle a, b\rangle$ will denote the interval $[a \wedge b, a \vee b]$ (the smallest interval containing both a and b). $\tilde{\mathfrak{L}}$ will denote the dual of $\mathfrak{L}$. Given two lattices $(L; \wedge, \vee)$ and $(L'; \cap, \cup)$, the corresponding order relations will be denoted by $\leqslant$ and $\subseteq$ respectively.

1. Main results

THEOREM 1. *Let $\mathfrak{L} = (L; \wedge, \vee)$ and $\mathfrak{L}' = (L'; \cap, \cup)$ be lattices and $f\colon L \to L'$ a bijection (denote $f(x) = x'$). The following conditions are equivalent:*

(i) *$x \leqslant y$ implies $f^{-1}(x' \cap z') \leqslant f^{-1}(y' \cap z')$, $f^{-1}(x' \cup z') \leqslant f^{-1}(y' \cup z')$, $x' \subseteq y'$ implies $(x \wedge z)' \subseteq (y \wedge z)'$, $(x \vee z)' \subseteq (y \vee z)'$.*

(ii) *$K \in C(\mathfrak{L})$ if and only if $f(K) = \{f(x)|\ x \in K\} \in C(\mathfrak{L}')$.*

(iii) *$I \in I(\mathfrak{L})$ if and only if $f(I) \in I(\mathfrak{L}')$.*

(iv) *There are subdirect representations* $g\colon \mathfrak{L}\to\mathfrak{A}\times\mathfrak{B}$ *and* $g'\colon \mathfrak{L}'\to\mathfrak{A}\times\tilde{\mathfrak{B}}$ *such that the diagram*

$$\begin{array}{ccc} \mathfrak{L} & \xrightarrow{\;f\;} & \mathfrak{L}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ \mathfrak{A}\times\mathfrak{B} & \xrightarrow{\;i\;} & \mathfrak{A}\times\tilde{\mathfrak{B}} \end{array}$$

commutes; *the mapping* i (*not a homomorphism*) *is given by* $i(a,b) = (a,b)$.

(v) *There are direct representations* $g\colon \mathfrak{L}\to\mathfrak{A}\times\mathfrak{B}$ *and* $g'\colon \mathfrak{L}'\to\mathfrak{A}\times\tilde{\mathfrak{B}}$ *such that the diagram drawn in* (iv) *commutes.*

Remarks. 1. Obviously the mapping f having one of properties (i)–(v) is an isomorphism of the (unoriented) graphs of $\mathfrak{L}$ and $\mathfrak{L}'$ (provided all bounded chains in $\mathfrak{L}$ and $\mathfrak{L}'$ are finite). From the results of [5] and [6] it follows that in case $\mathfrak{L}$ and $\mathfrak{L}'$ are finite and $\mathfrak{L}$ is modular also the converse is true. (In fact the methods of [5] and [6] may be used to lattices whose all bounded chains are finite.)

2. The equivalence of (ii) and (v) was proved in [9] (see also [7]). The proof in the present paper is much more simple.

3. In [9] there is shown that the following condition is equivalent to (ii).

(vi) $(a\wedge x)\vee(x\wedge b) = x = (a\vee x)\wedge(x\vee b)$ *if and only if*

$$(a'\cap x')\cup(x'\cap b') = x' = (a'\cup x')\cap(x'\cup b')$$

(i.e. the mappings f and f^{-1} preserve the betweenness relation (in the sense of [10])).

THEOREM 2. *Let* $\mathfrak{L}, \mathfrak{L}'$ *be lattices. There are bijections between the following three sets*: (1) *the set of all* (*lattice-*) *isomorphisms of* $C(\mathfrak{L})$ *to* $C(\mathfrak{L}')$; (2) *the set of all* (*lattice-*) *isomorphisms of* $I(\mathfrak{L})$ *to* $I(\mathfrak{L}')$; (3) *the set of all mappings* $f\colon L\to L'$ *having the properties in Theorem* 1.

2. Proofs of the theorems

2.1. *Proof of Theorem* 1.

2.1.1. (i) $\Rightarrow$ (ii). Suppose $K\in C(\mathfrak{L})$. Let $x', y'\in f(K)$. From $x\wedge y\leqslant x$, $y\leqslant x\vee y$ using (i) we get $x\wedge y = f^{-1}\big((x\wedge y)'\cap(x\wedge y)'\big)\leqslant f^{-1}(x'\cap y')\leqslant f^{-1}\big((x\vee y)'\cap(x\vee y)'\big) = x\vee y$, hence $f^{-1}(x'\cap y')\in K$ and $x'\cap y'\in f(K)$. Analogously, $x'\cup y'\in f(K)$. Next let $x', y'\in f(K)$, $x'\subseteq z'\subseteq y'$. Then $x, y\in K$, $(x\wedge y)'\subseteq(z\wedge y)'\subseteq(y\wedge y)' = y'$ and $x\wedge y = f^{-1}\big((x\wedge y)'\cap(z\wedge y)'\big)\leqslant f^{-1}\big(y'\cap(z\wedge y)'\big) = z\wedge y\leqslant y$, hence $z\wedge y\in K$. Analogously, $z\vee y\in K$, hence $z\in K$ and $z'\in f(K)$. Thus $f(K)\in C(\mathfrak{L}')$. The converse assertion follows by symmetry.

2.1.2. (ii) $\Rightarrow$ (iii). Let $I = [a, b]$ be an interval in $\mathfrak{L}$. Then $f(I)$ is a convex sublattice of $\mathfrak{L}'$ and if $f(K)$ is a convex sublattice of $\mathfrak{L}'$ containing both a' and b', then K is a convex sublattice of $\mathfrak{L}$ containing both a and b, hence $I\subset K$ and $f(I)\subset f(K)$. Hence $f(I) = \langle a', b'\rangle$.

2.1.3. The proof of the implication (iii) $\Rightarrow$ (iv) proceeds in several steps. Suppose f, $\mathfrak{L}$ and $\mathfrak{L}'$ satisfy (iii).

2.1.3.1. Obviously $z\in\langle x, y\rangle$ (in $\mathfrak{L}$) if and only if $z'\in\langle x', y'\rangle$ (in $\mathfrak{L}'$). Hence $(x\wedge y)'$, $(x\vee y)'\in[x'\cap y', x'\cup y']$ and $f^{-1}(x'\cap y')$, $f^{-1}(x'\cup y')\in[x\wedge y, x\vee y]$.

2.1.3.2. *Let* $a, b, c, d\in L$, $a\wedge d = c$, $a\vee d = b$. *Then* $a'\subseteq b'$ *if and only if* $c'\subseteq d'$, *and* $b'\subseteq a'$ *if and only if* $d'\subseteq c'$.

Proof. Obviously $\langle a, d\rangle = \langle b, c\rangle$. There are $u', v'\in L'$ such that in $\mathfrak{L}'$ $\langle a', d'\rangle = [u', v']$, hence $a'\cap d' = u' = b'\cap c'$, $a'\cup d' = v' = b'\cup c'$. If $a'\subseteq b'$, then (since $b'\subseteq v'$) $v' = b'\cup d'\in\langle b', d'\rangle$, hence $v\in\langle b, d\rangle = [d, b]$ and $d\in[c, v]$ which yields $d'\in\langle c', v'\rangle$ and, since $c'\subseteq v'$, $c'\subseteq d'$. The proof of the converse implication is analogous. The proof of the second equivalence is similar.

2.1.3.3. *Let* $a\leqslant b\leqslant c$. *Then* $a'\subseteq c'$ *implies* $a'\subseteq b'\subseteq c'$ *and* $c'\subseteq a'$ *implies* $c'\subseteq b'\subseteq a'$.

The proof is obvious.

2.1.3.4. Let us define relations θ_0, θ_1 in L and θ_0', θ_1' in L' as follows. $x\theta_1 y$ (and $x'\theta_1' y'$) if and only if there is $t\in L$ such that $t\leqslant x$, $t\leqslant y$ and $t'\subseteq x'$, $t'\subseteq y'$. $x\theta_0 y$ (and $x'\theta_0' y'$) if and only if there is $t\in L$ such that $t\leqslant x$, $t\leqslant y$ and $x'\subseteq t'$, $y'\subseteq t'$.

2.1.3.5. θ_i (θ_i') *are equivalence relations in* L (L') *for* $i = 0, 1$.

Proof for θ_1. Reflexivity and symmetry are obvious. Let $x\theta_1 y$, $y\theta_1 z$. Then there are $t, s\in L$ such that $t\leqslant x$, $t\leqslant y$, $t'\subseteq x'$, $t'\subseteq y'$, $s\leqslant y$, $s\leqslant z$, $s'\subseteq y'$, $s'\subseteq z'$. By 2.1.3.3, $(x\wedge y)'\subseteq x'$, $(y\wedge z)'\subseteq z'$, $(x\wedge y)'\subseteq\big((x\wedge y)\vee(y\wedge z)\big)'$, $(y\wedge z)'\subseteq\big((x\wedge y)\vee(y\wedge z)\big)'$. By 2.1.3.2, $(x\wedge y\wedge z)'\subseteq(x\wedge y)'$, $(x\wedge y\wedge z)'\subseteq(y\wedge z)'$. This yields $x\theta_1 z$. The assertion for θ_1' follows from the symmetry. The proofs for θ_0 and θ_0' are analogous.

2.1.3.6. θ_0, θ_1 (θ_0', θ_1') *are congruence relations on* $\mathfrak{L}$ ($\mathfrak{L}'$).

Proof for θ_1. Let $x, y, z\in L$, $x\theta_1 y$, and let t be the element in the definition 2.1.3.4. By 2.1.3.3, $t'\subseteq\big((z\wedge x)\vee t\big)'$ and by 2.1.3.2, $(z\wedge t)'\subseteq(z\wedge x)'$. Symmetrically, $(z\wedge t)'\subseteq(z\wedge y)'$. Hence $z\wedge x\,\theta_1\, z\wedge y$. The proof of $z\vee x\,\theta_1\, z\vee y$ is similar. The proof for θ_1' follows from the symmetry, and those for θ_0 and θ_0' are similar.

2.1.3.7. (a) $x\leqslant y$ *and* $x\theta_1 y$ *imply* $x'\subseteq y'$.

(b) $x\leqslant y$ *and* $x\theta_0 y$ *imply* $y'\subseteq x'$.

Proof. The proof follows from definition 2.1.3.4 and from 2.1.3.3.

2.1.3.8. $\theta_0 \wedge \theta_1 = \omega$ (the identity), $\theta'_0 \wedge \theta'_1 = \omega'$.

The first assertion follows from 2.1.3.7, the second follows by symmetry.

2.1.3.9. $\mathfrak{L}/\theta_0 \cong \mathfrak{L}'/\theta'_0$, $\mathfrak{L}/\theta_1 = \widetilde{\mathfrak{L}'/\theta'_1}$. *The corresponding isomorphisms are* $[x]\theta_i \mapsto [x']\theta'_i$ $(i = 0, 1)$.

Proof. According to 2.1.3.1, $x \wedge y \leqslant f^{-1}(x' \cap y')$, $(x' \cap y') \subseteq (x \wedge y)'$, hence $x \wedge y \, \theta_0 \, f^{-1}(x' \cap y')$, $(x \wedge y)' \, \theta'_0 \, x' \cap y'$. $[x]\theta_0 \leqslant [y]\theta_0$ implies $[x]\theta_0 = [x \wedge y]\theta_0$, $[x']\theta'_0 = [(x \wedge y)']\theta'_0 = [x' \cap y']\theta'_0 = [x']\theta'_0 \cap [y']\theta_0$, hence $[x']\theta'_0 \subseteq [y']\theta'_0$. The converse implication follows by symmetry. This proves $\mathfrak{L}/\theta_0 \cong \mathfrak{L}'/\theta'_0$. The proof of the second isomorphism is analogous.

2.1.3.10. $\mathfrak{L}$ *and* $\mathfrak{L}'$ *satisfy* (iv).

Proof. Using 2.1.3.8 and 2.1.3.9 we get subdirect representations

$$\mathfrak{L} \to \mathfrak{L}/\theta_0 \times \mathfrak{L}/\theta_1, \quad \mathfrak{L}' \to \mathfrak{L}'/\theta'_0 \times \mathfrak{L}'/\theta'_1 \cong \mathfrak{L}/\theta_0 \times \widetilde{\mathfrak{L}/\theta_1}$$

given by $g\colon x \mapsto ([x]\theta_0, [x]\theta_1)$ and $g'\colon x' \mapsto ([x]\theta_0, [x]\theta_1)$. For $x \in L$ we get $(g'f)x = g'(x') = g(x) = (ig)x$. This proves (iv).

2.1.4. (iv) $\Rightarrow$ (v). It suffices to show that g and g' in (iv) are surjective. Denote the lattice operations in $\mathfrak{A}$ and $\mathfrak{B}$ by $\wedge$ and $\vee$. From the commutativity of the diagram and from the definition of the mapping i it immediately follows $\mathrm{Im}(g) = \mathrm{Im}(g')$. First we claim that $(a_1, b_1), (a_2, b_2) \in \mathrm{Im}(g)$ implies $(a_1, b_2) \in \mathrm{Im}(g)$. Indeed, $(a_1, b_1) = g(c)$ and $(a_2, b_2) = g(d)$ for some $c, d \in L$. Using the commutativity of the diagram we get $g'(c') = (a_1, b_1)$, $g'(d') = (a_2, b_2)$. Then $g'(c' \cap d') = (a_1 \wedge a_2, b_1 \vee b_2)$, $g'(c' \cup d') = (a_1 \vee a_2, b_1 \wedge b_2)$, hence $(a_1 \wedge a_2, b_1 \vee b_2)$ and $(a_1 \vee a_2, b_1 \wedge b_2)$ belong to $\mathrm{Im}(g)$. Again, the elements of $\mathfrak{A} \times \mathfrak{B}$, $(a_1, b_1 \vee b_2) = (a_1 \wedge a_2, b_1 \vee b_2) \vee (a_1, b_1)$ and $(a_1 \vee a_2, b_2) = (a_1 \vee a_2, b_1 \wedge b_2) \vee (a_2, b_2)$ belong to $\mathrm{Im}(g)$, hence $(a_1, b_2) = (a_1, b_1 \vee b_2) \wedge (a_1 \vee a_2, b_2) \in \mathrm{Im}(g)$.

Now let $a \in A$, $b \in B$ be arbitrary. Since g is a subdirect representation there are $a_1 \in A$ and $b_1 \in B$ with (a, b_1), (a_1, b) belonging to $\mathrm{Im}(g)$. Using the above result we get $(a, b) \in \mathrm{Im}(g)$ which proves g surjective. The surjectivity of g' follows from $\mathrm{Im}(g) = \mathrm{Im}(g')$.

2.1.5. (v) $\Rightarrow$ (i). As in 2.1.4, $g(x) = (x_1, x_2) = g'(x')$, $g(y) = (y_1, y_2) = g'(y')$ and $x \leqslant y$ yields $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $g\big(f^{-1}(x' \cap z')\big) = g'(x' \cap z') = (x_1 \wedge z_1, x_2 \vee z_2) \leqslant (y_1 \wedge z_1, y_2 \vee z_2) = g\big(f^{-1}(y' \cap z')\big)$, hence $f^{-1}(x' \cap z') \leqslant f^{-1}(y' \cap z')$. The second relation follows in the same way. The remaining relations follow by symmetry.

2.2. *Proof of Theorem* 2. A one-element interval $\langle a, a \rangle$ will be denoted by $\langle a \rangle$. With any isomorphism $g\colon C(\mathfrak{L}) \to C(\mathfrak{L}')$ a mapping $g^*\colon L \to L'$ will be associated as follows. Given $a \in L$, $\langle a \rangle$ is an atom of $C(\mathfrak{L})$ hence $g\langle a \rangle$ is an atom $\langle a' \rangle$ in $C(\mathfrak{L}')$. Set $g^*(a) = a'$. Obviously $g^*\colon a \mapsto a'$ is a bijection of L to L'. We claim that g^* has property (ii). Given $K \in C(\mathfrak{L})$, $a \in K$ is equivalent successively to $\langle a \rangle \subset K$, $\langle a' \rangle = g\langle a \rangle \subset gK$, $a' \in gK$, hence $gK = \{g^*(a) \mid a \in K\} = g^*(K)$. Consequently $g^*(K) \in C(\mathfrak{L}')$. From the symmetry it follows that $M \in C(\mathfrak{L}')$ implies $g^{*-1}(M) \in C(\mathfrak{L})$, hence g^* satisfies (ii).

Now let $h\colon L \to L'$ be a mapping satisfying (ii). Setting, for $K \in C(\mathfrak{L})$, $h^+K = \{h(a) \mid a \in K\}$ we get a mapping $h^+\colon C(\mathfrak{L}) \to C(\mathfrak{L}')$ which is easy to check to be an isomorphism of lattices. Obviously $(g^*)^+ = g$ and $(h^+)^* = h$ which gives a bijection between the sets (1) and (3) of Theorem 2. In the same way a bijection between the set (2) and the set of all mappings $f\colon L \to L'$ satisfying (iii) can be established.

References

[1] G. Birkhoff, *Some applications of universal algebra*, Proc. Colloq. Univ. Algebra, Esztergom, 1977; Colloq. Math. Soc. J. Bolyai, vol. 29, Universal algebra, North-Holland, 1981, 107–128.

[2] C. C. Chen and K. M. Koh, *On the lattice of convex sublattices of a finite lattice*, Nanta Math. 5 (1972), 93–95.

[3] W. D. Duthie, *Segments of ordered sets*, Trans. Amer. Math. Soc. 51 (1942), 1–14.

[4] M. S. Gel'fand, *Segments in a modular lattice* (Russian), 71 (1953), 199–204.

[5] J. Jakubík, *On the graph-isomorphism of lattices* (Russian), Czechoslov. Math. J. 4(79) (1954), 131–141.

[6] —, *Unoriented graphs of modular lattices*, ibid. 25 (1975), 240–246.

[7] J. Jakubík and M. Kolibiar, *On some properties of pairs of lattices* (Russian), ibid. 4(79) (1954), 1–27.

[8] K. M. Koh, *On the lattice of convex sublattices of a lattice*, Nanta Math. 5 (1972), 18–37.

[9] M. Kolibiar, *On the relation "between" in lattices* (Slovak), Mat.-Fyz. Časopis. Slovensk. Akad. Vied. 5 (1955), 162–171.

[10] E. Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc. 52 (1942), 95–114.

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