The regularity classes of the semigroup of all binary relations have been described in full generality by K. A. Zarecki [7]. In this note we are concerned with special binary relations for which the diagrams immediately tell us whether or not the relation belongs to a certain regularity class.

In particular, we give a complete characterization of those binary relations on a finite set belonging to a non-trivial regularity class which happen to be the complements of quasi-orders. To do this, we may restrict ourselves to complete lattice orders. A special case was already established in [3]: the complement of a partial order is regular if and only if the associated completion by cuts is completely distributive.

Let $S$ be a semigroup with identity. Given non-negative integers $m$ and $n$, the regularity class $\mathcal{F}_g(m, n)$ is the set of all elements $a \in S$ for which there exists an element $x \in S$ such that $a^m x a^n = a$. The elements of $\mathcal{F}_g(1, 1)$ and $\mathcal{F}_g(2, 2)$ are called regular and completely regular, respectively. For $m + n \leq 1$, $\mathcal{F}_g(m, n)$ is trivial, that is $\mathcal{F}_g(m, n) = S$. Every regularity class is already equal to $\mathcal{F}_g(m, n)$ for some $m, n \leq 2$. The non-trivial classes other than $\mathcal{F}_g(0, 2)$, $\mathcal{F}_g(1, 1)$, and $\mathcal{F}_g(2, 0)$ are given by trivial intersections (cf. [3]).

The semigroup of all binary relations on a set $X$ is denoted by $\mathcal{B}(X)$. For $\sigma, \rho \in \mathcal{B}(X)$ the product is written as $\sigma \circ \rho = \{(x, y) \in X \times X | \exists z \in X, (x, z) \in \sigma \land (z, y) \in \rho\}$. $\sigma^\ominus$ and $\sigma'$ denote the converse of $\sigma$ and the complement of $\sigma$ respectively; $\sigma^{\ominus} \cdot \rho$ is written for $(\sigma^\ominus \cdot \rho) \circ (\sigma^\ominus \cdot \rho)^\ominus$.

Our first lemma is a straightforward extension of [5], Theorem 1. Lemma 2 is an immediate consequence of this (cf. [5], Theorem 2).

**Lemma 1.** Let $\rho, \rho_1, \rho_2 \in \mathcal{B}(X)$ be binary relations and let $\sigma = \{(x, y) | \rho_1 \circ \rho \circ \rho_2 \subseteq \rho\}$. Then $\sigma = (\rho_1 \circ \rho' \circ \rho_2)^\ominus$.

**Proof.** For $(x, y) \in X$, $(x, y) \notin \sigma$ if and only if there exists $(u, v) \notin \rho'$ such that $(u, x) \in \rho_1$ and $(y, v) \in \rho_2$ if and only if $(x, y) \in \rho_1 \circ \rho' \circ \rho_2$.

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LEMMA 2. For a binary relation \( \rho \in \mathcal{R}(X) \), \( \rho \in \mathcal{P}_{\mathcal{R}}(m, n) \) if and only if \( \rho \subseteq \rho^0 \circ (\rho^{-1} \circ \rho^0) \circ \rho^0 \).

E. W. Wolfs [6] characterized partial orders as regular, reflexive, antisymmetric relations (for a simple proof cf. [5]). Using Lemma 2 it is an easy exercise to prove the analogous result for any other non-trivial regularity class:

**Theorem 1.** A reflexive and antisymmetric relation \( \rho \in \mathcal{R}(X) \) belongs to a non-trivial regularity class of \( \mathcal{R}(X) \) if and only if \( \rho \) is a partial order.

**Proof.** Given \( m + n \geq 2 \), let \( \rho \in \mathcal{P}_{\mathcal{R}}(m, n) \) be reflexive and antisymmetric. We will show that \( \rho \) is transitive. Wolfs’s theorem settles the case \( m = n = 1 \). We may therefore assume that \( m = 0 \) and \( n = 2 \). Put \( \sigma = \rho^0 \circ (\rho^{-1} \circ \rho^0) \). By Lemma 2 \( \rho \subseteq \sigma \circ \rho^0 \). Since \( \rho \) is reflexive, \( \rho^0 \subseteq \rho^0 \circ \rho^0 \) (cf. [5]), whence \( \sigma \subseteq \rho^0 \). Further for each \( a \in X \), \((a, a) \in \rho^0 \circ \rho^0 \), whence there exists \( (x, y) \in \rho \) such that \((a, x) \in \rho \) and \((y, a) \in \rho \). Since \( \rho \) is antisymmetric, \( a \neq y \) implies \( (a, y) \in \rho^0 \). Thus \((a, x) \in \rho^0 \circ \rho^0 \), which is impossible. So we get \( a = y \) and therefore \((a, a) \in \rho \). As \((a, a) \in \rho \), again by antisymmetry we get \( a = x \) and thus \((a, x) \in \rho \). Hence \( \rho \) is reflexive and we infer from Lemma 1 that \( \rho \) is transitive, completing the proof.

Let \( \sigma \in \mathcal{R}(X) \) be an equivalence relation and let \( \mathcal{R}(X) \), the quotient group of \( \mathcal{R}(X) \) which consists of all elements having \( \sigma \) as their two-sided identity. Each \( \sigma \in \mathcal{R}(X) \) defines a binary relation \( \rho \) on the quotient set as follows: \((x, y) \in \rho \) if and only if \((x, y) \in \sigma \). By [3], Proposition 6.2, \( \rho \Rightarrow \sigma \) establishes an isomorphism between \( \mathcal{R}(X) \) and \( \mathcal{R}(\sigma) \). We therefore have: For \( \rho \in \mathcal{R}(X) \), \( \rho \in \mathcal{P}_{\mathcal{R}}(m, n) \) if and only if \( \rho \in \mathcal{P}_{\mathcal{R}}(m, n) \) if and only if \( \rho \in \mathcal{P}_{\mathcal{R}}(m, n) \) if and only if \( \rho \in \mathcal{P}_{\mathcal{R}}(m, n) \), a quasi-order \( \leq \) on \( X \) is a reflexive and transitive relation. It defines an equivalence relation \( \equiv \) by \( a \equiv b \) if \( a \leq b \) and \( b \leq a \) and finally a partial order \( \leq \) on \( X \) as in the above way. Since the complement of \( \equiv \) obviously belongs to \( \mathcal{R}(\sigma) \), we have proved the following:

**Lemma 3.** Let \( \leq \) be a quasi-order on \( X \) and let \( \leq \) denote the corresponding partial order on \( X \). Then non-\( \leq \) is \( \in \mathcal{P}_{\mathcal{R}}(m, n) \) if and only if \( \leq \) is in \( \mathcal{P}_{\mathcal{R}}(m, n) \).

Let \( X \) be a partially ordered set. Put \( \beta_X = \{(p, q) \mid \forall a \in X \text{ either } p \leq a \text{ or } q \leq a\} \), \( \mathcal{P}(X) = \{p \in X \mid p \leq q \Rightarrow \exists a \in X, (p, a) \in \beta_X, (r, q) \not\in \beta_X\} \).

\( \beta_X \) denotes the normal completion (alias completion by cuts) of \( X \). Let \( L \) be a complete lattice. A subset \( \mathcal{L} \subseteq X \) is called join dense (meet dense) in \( L \) if every element of \( L \) is the join (meet) of elements of \( \mathcal{L} \). It is known that \( L \) is isomorphic to \( X^* \) if and only if \( L \) is join dense and meet dense in \( X \) (cf. [1]). For \( p, q \in L \), \( q \leq p \) if and only if \( q \leq p \) if and only if \( q \leq p \) if and only if \( q \leq p \). The pair \((-,-)\) sets up a Galois connection between \( L \) and its dual and determines the relation \( \mathcal{L} \) (see [4], Theorem 6); in particular one infers from this the following:

**Lemma 4.** For a complete lattice \( L \), the image of the map \( - \Rightarrow - \) is equal to \( \mathcal{L}(X) \).

**Lemma 5.** For a partially ordered set \( X \), \( \leq \) belongs to \( \mathcal{P}_{\mathcal{R}}(0, 2) \) if and only if \( \mathcal{L}(X) = X \).

**Proof.** From [2], the lemma, it follows that the product \( \mathcal{P}_{\mathcal{R}}(0, 2) \) is equal to \( \mathcal{P}_{\mathcal{R}}(0, 2) \), which is evidently the complement of \( \beta_X \). Therefore by virtue of Lemma 2, \( \leq \) is in \( \mathcal{P}_{\mathcal{R}}(0, 2) \) if and only if \( \leq \) is contained in \( \beta_X(\beta_X^{-1}) \). By the definition of \( \mathcal{L}(X) \) this is equivalent to \( \mathcal{L}(X) \).

**Lemma 6.** For a partially ordered set \( X \), \( \leq \) belongs to \( \mathcal{P}_{\mathcal{R}}(0, 2) \) if and only if \( \leq \) belongs to \( \mathcal{P}_{\mathcal{R}}(0, 2) \).

**Proof.** Let \( X \) be join dense and meet dense in \( L \). Clearly \( \beta_X \subseteq \beta_X \subseteq \beta_X \). Suppose that \( \mathcal{L}(X) = X \). Since \( X \) is meet dense in \( L \), for each \( p \in X \) and \( r \in L \), there exists \( t \in X \) such that \( r \leq t \) and \( p \leq t \). Then there exists \( q \in X \) such that \( p \leq q \) and \( t \leq q \). Consequently \( r \leq q \) and \( q \not\in \beta_X \). Hence \( \mathcal{L}(X) \).

Combining Lemma 5 and 6, the theorem, we conclude that for a partially ordered set \( X \), \( \leq \) is in \( \mathcal{P}_{\mathcal{R}}(0, 2) \) if and only if \( \leq \) is in \( \mathcal{P}_{\mathcal{R}}(0, 2) \).

Note that it is also not difficult to derive Lemmas 3, 5, 6 from the results of [7].

Let \( L \) be a complete lattice. Then by \( \mathcal{L}(X) \) denote the set of all elements \( a \in L \) such that \( a \neq 1 \) and whenever \( a \neq 1 \), \( \mathcal{L}(X) \) is defined dually. By Lemma 4 \( \mathcal{L}(X) \) is \( \mathcal{L}(X) \) and therefore \( \mathcal{L}(X) \). This proves:

**Lemma 7.** If \( L \) is a complete lattice, then \( \leq \) is in \( \mathcal{P}_{\mathcal{R}}(0, 2) \) whenever \( \mathcal{L}(X) \) is join dense in \( L \).

Using Lemmas 4, 5, 7 the following assertions are readily verified:

**Examples.** 1. The lattice \( L \) of Figure 1 is not modular, though \( \leq \) is in \( \mathcal{P}_{\mathcal{R}}(0, 2) \).

2. For the chain \( N \) of non-negative integers, \( \leq \) belongs to \( \mathcal{P}_{\mathcal{R}}(1, 2) \) but not to \( \mathcal{P}_{\mathcal{R}}(2, 0) \).
3. Put $Z_n = \{(a, b) \in Z^2 | 0 \leq a - b \leq n\}$, where $Z$ denotes the chain of integers. For each $n \in N$ the complement of the partial order on $Z_n$ is completely regular. (Note that $\mathcal{H}(Z_n) = \mathcal{F}(Z_n) = \{(a, a) \in Z \cup \cup \{(a+n, a) | a \in Z\}$.)

4. For every chain $C$ which is dense in itself, $\leq$ is completely regular. Obviously for every lattice of Figure 2 the complement of the partial order is completely regular. That this infinite list in fact includes all finite examples is stated by our concluding theorem.

**Theorem 2.** Given a finite lattice $L$, the following are equivalent:

(i) $\leq$ is completely regular,

(ii) $\leq$ belongs to $\mathcal{N}(0, 2)$,

(iii) $L$ is a sum of four element Boolean lattices (see Figure 2).

**Proof.** (i) $\Rightarrow$ (ii) is trivial, (iii) $\Rightarrow$ (i) follows from Lemma 7. Suppose that (ii) holds. Clearly we may assume that $L$ has at least two elements. Assume that $L$ contains only one atom $p$. Since, by Lemmas 4, 5, $p = p_0$, $p_0$ is the unique atom of the interval $[p, 1]$. Continuing this argument we conclude that $L$ contains an infinite chain, a contradiction. Therefore $L$ has at least two atoms $p$ and $q$. Thus for each element $x \in L$ distinct from $0, p, q$ we get $p \geq q \lor p_0 = p \lor q$ and $x \geq p \lor q$. Hence, if $p \lor q < 1$, we may apply the above arguments to the interval $[p \lor q, 1]$. Thus it follows by induction that $L$ must occur in the list of Figure 2.

**References**