[3] H. Huß und K. Rosenbaum, *Über die symplektische Struktur von Gruppenalgebren*, Wiss. Z. PH Erfurt/Mühlhausen 2 (1974), 55–63.
[4] K. Rosenbaum, *Vorlesungen über symplektische Moduln*, Teil 4, Erfurt 1976.

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ON REGULARITY CLASSES OF BINARY RELATIONS

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The regularity classes of the semigroup of all binary relations have been described in full generality by K. A. Zareckiĭ [7]. In this note we are concerned with special binary relations for which the diagrams immediately tell us whether or not the relation belongs to a certain regularity class. In particular, we give a complete characterization of those binary relations on a finite set belonging to a non-trivial regularity class which happen to be the complements of quasi-orders. To do this, we may restrict ourselves to complete lattice orders. A special case was already established in [2]: the complement of a partial order is regular if and only if the associated completion by cuts is completely distributive.

Let S be a semigroup with identity. Given non-negative integers m and n, the regularity class $\mathscr{C}_S(m, n)$ is the set of all elements $a \in S$ for which there exists an element $x \in S$ such that $a^m x a^n = a$. The elements of $\mathscr{C}_S(1, 1)$ and $\mathscr{C}_S(2, 2)$ are called *regular* and *completely regular*, respectively. For $m+n \leqslant 1$, $\mathscr{C}_S(m, n)$ is *trivial*, that is $\mathscr{C}_S(m, n) = S$. Every regularity class is already equal to $\mathscr{C}_S(m, n)$ for some $m, n \leqslant 2$. The non-trivial classes other than $\mathscr{C}_S(0, 2)$, $\mathscr{C}_S(1, 1)$, and $\mathscr{C}_S(2, 0)$ are given by trivial intersections (cf. [3]).

The semigroup of all binary relations on a set X is denoted by $\mathscr{B}(X)$. For $\varrho, \sigma \in \mathscr{B}(X)$ the product is written as $\varrho \circ \sigma = \{(x, z) \mid \exists y \in X, (x, y) \in \varrho, (y, z) \in \sigma\}$. ϱ^{-1} and ϱ' denote the converse of ϱ and the complement of ϱ respectively; ϱ^{-n} is written for $(\varrho^{-1})^n$.

Our first lemma is a straightforward extension of [5], Theorem 1. Lemma 2 is an immediate consequence of this (cf. [5], Theorem 2).

LEMMA 1. *Let $\varrho, \varrho_1, \varrho_2 \in \mathscr{B}(X)$ be binary relations and let $\sigma = \{(x, y) \mid \varrho_1 \circ \{(x, y)\} \circ \varrho_2 \subseteq \varrho\}$. Then $\sigma = (\varrho_1^{-1} \circ \varrho' \circ \varrho_2^{-1})'$.*

Proof. For $x, y \in X$, $(x, y) \notin \sigma$ if and only if there exists $(u, v) \in \varrho'$ such that $(u, x) \in \varrho_1$ and $(y, v) \in \varrho_2$ if and only if $(x, y) \in \varrho_1^{-1} \circ \varrho' \circ \varrho_2^{-1}$.

LEMMA 2. *For a binary relation* $\varrho \in \mathscr{B}(X)$, $\varrho \in \mathscr{C}_{\mathscr{B}(X)}(m, n)$ *if and only if* $\varrho \subseteq \varrho^m \circ (\varrho^{-m} \circ \varrho' \circ \varrho^{-n})' \circ \varrho^n$.

E. S. Wolk [6] characterized partial orders as regular, reflexive, antisymmetric relations (for a simple proof cf. [5]). Using Lemma 2 it is an easy exercise to prove the analogous result for any other non-trivial regularity class:

THEOREM 1. *A reflexive and antisymmetric relation* $\varrho \in \mathscr{B}(X)$ *belongs to a non-trivial regularity class of* $\mathscr{B}(X)$ *if and only if* ϱ *is a partial order.*

Proof. Given $m+n \geqslant 2$, let $\varrho \in \mathscr{C}_{\mathscr{B}(X)}(m, n)$ be reflexive and antisymmetric. We will show that ϱ is transitive. Wolk's theorem settles the case $m = n = 1$. We may therefore assume that $m = 0$ and $n = 2$. Put $\sigma = (\varrho' \circ \varrho^{-2})'$. By Lemma 2 $\varrho \subseteq \sigma \circ \varrho^2$. Since ϱ is reflexive, $\varrho' \subseteq \varrho' \circ \varrho^{-2}$ (cf. [2]), whence $\sigma \subseteq \varrho$. Further for each $a \in X$, $(a, a) \in \sigma \circ \varrho^2$, whence there exists $(x, y) \in \varrho$ such that $(a, x) \in \sigma$ and $(y, a) \in \varrho$. Since ϱ is antisymmetric, $a \neq y$ implies $(a, y) \in \varrho'$ and thus $(a, x) \in \varrho' \circ \varrho^{-2}$, which is impossible. So we get $a = y$ and therefore $(x, a) \in \varrho$. As $(a, x) \in \varrho$, again by antisymmetry we get $a = x$ and thus $(a, a) \in \sigma$. Hence σ is reflexive and we infer from Lemma 1 that ϱ is transitive, completing the proof.

Let $\alpha \in \mathscr{B}(X)$ be an equivalence relation and let $\mathscr{B}_\alpha(X)$ denote the subsemigroup of $\mathscr{B}(X)$ which consists of all elements having α as their two-sided identity. Each $\varrho \in \mathscr{B}_\alpha(X)$ defines a binary relation ϱ_α on the quotient set as follows: $(x\alpha, y\alpha) \in \varrho_\alpha$ if and only if $(x, y) \in \varrho$. By [8], Proposition 6.2, $\varrho \mapsto \varrho_\alpha$ establishes an isomorphism between $\mathscr{B}_\alpha(X)$ and $\mathscr{B}(X/\alpha)$. We therefore have: For $\varrho \in \mathscr{B}_\alpha(X)$, $\varrho \in \mathscr{C}_{\mathscr{B}(X)}(m, n)$ if and only if $\varrho \in \mathscr{C}_{\mathscr{B}_\alpha(X)}(m, n)$ if and only if $\varrho_\alpha \in \mathscr{C}_{\mathscr{B}(X/\alpha)}(m, n)$. A quasi-order $\prec$ on X is a reflexive and transitive relation. It defines an equivalence relation $\equiv$ by $a \equiv b$ if $a \prec b$ and $b \prec a$ and finally a partial order $\leqslant$ on $X/\equiv$ in the above way. Since the complement of $\prec$ obviously belongs to $\mathscr{B}_\equiv(X)$, we have proved the following

LEMMA 3. *Let* $\prec$ *be a quasi-order on* X *and let* $\leqslant$ *denote the corresponding partial order on* $X/\equiv$. *Then* non-$\prec$ *is in* $\mathscr{C}_{\mathscr{B}(X)}(m, n)$ *if and only if* $\not\leqslant$ *is in* $\mathscr{C}_{\mathscr{B}(X/\equiv)}(m, n)$.

Let X be a partially ordered set. Put

$$\beta_X = \{(p, q) \mid \forall x \in X \text{ either } x \leqslant p \text{ or } q \leqslant x\},$$

$$\mathscr{R}(X) = \{p \in X \mid p \not\leqslant r \Rightarrow \exists q \in X,\ (p, q) \in \beta_X,\ (r, q) \notin \beta_X\}.$$

X^* denotes the normal completion (alias completion by cuts) of X. Let L be a complete lattice. A subset $X \subseteq L$ is called *join dense* (*meet dense*) in L if every element of L is the join (meet) of elements of X. It is known that L is isomorphic to X^* if and only if X is join and meet dense in L (cf. [1]). For $p, q \in L$ set $q_- = \bigvee\{t \in L \mid q \not\leqslant t\}$ and $p_+ = \bigwedge\{t \in L \mid t \not\leqslant p\}$. The pair $(-, +)$ sets up a Galois connection between L and its dual and determines the relation β_L (see [4], Theorem 6); in particular one infers from this the following

LEMMA 4. *For a complete lattice* L, *the image of the map* $-$ *is equal to* $\mathscr{R}(L)$.

LEMMA 5. *For a partially ordered set* X, $\not\leqslant$ *belongs to* $\mathscr{C}_{\mathscr{B}(X)}(0, 2)$ *if and only if* $\mathscr{R}(X) = X$.

Proof. From [2], the lemma, it follows that the product $\leqslant \circ \not\geqslant \circ \not\geqslant$ is equal to $\not\geqslant \circ \not\geqslant$, which is evidently the complement of β_X. Therefore by virtue of Lemma 2, $\not\leqslant$ is in $\mathscr{C}_{\mathscr{B}(X)}(0, 2)$ if and only if $\not\leqslant$ is contained in $\beta_X \circ (\beta_X^{-1})'$. By the definition of $\mathscr{R}(X)$ this is equivalent to $\mathscr{R}(X) = X$.

LEMMA 6. *For a partially ordered set* X, $\not\leqslant$ *belongs to* $\mathscr{C}_{\mathscr{B}(X)}(0, 2)$ *if and only if* $\not\leqslant$ *belongs to* $\mathscr{C}_{\mathscr{B}(X^*)}(0, 2)$.

Proof. Let X be join and meet dense in $L \cong X^*$. Clearly $\beta_X = \beta_L \cap X^2$. Suppose that $\mathscr{R}(X) = X$. Since X is meet dense in L, for each $p \in X$ and $r \in L$ with $p \not\leqslant r$, there exists $t \in X$ such that $r \leqslant t$ and $p \not\leqslant t$. Then there exists $q \in X$ such that $(p, q) \in \beta_X$ and $(t, q) \notin \beta_X$. Consequently $(r, q) \notin \beta_L$, whence $X \subseteq \mathscr{R}(L)$. Since X is join dense in L and by Lemma 4 $\mathscr{R}(L)$ is closed under arbitrary joins, we conclude $\mathscr{R}(L) = L$. Conversely, if $\mathscr{R}(L) = L$, then $(r, p_+) \notin \beta_L$ whenever $p, r \in X$ with $p \not\leqslant r$. Since X is join dense in L, there exists $q \in X$ such that $q \leqslant p_+$ and $(r, q) \notin \beta_X$, whence $p \in \mathscr{R}(X)$, proving $\mathscr{R}(X) = X$.

Combining Lemma 6 and [2], the theorem, we conclude that for a partially ordered set X, $\not\leqslant$ is in $\mathscr{C}_{\mathscr{B}(X)}(m, n)$ if and only if $\not\leqslant$ is in $\mathscr{C}_{\mathscr{B}(X^*)}(m, n)$.

Note that it is also not difficult to derive Lemmas 3, 5, 6 from the results of [7].

Let L be a complete lattice. Then by $\mathscr{M}(L)$ denote the set of all elements $x \in L$ such that $x \neq 1$ and whenever $x \geqslant \bigwedge S$ for $S \subseteq L$, then there exists $s \in S$ with $x \geqslant s$. $\mathscr{J}(L)$ is defined dually. By Lemma 4 $\mathscr{M}(L) \subseteq \mathscr{R}(L)$ and therefore $\mathscr{R}(L) = L$ whenever $\mathscr{M}(L)$ is join dense in L. This proves

LEMMA 7. *If* L *is a complete lattice, then* $\not\leqslant$ *is in* $\mathscr{C}_{\mathscr{B}(L)}(0, 2)$ *whenever* $\mathscr{M}(L)$ *is join dense in* L.

Using Lemmas 4, 5, 7 the following assertions are readily verified:

EXAMPLES. 1. The lattice L of Figure 1 is not modular, though $\not\leqslant$ is in $\mathscr{C}_{\mathscr{B}(L)}(0, 2)$.

2. For the chain N of non-negative integers, $\not\leqslant$ belongs to $\mathscr{C}_{\mathscr{B}(N)}(1, 2)$ but not to $\mathscr{C}_{\mathscr{B}(N)}(2, 0)$.

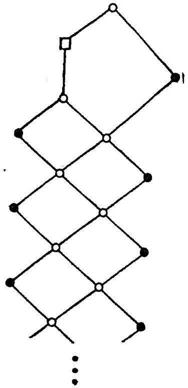

Fig. 1

3. Put $Z_n = \{(a, b) \in \boldsymbol{Z}^2 |\ 0 \leqslant a - b \leqslant n\}$, where $\boldsymbol{Z}$ denotes the chain of integers. For each $n \in \boldsymbol{N}$ the complement of the partial order on Z_n is completely regular. (Note that $\mathscr{M}(Z_n) = \mathscr{J}(Z_n) = \{(a, a) |\ a \in \boldsymbol{Z}\} \cup \cup \{(a+n, a) |\ a \in \boldsymbol{Z}\}$.)

4. For every chain C which is dense in itself, $\not\leqslant$ is completely regular.

Obviously for every lattice of Figure 2 the complement of the partial order is completely regular. That this infinite list in fact includes all finite examples is stated by our concluding theorem.

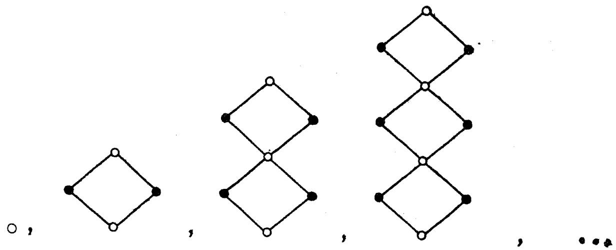

Fig. 2

THEOREM 2. *Given a finite lattice L, the following are equivalent:*

(i) $\not\leqslant$ *is completely regular,*

(ii) $\not\leqslant$ *belongs to* $\mathscr{C}_{\mathscr{B}(L)}(0, 2)$,

(iii) L *is a sum of four element Boolean lattices* (see Figure 2).

Proof. (i) $\Rightarrow$ (ii) is trivial, (iii) $\Rightarrow$ (i) follows from Lemma 7. Suppose that (ii) holds. Clearly we may assume that L has at least two elements. Assume that L contains only one atom p. Since, by Lemmas 4, 5, $p = p_{+-}$, p_+ is the unique atom of the interval $[p, 1]$. Continuing this argument we conclude that L contains an infinite chain, a contradiction. Therefore L has at least two atoms p and q. Then $p \not\leqslant p_+$ and $q \not\leqslant q_+$, whence $q = p_+$ and $p = q_+$. Thus for each element $x \in L$ distinct from 0, p, q, we get $x \geqslant q_+ \vee p_+ = p \vee q$ and $x_+ \geqslant p \vee q$. Hence, if $p \vee q < 1$, we may apply the above arguments to the interval $[p \vee q, 1]$. Thus it follows by induction that L must occur in the list of Figure 2.

References

[1] R. Balbes and P. Dwinger, *Distributive lattices*, University of Missouri Press, 1974.

[2] H.-J. Bandelt, *Regularity and complete distributivity*, Semigroup Forum 19 (1980), 123–126.

[3] E. S. Ljapin, *Semigroups*, 3rd ed., Amer. Math. Soc., 1974.

[4] G. N. Raney, *Tight Galois connections and complete distributivity*, Trans. Amer. Math. Soc. 97 (1960), 418–426.

[5] B. M. Schein, *Regular elements of the semigroup of all binary relations*, Semigroup Forum 13 (1976), 95–102.

[6] E. S. Wolk, *A characterization of partial order*, Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys. 17 (1969), 207–208.

[7] K. A. Zareckiĭ, *The semigroup of binary relations* (Russian), Mat. Sbornik 61 (1963), 291–305.

[8] —, *Maximal submonoids of the semigroup of binary relations*, Semigroup Forum 9 (1974), 196–208.

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