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# MODAL OPERATORS ON SYMMETRICAL HEYTING ALGEBRAS

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## 1. Introduction

The results discussed here are concerning an algebraic structure that is closely related to a many-valued propositional calculus.

Among the extensions of the intuitionistic logic, the general symmetrical modal propositional calculus has been introduced by Moisil in 1942 [16]. This calculus is obtained by the addition of one unary connective — the negation sign — to the alphabet of the intuitionistic propositional calculus. Two logical axioms and one rule of inference characterize this new connective: the double negation laws and the contraposition rule. In 1969, in order to study this calculus from an algebraic standpoint, Monteiro [24] developed the theory of the symmetrical Heyting algebras. These algebras are Heyting algebras or pseudo-Boolean algebras with a symmetry, i.e. with an involution that inverts the order given by the implication. Particular cases of symmetrical Heyting algebras are symmetrical three-valued Heyting algebras [8], three-valued Łukasiewicz algebras [14] and of course Boolean algebras.

On the other hand, in 1967, Rousseau [28], [29] formulated the classical and the intuitionistic $n$-valued propositional calculus, giving the first standard axiom system for these. Algebraic approaches to these calculi are respectively Post algebras and pseudo-Post algebras. Predicate calculi based on these propositional calculi have been studied by Rasiowa [26].

The aim of this expose is to present a many-valued logic, connected with the general symmetrical modal propositional calculus. We will concentrate our attention on the algebraic view-point. Thus we will introduce the notion of a symmetrical Heyting algebra of order $n$. Roughly speaking, it is a symmetrical Heyting algebra with $n-1$ unary operators satisfying suitable conditions. In a semantical model for this logic the mentioned operators may be interpreted as modal operators.

In a talk given at the University of Lyon in 1975 we have considered symmetrical Łukasiewicz algebras [9], i.e. Łukasiewicz algebras of order $n$ with an automorphism which is at the same time an involution. In the case $n = 3$ the notions of symmetrical Heyting algebra of order 3 and symmetrical three-valued Łukasiewicz algebra are equivalent. For one thing, this fact is a consequence of Moisil's remark [16] that, in the three-valued case, the Łukasiewicz negation can always be defined by means of the pseudo-complement, the dual pseudo-complement and the meet and the join. For another, it is well known that three-valued Łukasiewicz algebras are Heyting algebras [19]. Since Moisil's remark does not hold for $n > 3$, the structures mentioned above are only equivalent up to and including the three-valued case.

In order to obtain the definition of the abstract algebra that will be considered here, the characterization given in [10] has played an essential role.

Only the first part of the lectures given at the Seminar will be published in this volume. Two typical representation theorems — by means of sets and topological one — have been published in [11].(1)

## 2. Preliminaries

We recall the definition of structures and some properties needed for the understanding of the work.

According to [21], p. 151, a *Hilbert–Bernays algebra* is an abstract algebra $(A, 1, \wedge, \vee, \Rightarrow)$ such that $(A, 1, \wedge, \vee)$ is a lattice with unit 1 and for any two elements $x, y$ there is a greatest element $z = x \Rightarrow y$ such that $x \wedge z \leqslant y$.

It is well known that in a Hilbert–Bernays algebra the system $(A, 1, \wedge, \vee)$ is a distributive lattice with unit $1 = x \Rightarrow x$ [2].

A Hilbert–Bernays algebra with a zero element 0 will be said to be a *Heyting algebra*. In this case the element $\neg x = x \Rightarrow 0$ is called the pseudo-complement of $x$.

An abstract algebra $(A, 0, 1, \wedge, \vee, \sim)$ is said to be a *De Morgan algebra* or quasi-Boolean algebra ([1], [26], p. 44) if $(A, 0, 1, \wedge, \vee)$ is a distributive lattice with zero 0 and unit 1 and $\sim$ is a De Morgan negation on $A$. This last condition means that $\sim$ is a unary operation on $A$ — called the De Morgan negation — satisfying the following conditions:

$$\sim\sim x = x, \qquad \sim(x \vee y) = \sim x \wedge \sim y.$$

The following properties are true in any De Morgan algebra ([26], p. 44):

$$x \leqslant y \quad \text{if and only if} \quad \sim y \leqslant \sim x,$$

(1) Other selected parts will appear in Zeitschrift für Mathematische Logic und Grundlagen der Mathematik.



$$\sim 1 = 0 \quad \text{and} \quad \sim 0 = 1,$$
$$\sim(x \wedge y) = \sim x \vee \sim y.$$

A De Morgan negation $\sim$ on a lattice $A$ is called a Kleene negation if for all $x, y \in A$ the condition

$$x \wedge \sim x \leqslant y \vee \sim y$$

is satisfied. In this case the De Morgan algebra is said to be a Kleene algebra or normal $i$-lattice ([26], [12]).

By a *symmetrical Heyting algebra* we will mean an abstract algebra $(A, 0, 1, \wedge, \vee, \Rightarrow, \sim)$ where $(A, 0, 1, \wedge, \vee, \Rightarrow)$ is a Heyting algebra and $\sim$ is a De Morgan negation on $A$. We have borrowed this notion from [24]. In particular if $(A, 0, 1, \wedge, \vee, \Rightarrow)$ is a Boolean algebra the notion of a symmetrical Heyting algebra is similar to that of a symmetrical Boolean algebra or involutive Boolean algebra ([17], [22], [23]).

## 3. Symmetrical Heyting algebras of order $n$

Throughout this lectures we will be concerned with an abstract algebra, whose definition is given below.

On a symmetrical Heyting algebra $(A, 0, 1, \wedge, \vee, \Rightarrow, \sim)$ we are going to define $n-1$ unary operators ($n$ an integer $\geqslant 2$), noted $S_1, S_2, \ldots, S_{n-1}$. The required properties for these are the following:

— the operators $S_i$ are $(0, 1)$-lattice homomorphisms from $A$ onto the sublattice $B(A)$ of all complemented elements of $A$ such that $S_i S_j x = S_j x$ for all $i, j = 1, \ldots, n-1$;

— $S_1$ and $S_{n-1}$ are respectively an interior operator and a closure operator on $A$ ([26], p. 115–116);

— they are related to the operation $\Rightarrow$ and $\sim$ by the equation

$$S_i(x \Rightarrow y) = \bigwedge_{k=i}^{n-1} (S_k x \Rightarrow S_k y), \qquad S_i \sim x = \sim S_{n-i} x.$$

This situation suggests the following definition:

**3.1.** Definition. An abstract algebra $\mathfrak{A} = (A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim, S_1, \ldots, S_{n-1})$, $n$ an integer $\geqslant 2$, where 0, 1 are zero-argument operations, $\neg, \sim, S_1, \ldots, S_{n-1}$ are one argument operations and $\wedge, \vee, \Rightarrow$ are two-argument operations is said to be a *symmetrical Heyting algebra of order $n$* if

(S1) $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ is a symmetrical Heyting algebra,

and for every $x, y \in A$ and for all $i, j = 1, \ldots, n-1$ the following equations hold:

(S2) $S_i(x \wedge y) = S_i x \wedge S_i y$, (S3) $S_i(x \Rightarrow y) = \bigwedge_{k=i}^{n-1} (S_k x \Rightarrow S_k y)$,

(S4) $S_i S_j x = S_j x$,

(S5) $S_1 x \vee x = x$,

(S6) $S_i \sim x = \sim S_{n-i} x$,

(S7) $S_1 x \vee \neg S_1 x = 1$, with $\neg x = x \Rightarrow 0$.

It follows from the above definition and from the fact that the class of all symmetrical Heyting algebras is equationally definable [21] that the class of all symmetrical Heyting algebras of order $n$ is also equationally definable.

We will refer to a SH-algebra $A$ of order $n$, for short.

Let us note the following facts:

**3.2.** *If* $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim, S_1)$ *is* SH-*algebra of order* 2, *then* $S_1 x = x$ *for all* $x \in A$ *and* $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ *is a symmetrical Boolean algebra.*

Indeed, by (S6) $S_1 \sim x = \sim S_1 x$; by (S5) and (S1) $S_1 \sim x \leqslant \sim x$ and $\sim x \leqslant \sim S_1 x$. So $S_1 \sim x = \sim x$ and $S_1 x = x$ for all $x \in A$. Moreover, by (S7), $x \vee \neg x = 1$ for each $x \in A$, so $(A, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ is a symmetrical Boolean algebra.

It is possible to show that SH-algebras of order 3 are equivalent to involutive three-valued Heyting algebras studied in [7] and [8].

**3.3.** The definition of the Łukasiewicz algebra of order 3, or three-valued Łukasiewicz algebra, has been introduced by Moisil in 1940 [14], as an attempt to give an algebraic approach to the three-valued logic considered by Łukasiewicz in several mathematical logic works [13]. Łukasiewicz algebras of order $n$ are generalizations of the same algebras of order 3 and have also been introduced by Moisil in 1941 [15]. For a development of the theory of Łukasiewicz algebras of order $n$ see the works of Moisil himself [18], [20] and [3], [4], [5].

In [10] a characterization of Łukasiewicz algebras of order $n$ has been given in which the intuitionistic implication plays an essential role. This characterization allows us to conclude that Łukasiewicz algebras of order $n$ are SH-algebras of the same order.

**3.4.** *For every* SH-*algebra* $A$ *of order* $n$ *the following conditions are satisfied*:

(S8) $S_i 1 = 1, \quad S_i 0 = 0$, *for all* $i = 1, \ldots, n-1$,

(S9) $S_i(x \vee y) = S_i x \vee S_i y$, *for all* $i = 1, \ldots, n-1$,

(S10) *If* $S_i x = S_i y$ *for all* $i = 1, \ldots, n-1$, *then* $x = y$ (*determination principle*),



(S11) $x \leqslant y$ *if and only if* $S_i x \leqslant S_i y$,

(S12) $S_1 x \leqslant S_2 x \leqslant \ldots \leqslant S_{n-1} x$,

(S13) $x \leqslant S_{n-1} x$,

(S14) $S_i x \wedge \neg S_i x = 0$, *for all* $i = 1, \ldots, n-1$,

(S15) $S_i x \vee \neg S_i x = 1$, *for all* $i = 1, \ldots, n-1$,

(S16) $S_i(\neg x) = \neg S_{n-1} x$, *for all* $i = 1, \ldots, n-1$.

In fact, by (S1) and (S3), for all $i = 1, \ldots, n-1$

$$S_i 1 = S_i(x \Rightarrow x) = \bigwedge_{k=i}^{n-1} (S_k x \Rightarrow S_k x) = \bigwedge_{k=i}^{n-1} 1 = 1.$$

Consequently, by (S1), (S6) and the last result we get $S_i 0 = S_i(\sim 1) = \sim S_{n-i} 1 = \sim 1 = 0$ for all $i = 1, \ldots, n-1$. Thus (S8) holds. (S9) is a consequence of (S1), (S6) and (S2). The proof of (S10) can be found in [10], p. 135. Suppose $x \leqslant y$ for some $x, y \in A$. By (S2) it follows that $S_i x \leqslant S_i y$. On the other hand, if $S_i x \leqslant S_i y$, then by (S2) $S_i x = S_i x \wedge S_i y = S_i(x \wedge y)$. Hence by the determination principle $x = x \wedge y$ and $x \leqslant y$. Thus (S11) holds. The proof of (S12) can be found in [10], p. 135. By (S11) $x \leqslant S_{n-1} x$ is equivalent to $S_i x \leqslant S_i S_{n-1} x$, which is equivalent, by (S4), to $S_i x \leqslant S_{n-1} x$. This together with (S12) proves that (S13) holds. Since $A$ is a Heyting algebra, $x \wedge \neg x = 0$ for every $x \in A$. In particular $S_i x \wedge \neg S_i x = 0$ for every $i = 1, \ldots, n-1$ and $x \in A$. Thus (S14) holds. On the other hand by (S4) and (S7) $S_i x \vee \neg S_i x = S_1 S_i x \vee \neg S_1 S_i x = 1$, i.e. $S_i x \vee \neg S_i x = 1$ for all $i = 1, \ldots, n-1$ and every $x \in A$ and (S15) holds. Using (S1), (S3), (S8) and (S12)

$$S_i(\neg x) = S_i(x \Rightarrow 0) = \bigwedge_{k=i}^{n-1} (S_k x \Rightarrow S_k 0) = \bigwedge_{k=i}^{n-1} \neg S_k x = \neg S_{n-1} x.$$

The proof of 3.4 is finished.

**3.5.** Given an arbitrary SH-algebra $A$, we will denote by $B(A)$ the Boolean algebra $(B(A), 0, 1, \wedge, \vee, \Rightarrow, \neg)$ of all complemented elements in $A$. If $b \in B(A)$, its complement $b'$ is equal to $\neg b$.

Indeed, $b \wedge b' = 0$ and $b \vee b' = 1$. The first equality means that $b' \leqslant \neg b$, so by the second equality $1 = b \vee b' \leqslant b \vee \neg b$, i.e. $b \vee \neg b = 1$. Thus $\neg b = b'$.

**3.6.** For all $i = 1, \ldots, n-1$ let $S_i(A)$ be the image of $A$ under $S_i$. By (S4) mappings $S_i$ have a common image $S_1(A) = S_2(A) = \ldots = S_{n-1}(A)$ and $S_i(A) = \{x \in A\colon S_i x = x\}$.

Since by (S2), (S8) and (S9) these mappings are (0, 1)-lattice homomorphisms, $S_i(A)$ is a sublattice of $A$. We will show that $S_i(A)$ is closed under $\sim$. In fact, if $x \in S_i(A)$, then $S_i x = x$; by (S6) $\sim x = \sim S_i x = S_{n-i} \sim x$ so $\sim x \in S_{n-i}(A) = S_i(A)$. Thus $S_i(A)$ is a De Morgan sublattice of $A$.

**3.7.** For all $i = 1, \ldots, n-1$, $S_i(A) = B(A)$.

To see $S_i(A) \subseteq B(A)$ it is sufficient to note that by (S14) and (S15) every element in $S_i(A)$ is complemented. On the other hand, if $b \in B(A)$, then there exists $b' \in B(A)$ such that $b \vee b' = 1$ and $b \wedge b' = 0$; operating with $S_1$ and applying (S9), (S8) and (S2) we get $S_1 b \vee S_1 b' = S_1 1 = 1$ and $S_1 b \wedge S_1 b' = S_1 0 = 0$; thus $(S_1 b)' = S_1 b' \leqslant b'$ by (S5). This is equivalent to $b \leqslant S_1 b$. But by (S5) again $S_1 b \leqslant b$, so $S_1 b = b$ and $b \in S_1(A)$. We have shown that $S_1(A) = B(A)$. Combining this result with 3.6 we get 3.7.

**3.8.** *It follows from the results above that the image* $(S_i(A) = B(A), 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ *is a symmetrical Boolean subalgebra of* $A$, *for all* $i = 1, \ldots, n-1$.

**3.9.** Using the De Morgan and the intuitionistic negations denoted "$\sim$" and "$\neg$" respectively, we can consider, for notational convenience, the operation "$\ulcorner$" given by the equality

$$\text{(A)} \qquad \ulcorner x = \sim \neg \sim x.$$

The operation $\ulcorner$ has the dual properties of these of $\neg$. Moreover,

(a) $\neg x \leqslant \ulcorner x$,

(b) $\neg \ulcorner x = \ulcorner \ulcorner x$,

(c) $\neg \sim b = \sim \neg b$ for all $b \in B(A)$,

(d) $B(A) = \{x \in A : \neg x = \ulcorner x\}$.

In fact, $\ulcorner x = \ulcorner x \vee 0 = \ulcorner x \vee (x \wedge \neg x) = (\ulcorner x \vee x) \wedge (\ulcorner x \vee \neg x) = \ulcorner x \vee \neg x$ so (a) holds. Since $\ulcorner \ulcorner x \wedge \ulcorner x = 0$, it follows $\ulcorner \ulcorner x \leqslant \neg \ulcorner x$. On the other hand, by (a), $\neg \ulcorner x \leqslant \ulcorner \ulcorner x$ and thus (b) holds. By (a), $\neg \sim b \leqslant \ulcorner \sim b = \sim \neg b$. Since $b \in B(A)$, $b \vee \neg b = 1$ so $\sim b \wedge \sim \neg b = 0$ and $\sim \neg b \leqslant \neg \sim b$ and (c) holds. The last property follows easily.

**3.10.** The operation $\sim$ on the lattice $A$ permits us to consider a duality principle. Consequently every statement proved for $\wedge$, $\vee$ and $\sim$ remains true if $\wedge$ and $\vee$ are replaced by $\vee$ and $\wedge$ respectively.

The following example of a SH-algebra of order $n$ plays an important role.

**3.11.** Example. Let $L_n$ be the set of fractions $j/(n-1)$ with $j = 0, 1, \ldots \ldots, n-1$ considered as a sublattice of the real numbers and $S_{n^2}$ the Cartesian product $(L_n)^2$, i.e. the set of all $z = (x, y)$ with $x \in L_n$ and $y \in L_n$. $S_{n^2}$ with the pointwise defined operations $\wedge$, $\vee$ is a Heyting algebra.



Let us put on $S_{n^2}$

$$\sim(x, y) = (1-y, 1-x)$$

and for all $i = 1, \ldots, n-1$,

$$S_i(x, y) = (S_i x, S_i y),$$

where

$$S_i x = S_i\big(j/(n-1)\big) = \begin{cases} 1 & \text{if } i+j \geqslant n, \\ 0 & \text{if } i+j < n. \end{cases}$$

The system $(S_{n^2}, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim, S_1, \ldots, S_{n-1})$ is a SH-algebra of order $n$.

In the case $n = 3$, we get the example given in [7] and [8]. In addition, we can see that, in general, we have $x \vee \sim x \neq 1$, $x \wedge \sim x \neq 0$, $x \vee \neg x \neq 1$, $x \wedge \ulcorner x \neq 0$, $\neg x \nleqslant \sim x$, $\sim x \nleqslant \neg x$, $\sim x \nleqslant \ulcorner x$, $\ulcorner x \nleqslant \sim x$.

In general, SH-algebras of order $n$ are not Kleene algebras (see Example 3.11 above for $n = 3$). But we can prove that:

**3.12.** *For a* SH-*algebra* $A$ *of order* $n$ *the following conditions are equivalent*:

(i) $A$ *satisfies the Kleene law*,

(ii) $A$ *is a Łukasiewicz algebra of order* $n$.

Suppose that $A$ is a SH-algebra of order $n$ satisfying the Kleene law. Since $A$ is a De Morgan lattice, it is a Kleene algebra and this fact implies that for every $z \in B(A)$, if $z'$ is the Boolean complement, then $z' = \sim z$ ([25], p. 454). By (S7) and (S14) the last result implies that $S_1 x \vee \sim S_1 x = 1$. According to the characterization of Łukasiewicz algebras of order $n$ given in [10], p. 134, we conclude that $A$ is a Łukasiewicz algebra of order $n$. That (ii) implies (i) is a consequence of the fact that the negation defined in Łukasiewicz algebras satisfies in particular the Kleene property as it was proved by different ways in [30] and [3].

Recall that Post algebras of order $n$ are analogous with centered Łukasiewicz algebras of the same order, i.e., Łukasiewicz algebras of order $n$ with $n-2$ elements $e_1, e_2, \ldots, e_{n-2}$ such that ([3], p. 41):

$$\text{(B)} \qquad S_i(e_j) = \begin{cases} 0 & \text{if } i+j < n, \\ 1 & \text{if } i+j \geqslant n. \end{cases}$$

By notational convenience let us put $e_0 = 0$ and $e_{n-1} = 1$.

This fact suggests us to introduce the following definition:

**3.13.** Definition. A SH-algebra of order $n$ will be said to be *centered* if it has $n-2$ elements $e_1, e_2, \ldots, e_{n-2}$ satisfying condition (B) above.

Combining 3.12 and the result above we get

**3.14.** *For a* SH-*algebra of order* $n$ *the following conditions are equivalent*:

(i) $A$ *is centered and satisfies the Kleene law*,

(ii) $A$ *is a Post algebra of order* $n$.

**3.15.** *In a centered* SH-*algebra of order* $n$ *the following properties hold* ([31], p. 198, [32]):

(a) $0 = e_0 < e_1 < \ldots < e_{n-2} < e_{n-1} = 1$,

(b) $x = \bigvee_{j=1}^{n-1} (S_{n-j}x \wedge e_j)$,

(c) *If* $b \in B(A)$ *and* $b \wedge e_j \leqslant e_{j-1}$ *for some* $j = 1, \ldots, n-1$, *then* $b = 0$.

*Thus every centered* SH-*algebra of order* $n$ *is a Post algebra of the same order.*

In account of (S11) to prove (a) it is equivalent to prove

(1) $$S_k e_i < S_k e_{i+1}.$$

If $k+i < n$ by (B) we get $S_k e_i = 0$ and (1) holds. If $k+i \geqslant n$, then $S_k e_i = 1$. But $k+i+1 > k+i \geqslant n$ so $S_k e_{i+1} = 1$ and (1) holds. Operating on the right side of (b) with $S_i$, $i = 1, \ldots, n-1$, and applying (S9), (S2), (S4), (B) and (S12)

$$S_i\Big(\bigvee_{j=1}^{n-1} S_{n-j}x \wedge e_j\Big) = \bigvee_{j=1}^{n-1} (S_{n-j}x \wedge S_i e_j) = \bigvee_{j=n-i}^{n-1} S_{n-j}x = S_i x.$$

By (S10), (b) holds. Suppose $b \in B(A)$ and $b \wedge e_j \leqslant e_{j-1}$ for some $j$. Operating with $S_{n-j}$ and applying (S11), (S2), 3.6, 3.7 and (B) we obtain $b \wedge S_{n-j}e_j \leqslant S_{n-j}e_{j-1} = 0$. So $b \wedge 1 = b = 0$ and (c) holds.

Following [6] if $A$ is a centered SH-algebra of order $n$ let us put

(C) $$-x = \bigvee_{i=1}^{n-1} (\neg S_i x \wedge e_i).$$

**3.16.** *In a centered* SH-*algebra of order* $n$ *the following conditions are satisfied*:

(d) $--x = x$,

(e) $-(x \wedge y) = -x \vee -y$,

(f) $x \wedge -x \leqslant y \vee -y$,

(g) $-b = \neg b$ *for all* $b \in B(A)$,

(h) $\sim -x = -\sim x$,

(i) $S_i - x = -S_{n-i}x$



Indeed, by (S9), (S2), 3.7 and 3.5

$$--x = \bigvee_{i=1}^{n-1} (\neg S_i - x \wedge e_i) = \bigvee_{i=1}^{n-1} \Big[\Big(\neg S_i \bigvee_{j=1}^{n-1} (\neg S_j x \wedge e_j)\Big) \wedge e_i\Big]$$
$$= \bigvee_{i=1}^{n-1} \Big(\bigwedge_{j=1}^{n-1} (S_j x \vee \neg S_i e_j) \wedge e_i\Big)$$
$$= \bigvee_{i=1}^{n-1} \Big(\bigwedge_{j=1}^{n-1} \big((S_j x \wedge e_i) \vee (\neg S_i e_j \wedge e_i)\big)\Big).$$

If $i+j < n$

$$(S_j x \wedge e_i) \vee (\neg S_i e_j \wedge e_i) = (S_j x \wedge e_i) \vee e_i = e_i.$$

If $i+j \geqslant n$

$$(S_j x \wedge e_i) \vee (\neg S_i e_j \wedge e_i) = S_j x \wedge e_i$$

so

$$\bigwedge_{i=1}^{n-1} (S_j x \wedge e_i) \vee (\neg S_i e_j \wedge e_i) = S_{n-i}x \wedge e_i.$$

Hence

$$--x = \bigvee_{i=1}^{n-1} (S_{n-i}x \wedge e_i) = x$$

and (d) holds. By definition (C) and applying (S2), 3.7 and 3.5 we get

$$-(x \wedge y) = \bigvee_{i=1}^{n-1} (\neg S_i(x \wedge y) \wedge e_i) = \bigvee_{i=1}^{n-1} (\neg S_i x \vee \neg S_i y) \wedge e_i$$
$$= \bigvee_{i=1}^{n-1} (\neg S_i x \wedge e_i) \vee (\neg S_i y \wedge e_i)$$
$$= \bigvee_{i=1}^{n-1} (\neg S_i x \wedge e_i) \vee \bigvee_{i=1}^{n-1} (\neg S_i y \wedge e_i) = -x \vee -y$$

so (e) holds.

To prove (f) it is equivalent to prove that

(f′) $S_i(x \wedge -x) \leqslant S_i(y \vee -y)$ for all $i = 1, \ldots, n-1$.

In order to prove this last inequality assume that $n$ is an odd number. On one hand, if $1 \leqslant i \leqslant (n-1)/2$, we get

(1) $$S_i(x \wedge -x) = \bigvee_{j=1}^{n-1} (S_i x \wedge \neg S_j x \wedge S_i e_j)$$
$$= \bigvee_{j=n-i}^{n-1} (S_i x \wedge \neg S_j x) = S_i x \wedge \neg S_{n-i} x.$$

But $1 \leqslant i \leqslant (n-1)/2$ so $2i \leqslant n-1$ and $i < n-i$. Hence $S_i x \leqslant S_{n-i} x$ and

(2) $$\neg S_{n-i} x \leqslant \neg S_i x.$$

Combining (1) and (2) we obtain

$$S_i(x \wedge -x) \leqslant S_i x \wedge \neg S_i x = 0$$

and (f') holds.

On the other hand, if $i \geqslant (n+1)/2$, then $2i \geqslant n+1 > n$ and $i > n-i$. Thus

$$S_i(y \vee -y) = S_i y \vee \bigvee_{j=1}^{n-1} (\neg S_j y \wedge S_i e_j)$$

$$= S_i y \vee \bigvee_{j=n-i}^{n-1} \neg S_j y \geqslant S_i y \vee \neg S_i y = 1$$

and (f') holds.

In the case $n$ is even it is still necessary to consider the possibility $i = n/2$. Thus

$$S_{n/2}(x \wedge -x) = \bigvee_{j=1}^{n-1} (S_{n/2} x \wedge \neg S_j x \wedge S_{n/2} e_j)$$

$$= \bigvee_{j=n/2}^{n-1} (S_{n/2} x \wedge \neg S_j x) \leqslant S_{n/2} x \wedge \neg S_{n/2} x = 0$$

and (f) holds. Moreover,

$$-b = \bigvee_{i=1}^{n-1} (\neg S_i b \wedge e_i) = \bigvee_{i=1}^{n-1} (\neg b \wedge e_i) = \neg b$$

which gives (g). Using the determination principle

$$S_k \sim -x = \bigwedge_{i=1}^{n-1} (\sim \neg S_i x \vee S_k \sim e_i) = \bigwedge_{i=1}^{n-1} (\sim \neg S_i x \vee S_k e_{n-i-1})$$

$$= \bigwedge_{k=i}^{n-1} \sim \neg S_i x = \sim \neg S_k x,$$

$$S_k - \sim x = \bigvee_{i=1}^{n-1} (\sim \neg S_{n-i} x \wedge S_k e_i) = \bigvee_{k=n-i}^{n-1} \sim \neg S_{n-i} x = \sim \neg S_k x$$

we conclude that (h) is true. Finally

$$-S_i x = \bigvee_{j=1}^{n-1} (\neg S_j S_i x \wedge e_j) = \bigvee_{j=1}^{n-1} (\neg S_i x \wedge e_j) = \neg S_i x$$

and

$$S_{n-i} - x = \bigvee_{j=1}^{n-1} (\neg S_j x \wedge S_{n-i} e_j) = \bigvee_{j=i}^{n-1} \neg S_j x = \neg S_i x$$

thus (i) holds.

In a centered SH-algebra of order $n$ let us put

(D) $$ax = -\sim x = \sim -x.$$



**3.17.** *In a centered* SH-*algebra $A$ of order $n$ the map $a$ is an automorphism of $A$ which is at the same time an involution.*

It is a consequence of Definition 3.1 and the properties of 3.16.

**3.18.** *The following conditions are equivalent*:

(i) *$A$ is a centered* SH-*algebra of order $n$,*

(ii) *$A$ is a Post algebra of order $n$ with an automorphism which is at the same time an involution.*

That (i) implies (ii) is a consequence of 3.15 and 3.17. On the other hand Rousseau [28] has shown that every Post algebra of order $n$ is a Heyting algebra and Epstein [6] has proved that every Post algebra of order $n$ has a symmetry "$-$". The operation $\sim x = -ax$ is a De Morgan negation on $A$. Gathering these results together the proof is complete.

## 4. A propositional calculus based on SH-algebras of order $n$

In this section we present a propositional calculus extension of the intuitionistic propositional calculus. The axiom system is given in such a way that the set of all SH-algebras of order $n$ is characteristic, in the sense that will be given below.

In discussing the axiom system we will use some familiar notions about propositional calculi (see [27]).

Let $L = (A^0, F)$ be a formalized language where $A^0 = \{V, \wedge, \vee, \Rightarrow, \neg, \sim, S_1, \ldots, S_{n-1}, (,)\}$, $n$ an integer $\geqslant 2$, is the alphabet and $F$ the set of all formulas over $A^0$. Formation-rules are as usual. Elements $p$ in $V$ are called propositional variables; $\wedge, \vee, \Rightarrow, \neg, \sim, S_1, \ldots, S_{n-1}$ propositional connectives and the parentheses are auxiliary signs.

In the axiom system below $\wedge, \vee, \Rightarrow$ and $\neg$ may be interpreted as the conjunction, disjunction, intuitionistic implication and intuitionistic negation respectively, $\sim$ as a De Morgan negation and $S_1, \ldots, S_{n-1}$ as modal functors.

To avoid a clumsy statement of the rule of substitution, axiom schemas are considered instead of axioms. For any formulas $x, y$ of $L$ we will write for brevity $(x \equiv y)$ instead of $\big((x \Rightarrow y) \wedge (y \Rightarrow x)\big)$.

**4.1. Axiom Schema.**

(A1)–(A8) axioms of the positive propositional calculus of Hilbert and Bernays (see [26], p. 236),

(A9) $(\sim\sim x \Rightarrow x)$,

(A10) $(x \Rightarrow \sim\sim x)$,

(A11) $(S_i(x \wedge y) \equiv S_i x \wedge S_i y)$,

(A12) $\big(S_i(x \Rightarrow y) \equiv \big((\ldots(S_i x \Rightarrow S_i y) \wedge \ldots) \wedge (S_{n-1}x \Rightarrow S_{n-1}y)\big)\big)$,

(A13) $(S_i S_j x \equiv S_j x)$, $i = 1, \ldots, n-1$,

(A14) $(S_1 x \Rightarrow x)$,

(A15) $(S_i {\sim} x \equiv {\sim} S_{n-i} x)$, $i = 1, \ldots, n-1$,

(A16) $(S_1 x \vee \neg S_1 x)$.

Rules of inference

(R1) $\dfrac{x, (x \Rightarrow y)}{y}$ Modus Ponens,

(R2) $\dfrac{(x \Rightarrow y)}{({\sim} y \Rightarrow {\sim} x)}$ Contraposition rule,

(R3) $\dfrac{(x \Rightarrow y)}{(S_1 x \Rightarrow S_1 y)}$.

**4.2.** Let $D$ be the least set of formulas of $L$ containing the logical axioms (A1)–(A16) and closed under the rules (R1)–(R3). The formalized language $L$ with the selected subset of derivable formulas make up an *n-valued general symmetrical modal propositional calculus.*

Following Lindenbaum and Tarski the set of formulas $F$ of the formalized language can be considered as an abstract algebra $\mathfrak{F} = (F, D, \wedge, \vee, \Rightarrow, \neg, {\sim}, S_1, \ldots, S_{n-1})$; $V$ is the set of generators of $\mathfrak{F}$. For $\alpha, \beta \in \mathfrak{F}$ let $\alpha \equiv \beta$ if and only if $\alpha \Rightarrow \beta \in D$ and $\beta \Rightarrow \alpha \in D$. By (A1)–(A8), (R1)–(R3) it is well known that $\equiv$ is a congruence on $F$.

The propositional calculus here considered is an extension of the general symmetrical modal logic introduced by Moisil ([16], p. 411, [20]). This author has shown ([16], p. 412–413) that the most interesting theorems in this logic are those showing that the negation $\sim$ is a duality. That is

(A17) $\big(({\sim} x \vee {\sim} y) \Rightarrow {\sim}(x \wedge y)\big)$,

(A18) $\big({\sim}(x \vee y) \Rightarrow ({\sim} x \wedge {\sim} y)\big)$,

(A19) $\big(({\sim} x \wedge {\sim} y) \Rightarrow {\sim}(x \vee y)\big)$,

(A20) $\big({\sim}(x \wedge y) \Rightarrow ({\sim} x \vee {\sim} y)\big)$.

Let $|F|$ be the set of all equivalence classes $|\alpha|$ algebraized in a standard way. Furthermore, $\alpha$ is derivable if and only if $|\alpha|$ is the unit element of $|F|$. In this way the Lindenbaum algebra $\mathfrak{L} = \mathfrak{F}/\equiv\, = (|F|, {\sim}|D|, |D|, \wedge, \vee, \Rightarrow, \neg, {\sim}, S_1, \ldots, S_{n-1})$ is a SH-algebra of order $n$.



**4.3.** By a *valuation* of $L$ in a SH-algebra $A$ of order $n$ we will understand any mapping $v$: $V \to A$, that is, any point $v = (v_p)_{p \in V}$ of the Cartesian product $A^V$. Every propositional variable $p$ in $L$ determines a mapping $p_A$: $A^V \to A$ by means of the equality $p_A(v) = v(p)$. By induction on the length of a formula, each $\alpha$ in $L$ determines a mapping $\alpha_A$: $A^V \to A$.

**4.4.** Since $\mathfrak{L}$ is a SH-algebra of order $n$ we can interpret formulas of $L$ as mappings from $\mathfrak{L}^V$ into $\mathfrak{L}$. The valuation $v^0$: $V \to \mathfrak{L}$ such that $v^0(p) = |p|$ for every propositional variable $p$ of $L$ will be called the *canonical valuation* of $L$ in $\mathfrak{L}$.

For every formula $\alpha$ of $L$

$$\alpha_{\mathfrak{L}}(v^0) = |\alpha|$$

for the canonical valuation $v^0$. In fact, for every propositional variable $p$

$$p_{\mathfrak{L}}(v^0) = v^0(p) = |p|$$

and by induction on the length of $\alpha$ the result is obtained.

A formula $\alpha$ of $L$ is said to be *valid* in $A$ provided that $\alpha_A(v) = 1$ for every valuation $v$ of $L$ in $A$.

Finally, we get that the class of all SH-algebras of order $n$ is characteristic, i.e.,

**4.5.** COMPLETENESS THEOREM. *For every formula $\alpha$ of the n-valued general symmetrical modal propositional calculus the following conditions are equivalent:*

(i) *$\alpha$ is derivable in the propositional calculus,*

(ii) *$\alpha$ is valid in every* SH-*algebra of order $n$.*

The method of the proof is similar to that which can be found in [27] for other propositional calculi. It is routine to show that a derivable formula in the propositional calculus is valid in every SH-algebra of order $n$. On the other hand, suppose $\alpha$ is valid in $\mathfrak{L}$, i.e. $\alpha_{\mathfrak{L}}(v) = 1$ for every valuation $v \in \mathfrak{L}^V$. In particular, if $v$ is the canonical valuation $v^0 \in \mathfrak{L}^V$, $\alpha_{\mathfrak{L}}(v^0) = 1$. Because of a result above, $|\alpha| = 1$ so $\alpha \in D$.

**4.6.** *Remark.* The $n$-valued general symmetrical modal propositional calculus is consistent. In fact, since (i) $\to$ (ii) in 4.5, no propositional variable $p$ in $V$ is derivable in the propositional calculus.

**4.7.** Let $D(A)$ the set of formulas of the $n$-valued general symmetrical modal propositional calculus which are valid in a SH-algebra $A$ of order $n$. The algebra $A$ is said to be a *characteristic matrix* for the propositional calculus if $D = D(A)$.

It is possible to show that the $n$-valued general symmetrical modal propositional calculus has a finite characteristic matrix, more precisely, that $D = D(S_{n^2})$.

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