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AN ALGEBRAIC TREATMENT OF FLOW DIAGRAMS AND ITS APPLICATION TO GENERALIZED RECURSION THEORY

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Flow diagrams are used in programming as descriptions of some partial mappings of appropriate sets into themselves. In our algebraic treatment, the role of such partial mappings will be played by the elements of a partially ordered semi-group, provided with some additional algebraic structure. The algebraic systems obtained in this way will be called programming spaces.

DEFINITION 1. A *programming space* is a 6-tuple $\langle \mathcal{F}, A, B, C, \veebar, *\rangle$, where $\mathcal{F}$ is a partially ordered semi-group with an identity I, A, B and C are elements of $\mathcal{F}$, $\veebar$ and $*$ are a binary and a unary operation in $\mathcal{F}$ respectively and the following conditions are satisfied for all φ, ψ, γ and ϱ in $\mathcal{F}$:

(i) $CA = CB = I$;

(ii) $(\varphi \veebar \psi)A = \varphi A$, $(\varphi \veebar \psi)B = \psi B$;

(iii) $\varrho(\varphi \veebar \psi) = \varrho\varphi \veebar \varrho\psi$;

(iv) $\varphi \leqslant \psi \Rightarrow I \veebar \varphi \leqslant I \veebar \psi$;

(v) $I \veebar (*\varphi)\varphi \leqslant *\varphi$;

(vi) $\gamma \veebar \varrho\varphi \leqslant \varrho \Rightarrow \gamma(*\varphi) \leqslant \varrho$.

Remark 1. Condition (v) can be replaced by the condition

$$I \veebar (*\varphi)\varphi = *\varphi.$$

This can be seen in the following way. Let $\langle \mathcal{F}, A, B, C, \veebar, *\rangle$ be a programming space in the sense of Definition 1. If $\varrho = I \veebar (*\varphi)\varphi$, then $\varrho \leqslant *\varphi$; using condition (iv), we obtain the inequality $I \veebar \varrho\varphi \leqslant I \veebar (*\varphi)\varphi$, i.e. the inequality $I \veebar \varrho\varphi \leqslant \varrho$, and from this inequality and (vi) it follows that $*\varphi \leqslant \varrho$. Hence $\varrho = *\varphi$. Thus we see that in a programming space $\langle \mathcal{F}, A, B, C, \veebar, *\rangle$ the element $*\varphi$ is the least fixed point of the mapping

$\theta \mapsto I \veebar \theta\varphi$. Furthermore, using conditions (iii) and (vi) we see also that for each γ in $\mathscr{F}$ the element $\gamma(*\varphi)$ is the least fixed point of the mapping $\theta \mapsto \gamma \veebar \theta\varphi$.

Now we shall give some examples of programming spaces. In each of these examples, the operations $\veebar$ and $*$ can be considered as a kind of branching and iteration respectively.

EXAMPLE 1. Let $\mathscr{F}$ be a partially ordered semi-gorup of all partial mappings of the set $\boldsymbol{N}$ of the natural numbers into itself (the semi-group multiplication is defined as composition, i.e. $\varphi\psi(s) \simeq \varphi(\psi(s))$ for all s in $\boldsymbol{N}$, and $\varphi \leqslant \psi$ means that ψ is an extension of φ). Let

$$A(s) = 2s, \quad B(s) = 2s+1, \quad C(s) = [s/2],$$

$$(\varphi \veebar \psi)(s) \simeq \begin{cases} \varphi(s), & \text{if } s \text{ is even,} \\ \psi(s), & \text{if } s \text{ is odd,} \end{cases}$$

and let $(*\varphi)(s) = t$ iff there is a finite sequence $r_0, r_1, \ldots, r_m$ of natural numbers such that $r_0 = s$, $r_m = t$, $r_0, r_1, \ldots, r_{m-1}$ are odd, r_m is even and $r_{j+1} = \varphi(r_j)$ for $j = 0, 1, \ldots, m-1$.

EXAMPLE 2. Let $\mathscr{F}$ be the partially ordered semi-group of all continuous partial mappings with open domains of the set of the real numbers into itself. Let

$$A(s) = e^s, \quad B(s) = -e^s, \quad C(s) \simeq \ln|s|,$$

$$(\varphi \veebar \psi)(s) = t \Leftrightarrow \big(s > 0 \ \& \ \varphi(s) = t\big) \vee \big(s < 0 \ \& \ \psi(s) = t\big)$$

and let $(*\varphi)(s) = t$ iff there is a finite sequence $r_0, r_1, \ldots, r_m$ of real numbers such that $r_0 = s$, $r_m = t$, $r_0, r_1, \ldots, r_{m-1}$ are negative, r_m is positive and $r_{j+1} = \varphi(r_j)$ for $j = 0, 1, \ldots, m-1$.

EXAMPLE 3. Let Y, Δ, T, F, K, ${}^{\pi}(q)$ have the same meaning as in [4]. Let $\mathscr{F}$ be the partially ordered semi-group of all partial mappings of Y into Y, let $A = T$, $B = F$, $C = K$, $\varphi \veebar \psi = \Delta(\omega, \varphi, \psi)$ and let $*\varphi = {}^{\omega}(\varphi)$.

The next examples can be considered as non-deterministic modifications of Example 1 (in Example 6, the idea of complexity of data processing is also involved).

EXAMPLE 4. Let $\mathscr{F}$ be the partially ordered semi-group of all multi-valued mappings of $\boldsymbol{N}$ into $\boldsymbol{N}$ ($\varphi\psi$ is defined by $\varphi\psi(s) = \bigcup \{\varphi(r)\colon r \in \psi(s)\}$ and $\varphi \leqslant \psi$ means that $\varphi(s) \subseteq \psi(s)$ for all s in $\boldsymbol{N}$). Let

$$A(s) = \{2s\}, \quad B(s) = \{2s+1\}, \quad C(s) = \{[s/2]\},$$

$$(\varphi \veebar \psi)(s) = \begin{cases} \varphi(s) & \text{if } s \text{ is even,} \\ \psi(s), & \text{if } s \text{ is odd,} \end{cases}$$

and let $t \in (*\varphi)(s)$ iff there is a finite sequence $r_0, r_1, \ldots, r_m$ of natural numbers such that $r_0 = s$, $r_m = t$, $r_0, r_1, \ldots, r_{m-1}$ are odd, r_m is even and $r_{j+1} \in \varphi(r_j)$ for $j = 0, 1, \ldots, m-1$.

EXAMPLE 5. Let $\mathscr{F}$ be the set of all mappings φ of $\boldsymbol{N}^2$ into the interval $[0, 1]$ such that $\sum_{t=0}^{\infty} \varphi(s, t) \leqslant 1$ for all s in $\boldsymbol{N}$, let $\varphi\psi$ be defined by

$$\varphi\psi(s, t) = \sum_{r=0}^{\infty} \varphi(r, t)\psi(s, r)$$

and let $\varphi \leqslant \psi$ means that $\varphi(s, t) \leqslant \psi(s, t)$ for all s and t in $\boldsymbol{N}$. Let

$$A(s, t) = \delta_{2s}^{t}, \quad B(s, t) = \delta_{2s+1}^{t}, \quad C(s, t) = \delta_{[s/2]}^{t},$$

$$(\varphi \veebar \psi)(s, t) = \begin{cases} \varphi(s, t), & \text{if } s \text{ is even,} \\ \psi(s, t), & \text{if } s \text{ is odd,} \end{cases}$$

and let $*\varphi$ be defined in the following way:

$$(*\varphi)(2k, t) = \delta_{2k}^{t},$$

$$(*\varphi)(2k+1, 2l+1) = 0,$$

$$(*\varphi)(2k+1, 2l) = \varphi(2k+1, 2l) + \sum_{q_1=0}^{\infty} \varphi(2k+1, 2q_1+1)\varphi(2q_1+1, 2l) + \\ + \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \varphi(2k+1, 2q_1+1)\varphi(2q_1+1, 2q_2+1)\varphi(2q_2+1, 2l) + \ldots$$

EXAMPLE 6. Let K be an arbitrary semi-group with an identity Λ and let $\mathscr{F}$ be the set of all subsets of the Cartesian product $\boldsymbol{N} \times K \times \boldsymbol{N}$. Let $\mathscr{F}$ be partially ordered by the inclusion relation and let

$$\varphi\psi = \{\langle s, kl, t\rangle\colon \exists r(\langle r, l, t\rangle \in \varphi \ \& \ \langle s, k, r\rangle \in \psi)\}.$$

Let

$$A = \{\langle s, \Lambda, 2s\rangle\colon s \in \boldsymbol{N}\},$$
$$B = \{\langle s, \Lambda, 2s+1\rangle\colon s \in \boldsymbol{N}\},$$
$$C = \{\langle s, \Lambda, [s/2]\rangle\colon s \in \boldsymbol{N}\},$$

$$\varphi \veebar \psi = \{\langle s, k, t\rangle\colon \langle s, k, t\rangle \in \varphi \ \& \ s \text{ is even}\} \cup \\ \cup \{\langle s, k, t\rangle\colon \langle s, k, t\rangle \in \psi \ \& \ s \text{ is odd}\}$$

and let $\langle s, k, t\rangle \in *\varphi$ iff there are a finite sequence $r_0, r_1, \ldots, r_m$ of natural numbers and a finite sequence $k_0, k_1, \ldots, k_{m-1}$ of elements of K such that $r_0 = s$, $r_m = t$, $r_0, r_1, \ldots, r_{m-1}$ are odd, r_m is even, $\langle r_j, k_j, r_{j+1}\rangle \in \varphi$ for $j = 0, 1, \ldots, m-1$ and $k_0 k_1 \ldots k_{m-1} = k$.

EXAMPLE 7. Let $\mathscr{F}$ be the set of all partial mappings of $\boldsymbol{N}$ into the set of subsets of $\boldsymbol{N}$, let $\varphi\psi$ be defined by the conditions

$$\mathrm{Dom}(\varphi\psi) = \{s\colon s \in \mathrm{Dom}\,\psi \ \& \ \psi(s) \subseteq \mathrm{Dom}\,\varphi\},$$
$$s \in \mathrm{Dom}(\varphi\psi) \Rightarrow \varphi\psi(s) = \bigcup \{\varphi(r)\colon r \in \psi(s)\}$$

and let $\varphi \leqslant \psi$ means that ψ is an extension of φ. Let

$$A(s) = \{2s\}, \quad B(s) = \{2s+1\}, \quad C(s) = \{[s/2]\}$$

for all s in N, let

$$(\varphi \veebar \psi)(s) \simeq \begin{cases} \varphi(s), & \text{if } s \text{ is even,} \\ \psi(s), & \text{if } s \text{ is odd,} \end{cases}$$

and let $*\varphi$ be defined by the conditions: $\mathrm{Dom}(*\varphi)$ is the least set Q of natural numbers containing all even numbers and such that $s \in Q$, whenever s is an odd number, $s \in \mathrm{Dom}\,\varphi$ and $\varphi(s) \subseteq Q$; if $s \in \mathrm{Dom}(*\varphi)$, then $t \in (*\varphi)(s)$ iff there is a finite sequence $r_0, r_1, \ldots, r_m$ of natural numbers such that $r_0 = s$, $r_m = t$, $r_0, r_1, \ldots, r_{m-1}$ are odd, r_m is even, $r_j \in \mathrm{Dom}\,\varphi$ and $r_{j+1} \in \varphi(r_j)$ for $j = 0, 1, \ldots, m-1$.

We shall complete this list of examples by giving a less concrete one (concrete instances of this example can be obtained using the examples of iterative combinatory spaces given in [6]–[10]).

Example 8. Let $\langle \mathscr{F}_0, \mathscr{C}, \Pi, L, R, \Sigma, T, F, \leqslant \rangle$ be an iterative combinatory space in the sense of [7]. Denote by $\mathscr{F}$ the semi-group $\mathscr{F}_0$ considered together with the partial ordering $\leqslant$ and let I be the identity of this semi-group, $A = (T, I)$, $B = (F, I)$, $C = R$, $\varphi \veebar \psi = (L \supset \varphi, \psi)$, $*\varphi = [\varphi, L]$. It is worth mentionning that conditions (vi), (vii) and (xi) of Definition 1 of [7] are not necessary for this example. Also the condition on ψ in Definition 5 of [7] can be replaced by the following weaker condition: $\psi = (\chi \supset I, \psi\varphi)$ and ψ belongs to each subset of $\mathscr{F}$ having the form $\{\theta\colon \gamma\theta \leqslant \varrho\}$ and closed under the mapping $\theta \mapsto (\chi \supset I, \theta\varphi)$ (the second part of this condition can be considered as a very special case of D. Scott's μ-induction rule).

Suppose now $\langle \mathscr{F}, A, B, C, \veebar, * \rangle$ be an arbitrary programming space. We shall prove some lemmas.

Lemma 1. *For all φ in $\mathscr{F}$, we have $(*\varphi)A = A$, $(*\varphi)B = (*\varphi)\varphi B$.*

Proof. We use the equality $*\varphi = I \veebar (*\varphi)\varphi$ and condition (ii) of Definition 1.

Lemma 2. *The mapping $\varphi \mapsto *\varphi$ is increasing.*

Proof. Let $\varphi_1 \leqslant \varphi_2$. Then $I \veebar (*\varphi_2)\varphi_1 \leqslant I \veebar (*\varphi_2)\varphi_2 = \varphi_2$ (by condition (iv) of Definition 1). Hence, by condition (vi) of Definition 1, we have the inequality $*\varphi_1 \leqslant *\varphi_2$.

Definition 2. A mapping Γ of $\mathscr{F}^n$ into $\mathscr{F}$ is called *left-homogeneous* if $\Gamma(\varrho\theta_1, \ldots, \varrho\theta_n) = \varrho\Gamma(\theta_1, \ldots, \theta_n)$ for all $\theta_1, \ldots, \theta_n$ and ϱ in $\mathscr{F}$.

The operation $\veebar$ is a left-homogeneous mapping of $\mathscr{F}^2$ into $\mathscr{F}$. If $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and δ is some element of $\mathscr{F}$, then the mapping

$\Gamma(\theta_1, \ldots, \theta_n) = (\theta_i \veebar \theta_j)\delta$ is a left-homogeneous mapping of $\mathscr{F}^n$ into $\mathscr{F}$. Trivial examples of left-homogeneous mappings are the mappings of the form $\Gamma(\theta_1, \ldots, \theta_n) = \theta_i\delta$, where $1 \leqslant i \leqslant n$ and δ is some element of $\mathscr{F}$. The study of left-homogeneous mappings gives a way for the algebraic treatment of flow diagrams. Namely, the tail functions (cf. [5] and [1]–[3]) of a flow diagram satisfy a system of equations (called the characteristic system of the diagram) which has the form

$$\theta_i = \Gamma_i(\theta_1, \ldots, \theta_m, I), \quad i = 1, \ldots, m,$$

where $\Gamma_1, \ldots, \Gamma_m$ are left-homogeneous mappings in the corresponding semi-group of partial mappings (as a matter of fact $\Gamma_1, \ldots, \Gamma_m$ have a very special form and each of them depends only on one or two of its arguments).

Lemma 3. *Let Γ be a left-homogeneous mapping of $\mathscr{F}^2$ into $\mathscr{F}$. Then*

$$\Gamma(\varphi, \psi) = C*(A\psi C)C*(A^2\varphi C^2)\Gamma(BA, AB)$$

for all φ and ψ in $\mathscr{F}$.

Proof. Using Lemma 1 we prove that

$$C*(A\psi C)C*(A^2\varphi C^2)BA = \varphi, \quad C*(A\psi C)C*(A^2\varphi C^2)AB = \psi.$$

Then we use the left-homogenity of Γ.

Remark 2. Lemma 3 gives a generalization of the expression for conditionals by means of composition and iteration given in [4].

Lemma 4. *If $\varphi_1 \leqslant \varphi_2$ and $\psi_1 \leqslant \psi_2$, then $\varphi_1 \veebar \psi_1 \leqslant \varphi_2 \veebar \psi_2$.*

Proof. We apply Lemma 3 to the mapping $\Gamma(\varphi, \psi) = \varphi \veebar \psi$ and then we use Lemma 2.

Definition 3. Let

$$\langle\!\langle \theta \rangle\!\rangle = \theta, \quad \langle\!\langle \theta_1, \ldots, \theta_m, \theta_{m+1} \rangle\!\rangle = \theta_{m+1}C \veebar \langle\!\langle \theta_1, \ldots, \theta_m \rangle\!\rangle C,$$

$$\zeta_{1,1} = I, \quad \zeta_{m+1,i} = B\zeta_{m,i} \quad \text{for} \quad i = 1, \ldots, m, \quad \zeta_{m+1,m+1} = A.$$

Lemma 5. *If $\varphi_1 \leqslant \psi_1, \ldots, \varphi_m \leqslant \psi_m$, then*

$$\langle\!\langle \varphi_1, \ldots, \varphi_m \rangle\!\rangle \leqslant \langle\!\langle \psi_1, \ldots, \psi_m \rangle\!\rangle.$$

Proof. By induction, using Lemma 4.

Lemma 6. *For each positive natural number m, the mapping*

$$\Gamma(\theta_1, \ldots, \theta_m) = \langle\!\langle \theta_1, \ldots, \theta_m \rangle\!\rangle$$

is left-homogeneous.

Proof. By induction, using condition (iii) of Definition 1.

LEMMA 7. *If* $1 \leqslant i \leqslant m$, *then* $\langle\!\langle \theta_1, \ldots, \theta_m \rangle\!\rangle \zeta_{m,i} = \theta_i$ *for all* $\theta_1, \ldots, \theta_m$ *in* $\mathcal{F}$.

Proof. By induction, using conditions (ii) and (i) of Definition 1.

LEMMA 8. *Let* Γ *be a left-homogeneous mapping of* $\mathcal{F}^m$ *into* $\mathcal{F}$. *Then*

$$\Gamma(\theta_1, \ldots, \theta_m) = \langle\!\langle \theta_1, \ldots, \theta_m \rangle\!\rangle \Gamma(\zeta_{m,1}, \ldots, \zeta_{m,m})$$

for all $\theta_1, \ldots, \theta_m$ *in* $\mathcal{F}$.

Proof. By application of Lemma 7.

From Lemmas 5 and 8 we can obtain the following generalization of Lemma 4: *If* Γ *is a left-homogeneous mapping of* $\mathcal{F}^m$ *into* $\mathcal{F}$ *and* $\varphi_1 \leqslant \psi_1, \ldots, \varphi_m \leqslant \psi_m$, *then* $\Gamma(\varphi_1, \ldots, \varphi_m) \leqslant \Gamma(\psi_1, \ldots, \psi_m)$.

The main result of this paper reads as follows:

THEOREM 1. *Let* $\Gamma_1, \ldots, \Gamma_m$ *be left-homogeneous mappings of* $\mathcal{F}^{m+1}$ *into* $\mathcal{F}$ *and let* $\gamma \in \mathcal{F}$. *Let*

$$\lambda_i = \Gamma_i(\zeta_{m+1,1}, \ldots, \zeta_{m+1,m+1}), \quad i = 1, \ldots, m,$$
$$\sigma = \langle\!\langle \lambda_1, \ldots, \lambda_m \rangle\!\rangle C,$$
$$\varphi_i = \gamma C(*\sigma)\zeta_{m+1,i}, \quad i = 1, \ldots, m.$$

Then

$$\varphi_i = \Gamma_i(\varphi_1, \ldots, \varphi_m, \gamma), \quad i = 1, \ldots, m,$$

and if the elements $\psi_1, \ldots, \psi_m$ *of* $\mathcal{F}$ *satisfy the inequalities*

$$\psi_i \geqslant \Gamma_i(\psi_1, \ldots, \psi_m, \gamma), \quad i = 1, \ldots, m,$$

then

$$\varphi_1 \leqslant \psi_1, \ldots, \varphi_m \leqslant \psi_m.$$

Proof. By Lemma 1, we have $\gamma C(*\sigma)\zeta_{m+1,m+1} = \gamma C(*\sigma)A = \gamma CA = \gamma$. Hence, using Lemmas 1 and 7, we can obtain

$$\begin{aligned}\varphi_i = \gamma C(*\sigma)B\zeta_{m,i} &= \gamma C(*\sigma)\sigma B\zeta_{m,i} \\ &= \gamma C(*\sigma)\langle\!\langle \lambda_1, \ldots, \lambda_m \rangle\!\rangle \zeta_{m,i} = \gamma C(*\sigma)\lambda_i \\ &= \Gamma_i\big(\gamma C(*\sigma)\zeta_{m+1,1}, \ldots, \gamma C(*\sigma)\zeta_{m+1,m+1}\big) \\ &= \Gamma_i(\varphi_1, \ldots, \varphi_m, \gamma).\end{aligned}$$

Now, let $\psi_1, \ldots, \psi_m$ be arbitrary elements of $\mathcal{F}$ satisfying the inequalities $\psi_i \geqslant \Gamma_i(\psi_1, \ldots, \psi_m, \gamma)$, $i = 1, \ldots, m$. Set $\delta = \langle\!\langle \psi_1, \ldots, \psi_m, \gamma \rangle\!\rangle$. If $1 \leqslant i \leqslant m$, then, using Lemma 7, we obtain

$$\delta\lambda_i = \Gamma_i(\delta\zeta_{m+1,1}, \ldots, \delta\zeta_{m+1,m+1}) = \Gamma_i(\psi_1, \ldots, \psi_m, \gamma) \leqslant \psi_i.$$

From here, applying Lemmas 6 and 5, we have

$$\begin{aligned}\gamma C \veebar \delta\sigma &= \gamma C \veebar \langle\!\langle \delta\lambda_1, \ldots, \delta\lambda_m \rangle\!\rangle C \leqslant \gamma C \veebar \langle\!\langle \psi_1, \ldots, \psi_m \rangle\!\rangle C \\ &= \langle\!\langle \psi_1, \ldots, \psi_m, \gamma \rangle\!\rangle = \delta.\end{aligned}$$

Using condition (vi) of Definition 1, we conclude that $\gamma C(*\sigma) \leqslant \delta$ and hence $\varphi_i \leqslant \delta\zeta_{m+1,i} = \psi_i$ for $i = 1, \ldots, m$.

In the programming space considered in Example 3, the elements φ_i constructed in Theorem 1 can be expressed by means of composition, conditionals, iteration and the elements $\gamma, A, B, C, \lambda_1, \ldots, \lambda_m$. If $\theta_i = \Gamma_i(\theta_1, \ldots, \theta_m, \gamma)$, $i = 1, \ldots, m$, is the characteristic system of some flow diagram, then $\gamma = I$ and the operations $\Gamma_1, \ldots, \Gamma_m$ are expressed by means of composition and conditionals. But conditionals can be expressed by means of composition, iteration and some fixed elements of $\mathcal{F}$, as shown in [4] (cf. Remark 2 above). Since the tail functions of a flow diagram form the least solution of its characteristic system, we see that Theorem 1 contains the essential part of Böhm and Jacopini's result about reduction of flow diagrams which is given in [4].

From Theorem 1 we easily obtain the following

COROLLARY. *Let* $\Gamma_1, \ldots, \Gamma_m$ *be left-homogeneous mappings of* $\mathcal{F}^{m+p}$ *into* $\mathcal{F}$ *and let* $\gamma_1, \ldots, \gamma_p$ *belong to* $\mathcal{F}$. *Let*

$$\lambda_i = \Gamma_i(\zeta_{m+1,1}, \ldots, \zeta_{m+1,m}, \zeta_{m+1,m+1}\zeta_{p,1}, \ldots, \zeta_{m+1,m+1}\zeta_{p,p}),$$
$$\sigma = \langle\!\langle \lambda_1, \ldots, \lambda_m \rangle\!\rangle C,$$
$$\varphi_i = \langle\!\langle \gamma_1, \ldots, \gamma_p \rangle\!\rangle C(*\sigma)\zeta_{m+1,i}, \quad i = 1, \ldots, m.$$

Then

$$\varphi_i = \Gamma_i(\varphi_1, \ldots, \varphi_m, \gamma_1, \ldots, \gamma_p), \quad i = 1, \ldots, m,$$

and if the elements $\psi_1, \ldots, \psi_m$ *of* $\mathcal{F}$ *satisfy the inequalities*

$$\psi_i \geqslant \Gamma_i(\psi_1, \ldots, \psi_m, \gamma_1, \ldots, \gamma_p), \quad i = 1, \ldots, m,$$

then

$$\varphi_1 \leqslant \psi_1, \ldots, \varphi_m \leqslant \psi_m.$$

For obtaining this corollary, it is sufficient to apply Theorem 1 to the mappings $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_m$ defined by

$$\bar{\Gamma}_i(\theta_1, \ldots, \theta_m, \theta_{m+1}) = \Gamma_i(\theta_1, \ldots, \theta_m, \theta_{m+1}\zeta_{p,1}, \ldots, \theta_{m+1}\zeta_{p,p})$$

and to the element $\gamma = \langle\!\langle \gamma_1, \ldots, \gamma_p \rangle\!\rangle$.

Using the corollary of Theorem 1 we can prove that least solutions of systems of the form

$$\theta_i = \Gamma_i(\theta_1, \ldots, \theta_m, \gamma_1, \ldots, \gamma_p), \quad i = 1, \ldots, m,$$

where $\Gamma_1, \ldots, \Gamma_m$ are left-homogeneous, can be found by the method of elimination. The proof can be based on the following proposition about increasing mappings in partially ordered sets (cf. [10], Proposition 1.1.7 of Ch. III):

Let $\mathscr{F}'$ and $\mathscr{F}''$ be partially ordered sets. Consider the set $\mathscr{F}' \times \mathscr{F}''$ with the partial ordering defined in the following way: $\langle \varphi', \varphi'' \rangle \leqslant \langle \psi', \psi'' \rangle$ iff $\varphi' \leqslant \psi'$ and $\varphi'' \leqslant \psi''$. Let Γ' and Γ'' be increasing mappings of the partially ordered set $\mathscr{F}' \times \mathscr{F}''$ into the partially ordered sets $\mathscr{F}'$ and $\mathscr{F}''$ respectively. Let for each θ'' in $\mathscr{F}''$ the inequality $\Gamma'(\theta', \theta'') \leqslant \theta'$ has a least solution $\theta' = B(\theta'')$ in $\mathscr{F}'$. Let $\theta'' = \varphi''$ be the least solution of the inequality $\Gamma''\big(B(\theta''), \theta''\big) \leqslant \theta''$ in $\mathscr{F}''$. Then the pair $\langle B(\varphi''), \varphi'' \rangle$ is the least solution of the inequality $\Gamma(\theta) \leqslant \theta$ in $\mathscr{F}' \times \mathscr{F}''$ and the least fixed point of the mapping Γ, where Γ is defined by the equality

$$\Gamma(\theta', \theta'') = \langle \Gamma'(\theta', \theta''), \Gamma''(\theta', \theta'') \rangle.$$

Now we shall show an application of Theorem 1 to the generalized recursion theory developped in [7] and [10]. In order to do this, we shall first prove the following statement:

LEMMA 9. *Let $\pi \in \mathscr{F}$ and Γ be a left-homogeneous mapping of $\mathscr{F}^3$ into $\mathscr{F}$. Let θ_0 be the least θ such that $\theta = \Gamma(I, \theta, \theta\pi B)$ and τ_0 be the least τ such that $\tau = \Gamma(A, \tau, B)$. Then $\theta_0 = C*(\tau_0\pi)\tau_0$.*

Proof. Let $\iota = *(\tau_0\pi)$. By the left-homogenity of Γ and Lemma 1, we have

$$C\iota\tau_0 = \Gamma(C\iota A, C\iota\tau_0, C\iota B) = \Gamma(I, C\iota\tau_0, C\iota\tau_0\pi B).$$

Hence $\theta_0 \leqslant C\iota\tau_0$. For proving an inequality in the opposite direction, note that by Theorem 1 we have

$$\tau_0 = C(*\sigma)\zeta_{2,1},$$

where

$$\sigma = \Gamma(\zeta_{2,2}A, \zeta_{2,1}, \zeta_{2,2}B)C.$$

Let $\varkappa = C \veebar \theta_0\pi$. Then $\varkappa A = CA = I$, $\varkappa B = \theta_0\pi B$. Hence

$$\begin{aligned}\langle\!\langle \theta_0, \varkappa \rangle\!\rangle \sigma &= \Gamma(\langle\!\langle \theta_0, \varkappa \rangle\!\rangle \zeta_{2,2}A, \langle\!\langle \theta_0, \varkappa \rangle\!\rangle \zeta_{2,1}, \langle\!\langle \theta_0, \varkappa \rangle\!\rangle \zeta_{2,2}B)C \\ &= \Gamma(\varkappa A, \theta_0, \varkappa B)C = \Gamma(I, \theta_0, \theta_0\pi B)C = \theta_0 C\end{aligned}$$

and therefore

$$\varkappa C \veebar \langle\!\langle \theta_0, \varkappa \rangle\!\rangle \sigma = \varkappa C \veebar \theta_0 C = \langle\!\langle \theta_0, \varkappa \rangle\!\rangle.$$

By condition (vi) of Definition 1 we conclude that

$$\varkappa C(*\sigma) \leqslant \langle\!\langle \theta_0, \varkappa \rangle\!\rangle.$$

Therefore

$$\varkappa\tau_0 \leqslant \langle\!\langle \theta_0, \varkappa \rangle\!\rangle \zeta_{2,1} = \theta_0$$

and hence

$$C \veebar \varkappa\tau_0\pi \leqslant C \veebar \theta_0\pi = \varkappa.$$

Applying once more condition (vi) of Definition 1, we get $C\iota \leqslant \varkappa$. Hence $C\iota\tau_0 \leqslant \varkappa\tau_0 \leqslant \theta_0$.

In the rest of the paper, we shall suppose that $\langle \mathscr{F}_0, \mathscr{C}, \Pi, L, R, \Sigma, T, F, \leqslant \rangle$ is an iterative combinatory space in the sense of [7] (we could also use the less restrictive assumptions given in Example 8 above, together with the condition that $(x, I)\theta = (x, \theta)$ for all x in $\mathscr{C}$ and all θ in $\mathscr{F}_0$). We shall also suppose that Ψ is some subset of $\mathscr{F}_0$. A mapping H of $\mathscr{F}_0$ into $\mathscr{F}_0$ will be called *normally representable with respect to Ψ* if there are elements $\alpha, \beta, \gamma, \chi$ and ν of $\mathscr{F}_0$ recursive in Ψ in the sense of [7] and such that

$$H(\omega) = \gamma[\beta(I, \omega\nu), \chi]\alpha$$

for all ω in $\mathscr{F}_0$. The promised application to generalized recursion theory reads as follows:

THEOREM 2. *Let H be a mapping of $\mathscr{F}_0$ into $\mathscr{F}_0$ which is normally representable with respect to Ψ and let $\varkappa$ be an element of $\mathscr{F}_0$ which is recursive in Ψ. Then the mapping $\omega \mapsto [H(\omega), \varkappa]$ is also normally representable with respect to Ψ.*

Proof. Let $\alpha, \beta, \gamma, \chi$ and ν be such as in the definition of normal representability formulated above. Having in mind the programming space described in Example 8, we shall apply Lemma 9 to the mapping Γ defined by

$$\Gamma(\theta_1, \theta_2, \theta_3) = \big(\chi \supset (\varkappa \supset \theta_1, \theta_2\alpha)\gamma, \theta_3\big)$$

and to the element $\pi = \beta_1(I, \omega\nu R)$, where ω is an arbitrary element of $\mathscr{F}_0$ and $\beta_1 = \beta(RL, R)$. We shall first show that $\pi B = \beta(I, \omega\nu)$. Let x be an arbitrary element of $\mathscr{C}$. Then

$$\pi Bx = \pi(F, x) = \beta_1\big((F, x), \omega\nu x\big) = \beta_1\big((F, x), I\big)\omega\nu x.$$

But $\beta_1\big((F, x), I\big)' = \beta(x, I)$ since for all y in $\mathscr{C}$ we have

$$\beta_1\big((F, x), I\big)y = \beta_1\big((F, x), y\big) = \beta(x, y) = \beta(x, I)y.$$

Therefore

$$\pi Bx = \beta(x, I)\omega\nu x = \beta(x, \omega\nu x) = \beta(I, \omega\nu)x.$$

The equality $\pi B = \beta(I, \omega\nu)$ is thus established. Denote by τ_0 the least τ such that $\tau = \Gamma(A, \tau, B)$. From the expression for τ_0 given by Theorem 1 we see that τ_0 is recursive in Ψ. We shall prove that

$$[H(\omega), \varkappa] = R[\tau_0\beta_1(I, \omega\nu R), L](\varkappa \supset A, \tau_0\alpha).$$

Let $\iota = [\tau_0\beta_1(I, \omega\nu R), L]$, i.e. $\iota = [\tau_0\pi, L]$. By Lemma 9 and Theorem 1, the element $R\iota\tau_0$ will be the least θ such that $\theta \geqslant \Gamma(I, \theta, \theta\pi B)$. Consider now the system of inequalities

$$(\varkappa \supset I, \theta\alpha) \leqslant \bar{\theta},$$

$$(\chi \supset \bar{\theta}\gamma, \theta\pi B) \leqslant \theta.$$

For each θ in $\mathscr{F}_0$, the element $(\varkappa \supset I, \theta\alpha)$ is the least solution of the first inequality with respect to $\bar{\theta}$. By substitution of this element instead of $\bar{\theta}$ in the second inequality we get the inequality $\Gamma(I, \theta, \theta\pi B) \leqslant \theta$ and its least solution with respect to θ is $R\iota\tau_0$. By the proposition formulated after the corollary of Theorem 1, the least solution of the considered system of equations simultaneously with respect to $\bar{\theta}$ and θ is

$$\bar{\theta} = (\varkappa \supset I, R\iota\tau_0\alpha), \qquad \theta = R\iota\tau_0.$$

On the other hand, we could solve the same system also by elimination of θ. For each $\bar{\theta}$ in $\mathscr{F}_0$, the element $\bar{\theta}\gamma[\pi B, \chi]$ is the least θ such that $(\chi \supset \bar{\theta}\gamma, \theta\pi B) \leqslant \theta$ (this follows easily from the definition of iteration). By substitution of $\bar{\theta}\gamma[\pi B, \chi]$ instead of θ in the inequality $(\varkappa \supset I, \theta\alpha) \leqslant \bar{\theta}$ we obtain the inequality $\big(\varkappa \supset I, \bar{\theta}H(\omega)\big) \leqslant \bar{\theta}$ and its least solution with respect to $\bar{\theta}$ is $[H(\omega), \varkappa]$. Thus the least solution of the considered system of equations simultaneously with respect to $\bar{\theta}$ and θ can be written also in the form

$$\bar{\theta} = [H(\omega), \varkappa], \qquad \theta = [H(\omega), \varkappa]\gamma[\pi B, \chi].$$

By comparing the two expressions for $\bar{\theta}$ we obtain that

$$[H(\omega), \varkappa] = (\varkappa \supset I, R\iota\tau_0\alpha) = R\iota(\varkappa \supset A, \tau_0\alpha).$$

The proof of Theorem 2 is completed.

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