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WEAK AUTOMORPHISMS OF 1-UNARY ALGEBRAS

LIBOR POLÁK

Department of Algebra and Geometry, Faculty of Science of JEP University, 66295 Brno, Czechoslovakia

Groups of weak automorphisms of 1-unary algebras have been described only in the case of free algebras (see [5] and [1]). Proposition 1 of this note generalizes these results and gives a description for the general case. Proposition 2 tells which groups are of the form "group of all weak automorphisms modulo group of all automorphisms" for 1-unary algebras, Proposition 3 shows that the class of all groups of weak automorphisms of 1-unary algebras is at least as rich as the class of all groups of automorphisms of these, and Proposition 4 ensures that this class does not contain all groups.

I would like to express my thanks for inspiration to my students R. Ptáčník and Z. Svoboda, the authors of [4]. In fact, they have studied groups of weak automorphisms for some 1-unary algebras but without the help of groups of automorphisms of those algebras.

Note that groups of automorphisms of 1-unary algebras were characterized in [2].

For a 1-unary algebra $\mathfrak{A} = (A, f)$ let $\mathscr{A}\mathfrak{A} = (A\mathfrak{A}, \cdot)$ and $\mathscr{W}\mathfrak{A} = (W\mathfrak{A}, \cdot)$ denote the group of all automorphisms and the group of all weak automorphisms of it, respectively. Let f^n stand for the nth iteration of f (i.e. $f^0 = \mathrm{id}_A$, $f^{n+1}a = f(f^n a)$ for all $a \in A$, $n = 1, 2, \ldots$). By N we denote the set of all positive natural numbers and let $\mathscr{Z}_d = (Z_d = \{0, 1, \ldots \ldots, d-1\}, \cdot)$ be a semigroup, where the operation $\cdot$ is the usual multiplication modulo d, $d \in N$.

PROPOSITION 1. *Let $\mathfrak{A} = (A, f)$ be a 1-unary algebra.*

(1) *If for no $n \in N$ $f^{n+1} = f$, then any weak automorphism of $\mathfrak{A}$ is an automorphism of $\mathfrak{A}$.*

(2) *Let d be the smallest $n \in N$ such that $\mathfrak{A}$ satisfies $f^{n+1} = f$. Then the set*

$Z\mathfrak{A} = \{n \in Z_d |$ *n is invertible in* $\mathscr{Z}_d$ *and algebras* (A, f) *and* (A, f^n) *are isomorphic*$\}$

forms, with respect to multiplications, a subgroup $\mathscr{Z}\mathfrak{A}$ *of semigroup* $\mathscr{Z}_d$, *and* $\mathscr{W}\mathfrak{A}$ *is isomorphic to a semidirect product of* $\mathscr{A}\mathfrak{A}$ *by* $\mathscr{Z}\mathfrak{A}$.

Proof. (1) If $\alpha \in W\mathfrak{A} - A\mathfrak{A}$, then there exist integers $m, n \geqslant 2$ such that

$$(*) \qquad \alpha f = f^n \alpha \quad \text{and} \quad f\alpha = \alpha f^m.$$

Thus we have $f^{mn} = f$ and (1) is proved.

(2) $\mathscr{Z}\mathfrak{A}$ is a finite abelian group. Let $\mathscr{Z}\mathfrak{A} = \mathscr{G}_1 \times \ldots \times \mathscr{G}_k$ be its decomposition into a direct product of its cyclic subgroups, $\mathscr{G}_i = (g_i)$ and let $\varrho_{g_i}\colon (A, f) \to (A, f^{g_i})$ be an isomorphism such that $\varrho_{g_i}^{g_i^{-1}} = 1_A$ and $\varrho_{g_i}\varrho_{g_j} = \varrho_{g_j}\varrho_{g_i}$. For $n = g_1^{n_1} \cdot \ldots \cdot g_k^{n_k} \in Z\mathfrak{A}$ is $\varrho_n = \varrho_1^{n_1} \cdot \ldots \cdot \varrho_k^{n_k}$ an isomorphism of (A, f) onto (A, f^n) (i.e. a weak automorphism of $\mathfrak{A}$ satisfying $(*)$). Thus, $\mathscr{W}\mathfrak{A}$ is isomorphic to $(A\mathfrak{A} \times Z\mathfrak{A}, \circ)$, where $(\alpha, m) \circ (\beta, n) = (\alpha\varrho_m\beta\varrho_m^{-1}, mn)$.

PROPOSITION 2. *For any subgroup* $\mathscr{G}$ *of the semigroup* $\mathscr{Z}_d$, $d \in N$, *there exists a* 1-*unary algebra* $\mathfrak{A}$ *such that* $\mathscr{W}\mathfrak{A}/\mathscr{A}\mathfrak{A}$ *is isomorphic to* $\mathscr{G}$.

Proof. Let

$$\mathscr{G} = (\{n_1, \ldots, n_k\}, \cdot) \quad \text{and} \quad B = \{0, 1, \ldots, d-1, 0_0, 1_0, 1_1, 2_0, 2_1, 2_2, \ldots, (d-1)_{d-1}\},$$

where all symbols are considered pairwise different.

We define

$$g(i) = i+1 \pmod d, \quad \text{and} \quad g(i_j) = i, \quad \text{for all } i, j \in Z_d,\ i \geqslant j.$$

Let $\mathfrak{A} = (A, f)$ be a 1-unary algebra with connected components isomorphic to $(B, g^{n_1}), (B, g^{n_2}), \ldots, (B, g^{n_k})$.

It is easy to check that $\mathscr{Z}\mathfrak{A}$ is isomorphic to $\mathscr{G}$.

PROPOSITION 3. *For any* 1-*unary algebra* $\mathfrak{A}$ *there exists a* 1-*unary algebra* $\mathfrak{B}$ *such that* $\mathscr{W}\mathfrak{B}$ *is isomorphic to* $\mathscr{A}\mathfrak{A}$.

Proof. Let $\mathfrak{A} = (A, f)$ be a 1-unary algebra with $W\mathfrak{A} \neq A\mathfrak{A}$. For any $a \in A$ we add to A two new elements a' and a'' and define $fa'' = a'$, $fa' = a$. Denoting the resulting algebra by $\mathfrak{B}$, we have $\mathscr{W}\mathfrak{B} = \mathscr{A}\mathfrak{B} \cong \mathscr{A}\mathfrak{A}$.

PROPOSITION 4. *There is no* 1-*unary algebra* $\mathfrak{A}$ *with* $\mathscr{W}\mathfrak{A}$ *isomorphic to an additive group of rationals*.

Proof. By [3], Corollary 3.2.3, there is on 1-unary algebra $\mathfrak{A}$ with $\mathscr{A}\mathfrak{A} \cong (Q, +)$. Moreover, $(Q, +)$ is not a semidirect product of its subgroup by a non-tirival finite group. Now, the assertion follows from Proposition 1.

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